Nonlinear problems with blow-up solutions: Numerical integration based on differential and nonlocal transformations, and differential constraints *

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Abstract

Several new methods of numerical integration of Cauchy problems with blow-up solutions for nonlinear ordinary differential equations of the first- and second-order are described. Solutions of such problems have singularities whose positions are unknown a priori (for this reason, the standard numerical methods for solving problems with blow-up solutions can lead to significant errors). The first proposed method is based on the transition to an equivalent system of equations by introducing a new independent variable chosen as the first derivative, $t = y'_x$, where x and y are independent and dependent variables in the original equation. The second method is based on introducing a new auxiliary nonlocal variable of the form $\xi = \int_{x_0}^{x} g(x, y, y'_x)\, dx$ with the subsequent transformation to the Cauchy problem for the corresponding system of ODEs. The third method is based on adding to the original equation of a differential constraint, which is an auxiliary ODE connecting the given variables and a new variable. The proposed methods lead to problems whose solutions are represented in parametric form and do

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not have blowing-up singular points; therefore the transformed problems admit the application of standard fixed-step numerical methods. The efficiency of these methods is illustrated by solving a number of test problems that admit an exact analytical solution. It is shown that: (i) the methods based on nonlocal transformations of a special kind are more efficient than several other methods, namely, the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation, and (ii) among the proposed methods, the most general method is the method based on the differential constraints. Some examples of nonclassical blow-up problems are considered, in which the right-hand side of equations has fixed singular points or zeros. Simple theoretical estimates are derived for the critical value of an independent variable bounding the domain of existence of the solution. It is shown by numerical integration that the first and the second Painlevé equations with suitable initial conditions have non-monotonic blow-up solutions. It is demonstrated that the method based on a nonlocal transformation of the general form as well as the method based on the differential constraints admit generalizations to the n th-order ODEs and systems of coupled ODEs.

Keywords: blow-up solutions, nonlinear ordinary differential equations, Cauchy problem, numerical integration, nonlocal transformations, Painlevé equations

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1. Introduction

1.1. Preliminary remarks. Blow-up solutions

We will consider Cauchy problems for ordinary differential equations (briefly, ODEs), whose solutions tend to infinity at some finite value of the independent variable $x = x_*$, where x_* does not appear explicitly in the differential equation under consideration and it is not known in advance. Similar solutions exist on a bounded interval (hereinafter in this article we assume that $x_0 \le x < x_*$) and are called blow-up solutions. This raises the important question for practice: how to determine the position of a singular point x_* and the solution in its neighborhood using numerical methods.

In the general case, the blow-up solutions that have a power singularity can be represented in a neighborhood of the singular point x_* in the form

$$y \simeq A(x_* - x)^{-\beta}, \quad \beta > 0, \tag{1}$$

where A and β are some constants. For these solutions we have $\lim\limits_{x\to x_*} |y| = \infty$ and $\lim\limits_{x\to x_*} |y'_x| = \infty$.

Differentiating (1), we obtain the derivatives near the singular point

$$y'_x \simeq A\beta(x_* - x)^{-\beta-1}, \quad y''_{xx} \simeq A\beta(\beta+1)(x_* - x)^{-\beta-2}. \tag{2}$$

It follows from (1) and (2) that the approximate relations

$$\frac{y'_x}{y} \simeq \frac{\beta}{x_* - x}, \qquad \frac{yy''_{xx}}{(y'_x)^2} \simeq \frac{\beta+1}{\beta} \tag{3}$$

are valid near the singular point x_*. From the first relation in (3) we have the limiting property $\lim\limits_{x \to x_*} (y'_x/y) = \infty$, which is common for any blow-up solution. The second relation in (3) can be used for computing the exponent β in performing numerical calculations.

The formulas (1)–(3) remain valid also for non-monotonic blow-up solutions if there is a neighborhood on the left of the singular point ($x_1 \le x < x_*$, where $x_0 \le x_1$), in which the solution is monotonic.

Example 1. Consider the test Cauchy problem for the first-order nonlinear ODE with separable variables

$$y'_x = y^2 \quad (x > 0), \qquad y(0) = 1. \tag{4}$$

The exact solution of this problem has the form

$$y = \frac{1}{1-x}. \tag{5}$$

It has a power-type singularity (a first-order pole) at the point $x_* = 1$ and does not exist for $x > x_*$.

The Cauchy problem (4) is a particular case of the three-parameters problem

$$y'_x = by^\gamma \quad (x > 0), \quad y(0) = a, \tag{6}$$

where a, b, and γ are arbitrary constants. If the inequalities

$$a > 0, \quad b > 0, \quad \gamma > 1 \tag{7}$$

are valid, then the exact solution of the problem (6) is given by the formula

$$y = A(x_* - x)^{-\beta}, \tag{8}$$

where

$$A = [b(\gamma-1)]^{\frac{1}{1-\gamma}}, \quad x_* = \frac{1}{a^{\gamma-1}b(\gamma-1)}, \quad \beta = \frac{1}{\gamma-1} > 0.$$

This solution exists on a bounded interval $0 \le x < x_*$, where x_* is a singular point of the pole-type solution, and does not exist for $x \ge x_*$. In this case, the solution (8) coincides with its asymptotic behavior in a neighborhood of the singular point (compare (1) with (8)).

There exist problems that have blow-up solutions with a different type of singularity (that differs from (1)). In particular, solutions with a logarithmic singularity at the point x_* have the form

$$y \approx A \ln\bigl[B(x_* - x)\bigr],$$

where A and $B > 0$ are some constants.

Example 2. The test Cauchy problem with exponential nonlinearity

$$y'_x = be^y \quad (x > 0), \qquad y(0) = a \tag{9}$$

admits the exact solution with a logarithmic singularity

$$y = -\ln(e^{-a} - bx) \tag{10}$$

for $a \geq 0$ and $b > 0$. This solution exists on the interval $0 \leq x < x_* = e^{-a}/b$ and does not exist for $x \geq x_*$.

1.2. Problems arising in numerical solutions of blow-up problems

The direct application of the standard fixed-step numerical methods to blow-up problems leads to certain difficulties because their solutions have a singularity and the range of variation of the independent variable is unknown in advance [1]. The difficulties arising in the application of the classical Runge–Kutta methods for solving the test Cauchy problem (6) are described below (the results of [2] are used).

The qualitative behavior of the numerical blow-up solution for equations of the form (6) for $a > 0$, $b > 0$, and $\gamma > 1$ is significantly different for the explicit and implicit Runge–Kutta methods (the explicit methods up to the fourth-order of approximation and the Euler implicit method have been tested in [2]).

All the explicit methods provide monotonically increasing solutions; and the higher order of the approximation method, the faster growth of the numerical solution. Soon after passing through the singular point x_*, in which the exact solution has a pole, an overflow occurs in the calculation and further computing is impossible. Such qualitative behavior is unpleasant, since it is difficult for the researcher to determine the cause of the overflow.

For the implicit methods, the picture is different. First, the solution increases, but even before the pole it breaks down to the region of negative values. The calculation of the right-hand side of (6) for fractional values of γ becomes impossible (because a fractional power of a negative number occurs).

Various special methods have been proposed in the literature for numerical integration of problems that have blow-up solutions.

One of the basic ideas of numerical integration of blow-up problems consists in the application of an appropriate transformation at the initial stage, which leads to the equivalent problem for one differential equation or a system of coupled equations whose solutions have no singularities at a priori unknown point (after such transformations, the unknown singular point $x = x_*$ usually goes to the infinitely remote point for the new independent variable).

Currently, two methods based on this idea are most commonly used. The first method, based on the hodograph transformation, $x = \bar{y}$, $y = \bar{x}$ (where the independent and dependent variables are interchanged) was proposed in [3]. The second method of this kind, called the method of the arc-length transformation, is described in [4] (for details, see below Item 2° in Sections 3.1 and 7.1, as well as reference [5]). This method is rather general and it can be applied for numerical integration of systems of ordinary differential equations.

The methods based on the hodograph and arc-length transformations for blow-up solutions with a power singularity of the form (1) lead to the Cauchy problems whose solutions tends to the asymptote with respect to the power law for large values of the new independent variable. This creates certain difficulties in some problems, since one has to consider large intervals of variation of the independent variable in numerical integration.

Based on other ideas, some special methods of numerical integration of blow-up problems are described, for example, in [1, 2, 5–9, 11–14]. In particular, it was suggested in [8, 13] to investigate such problems via compactifications, which are point transformations of the special form (whose inverse transformations have singularities).

In this paper, we propose several new methods of numerical integration of Cauchy problems for the first- and second-order nonlinear equations, which have blow-up solutions. These methods are based on differential and nonlocal transformations, and also on differential constraints, allowing us to obtain the equivalent problems for systems of equations whose solutions do not have singularities at a priori unknown point. Some special methods based on nonlocal transformations and differential constraints lead to the Cauchy problems whose solutions, which are found in parametric form by numerical integration, tend exponentially to the asymptote for large values of the new independent variable. Therefore, these methods are more effective than the methods based on the hodograph and arc-length transformations, which lead to solutions that are quite slowly (by the power law) tend to the asymptote for large values of the independent variable.

The presentation of the material is widely illustrated with test problems that admit an exact solution. Two-sided theoretical estimates are established for the critical value of the independent variable $x = x_*$, when an unlimited growth of the solution occurs as approaching it. It is shown that the method based on a nonlocal transformation of the general form as well as the method based on the differential constraints admit generalizations to the n th-order ordinary differential equations and systems of differential equations.

Remark 1. The works [14, 15] are devoted to investigation of problems with oscillating blow-up solutions (having an infinite set of local extrema) for some classes of the second-order equations. More complicated problems (described by fourth-order nonlinear ordinary differential equations) with oscillating blow-up solutions were considered in [16, 17]. The solutions of these problems in the neighborhood of the blow-up points have asymptotic behavior that substantially differs from (1). In this article, such problems are not considered.

2. Problems for first-order equations. Differential transformations

2.1. Solution method based on introducing a differential variable

The Cauchy problem for the first-order differential equation has the form

$$y'_x = f(x, y) \quad (x > x_0), \tag{11}$$

$$y(x_0) = y_0. \tag{12}$$

In what follows we assume that $f = f(x, y) > 0$, $x_0 \geq 0$, $y_0 > 0$, and also $f/y^{1+\varepsilon} \to \infty$ as $y \to \infty$, where $\varepsilon > 0$. In such problems, blow-up solutions arise when the right-hand side of a nonlinear equation is quite rapidly growing as $y \to \infty$.

First, we represent the nonlinear ODE (11) in the form of an equivalent system of differential-algebraic equations

$$t = f(x, y), \qquad y'_x = t, \tag{13}$$

where $y = y(x)$ and $t = t(x)$ are unknown functions to be determined.

By applying (13) and assuming that $y = y(t)$ and $x = x(t)$, we derive a system of ODEs of the standard form. By taking the full differential of the first equation of (13) and multiplying the second equation by dx, we get

$$dt = f_x\,dx + f_y\,dy, \qquad dy = t\,dx, \tag{14}$$

where f_x and f_y denote the corresponding partial derivatives of the function $f = f(x, y)$. Eliminating first dy and then dx from (14), we arrive at the ODE system of the first order

$$x'_t = \frac{1}{f_x + tf_y}, \quad y'_t = \frac{t}{f_x + tf_y} \quad (t > t_0), \tag{15}$$

which must be supplemented by the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = f(x_0, y_0). \tag{16}$$

Conditions (16) are derived from (12) and the first equation of (13).

Assuming that the conditions $f_x + tf_y > 0$ are valid at $t_0 < t < \infty$, the Cauchy problem (15)–(16) can be integrated numerically, for example, by applying the Runge–Kutta method or other standard fixed-step numerical methods (see, for example, [18–26]). In this case, the difficulties (described in Section 1.2) will not occur because of the presence of a singularity in the solutions (since $x'_t \to 0$ as $t \to \infty$). In view of (13), the singular point x_* of the solution corresponds to $t = \infty$, therefore the required value x_* is determined by the asymptotic behavior of the function $x = x(t)$ for large t.

Remark 2. Taking into account the first equation of (13), the system (15) can be represented in the form

$$x'_t = \frac{1}{f_x + tf_y}, \quad y'_t = \frac{f}{f_x + tf_y} \quad (t > t_0). \tag{17}$$

Another equivalent system of ODEs can be obtained by replacing t by f in (17).

2.2. *Test problem and numerical solutions*

Let us illustrate the method described in Section 2.1 with a simple example.

Example 3. Consider the test Cauchy problem (6)–(7). By introducing a new variable $t = y'_x$ in (6), we obtain the following Cauchy problem for the system of equations:

$$\begin{gathered} x'_t = \frac{1}{b\gamma t y^{\gamma-1}}, \quad y'_t = \frac{1}{b\gamma y^{\gamma-1}} \quad (t > t_0); \\ x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = a^\gamma b, \end{gathered} \tag{18}$$

which is a particular case of the problem (15)–(16) for $f = by^\gamma$, $x_0 = 0$, and $y_0 = a$. The exact solution of the problem (18) has the form

$$x = \frac{1}{b(\gamma - 1)}\left[a^{1-\gamma} - \left(\frac{b}{t}\right)^{\frac{\gamma-1}{\gamma}}\right], \quad y = \left(\frac{t}{b}\right)^{\frac{1}{\gamma}} \quad (t \ge a^\gamma b). \tag{19}$$

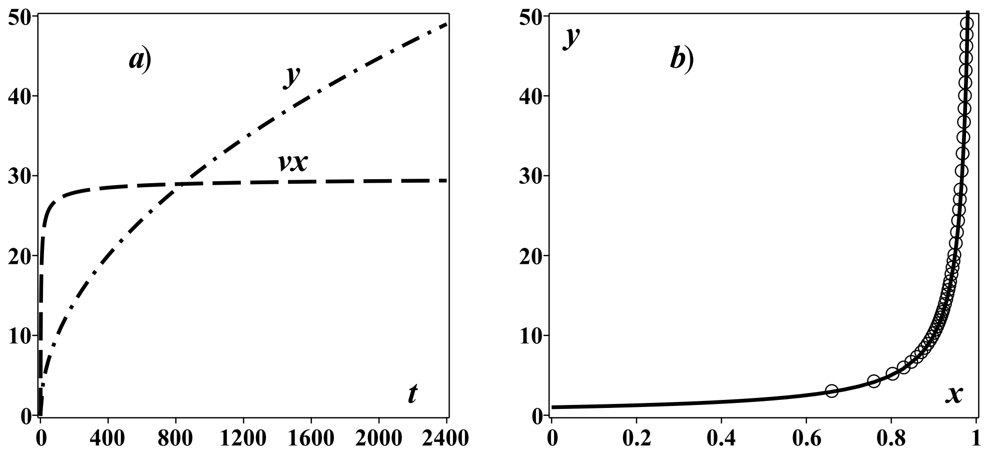

Figure 1: a)—the dependences $x = x(t)$ and $y = y(t)$ obtained by numerical solution of the problem (18) for $a = b = 1$ and $\gamma = 2$ ($\nu = 30$); b)—exact solution (5) (solid line) and numerical solution of the problem (18) for $a = b = 1$ and $\gamma = 2$ (circles).

It has no singularities; the function $x = x(t)$ increases monotonically with $t > a^\gamma b$, tending to the desired limiting value $x_* = \lim\limits_{t\to\infty} x(t) = \dfrac{1}{a^{\gamma-1}b(\gamma-1)}$, and the function $y = y(t)$ monotonously increases with increasing t. The solution (19) for the system (18) is a solution of the original problem (6)–(7) in parametric form.

In Fig. 1, we compare the exact solution (5) of the Cauchy problem for one equation (4) with the numerical solution of the transformed problem for the system of equations (18) for $a = b = 1$ and $\gamma = 2$, obtained by the classical numerical method, e.g. the Runge–Kutta method of the fourth-order of approximation with a fixed step of integration, equal to 0.2 (here and in what follows in the figures, for the sake of clarity, a scale factor $\nu = 30$ is introduced for the functions $x = x(t)$ or $x = x(\xi)$). In this case, the maximum error of the numerical solution does not exceed 0.017% for $y \le 50$.

Remark 3. Here and in what follows, the numerical integration interval for the new variable t (or ξ) is usually determined, for demonstration calculations, from the condition $\Lambda_{\rm m} = 50$, where

$$\Lambda_{\rm m} = \min[y,\, y'_x/y] \quad \text{(for } y_0 \sim 1 \text{ and } y_1 = y'_x(x_0) \sim 1). \tag{20}$$

In a few cases, the condition $\Lambda_{\rm m} = 100$ or $\Lambda_{\rm m} = 150$ is used, which is specially stipulated. For first-order ODE problems of the form (11)–(12), this definition of $\Lambda_{\rm m}$ can be replaced by the equivalent definition $\Lambda_{\rm m} = \min[y,\, f/y]$.

Conditions $y_0 \sim 1$ and $y_1 \sim 1$ in (20) are not strongly essential, since the substitution $y = y_0 - 1 + (y_1 - 1)(x - x_0) + \bar{y}$ leads to an equivalent problem with the initial conditions $\bar{y}(x_0) = \bar{y}'_x(x_0) = 1$.

2.3. Modified differential transformation

The solution (19) tends rather slowly to the asymptotic values $x \to x_*$ as $t \to \infty$ (in particular, for $s = 2$ and large t we have $x_* - x \sim t^{-1/2}$). To speed up the process of approaching the asymptotic behavior with respect to x in the system (15) is useful additionally to make the exponential-type substitution

$$t = t_0 \exp(\lambda\tau), \qquad \tau \geq 0, \tag{21}$$

where

$$\tau = \frac{1}{\lambda} \ln \frac{t}{t_0} = \frac{1}{\lambda} \ln \frac{y'_x}{t_0} \tag{22}$$

is a new independent variable and $\lambda > 0$ is a numerical parameter that can be varied. Transformations with a new independent variable of the form (22) will be called the modified differential transformations.

Example 4. As a result of the substitution (21), the Cauchy problem (18) is transformed to the form

$$\begin{gathered} x'_\tau = \frac{\lambda}{b\gamma y^{\gamma-1}}, \quad y'_\tau = \frac{a^\gamma \lambda e^{\lambda\tau}}{\gamma y^{\gamma-1}} \quad (\tau > 0); \\ x(0) = 0, \quad y(0) = a, \end{gathered} \tag{23}$$

and its exact solution is given by the formulas

$$x = \frac{1}{a^{\gamma-1} b(\gamma-1)} \left\{ 1 - \exp\left[-\frac{\lambda(\gamma-1)}{\gamma}\tau \right] \right\}, \quad y = a \exp\left(\frac{\lambda}{\gamma}\tau \right), \qquad \tau \geq 0. \tag{24}$$

Let $a = b = 1$, $\gamma = 2$ and the stepsize is equal to 0.4. For numerical integration of the test problem (23) for $\lambda = 1$ and $\lambda = 2$ with the maximum error 0.002%, it is required to take, respectively, the interval $[0, 8]$ and $[0, 4]$ with respect to τ to approach the asymptote (however, for numerical integration of the related problem (18) with the maximum error 0.016%, it is required to take an essentially larger interval $[0, 2980]$ with respect to t).

3. Problems for first-order equations. Nonlocal transformations and differential constraints

3.1. Solution method based on introducing a nonlocal variable

Introducing a new nonlocal variable [27–29] according to the formula,

$$\xi = \int_{x_0}^{x} g(x,y)\,dx, \qquad y = y(x), \tag{25}$$

leads the Cauchy problem for one equation (11)–(12) to the equivalent problem for the autonomous system of equations

$$\begin{gathered} x'_\xi = \frac{1}{g(x,y)}, \qquad y'_\xi = \frac{f(x,y)}{g(x,y)} \qquad (\xi > 0); \\ x(0) = x_0, \qquad y(0) = y_0. \end{gathered} \tag{26}$$

Here $g = g(x,y)$ is a regularizing function that depends on the solution of the problem (11)–(12) and can be varied. This function has to satisfy the following conditions:

$$g > 0 \text{ for } x \geq x_0,\ y \geq y_0; \qquad g \to \infty \text{ as } y \to \infty; \qquad f/g = k \text{ as } y \to \infty, \tag{27}$$

where $k = \text{const} > 0$ (moreover, the limiting case $k = \infty$ is also allowed).

From (25) and the second condition (27) it follows that $x'_\xi \to 0$ as $\xi \to \infty$. The Cauchy problem (26) can be integrated numerically applying the Runge–Kutta method or other standard numerical methods with a fixed stepsize in ξ.

Let us consider some possibilities for choosing the regularizing function $g = g(x,y)$ in the Cauchy problem (26) on concrete examples.

1°. The special case

$$g = f$$

is equivalent to the hodograph transformation with an additional shift of the dependent variable, which gives $\xi = y - y_0$.

2°. Setting

$$g = \sqrt{1 + f^2},$$

we arrive at the method of the arc-length transformation [4]. In this case, the Cauchy problem (26) takes the form

$$\begin{gathered} x'_\xi = \frac{1}{\sqrt{1 + f^2(x,y)}}, \qquad y'_\xi = \frac{f(x,y)}{\sqrt{1 + f^2(x,y)}}; \\ x(0) = x_0, \qquad y(0) = y_0. \end{gathered} \tag{28}$$

3°. Choosing

$$g = 1 + |f|,$$

we obtain the Cauchy problem

$$x'_\xi = \frac{1}{1+|f(x,y)|}, \quad y'_\xi = \frac{f(x,y)}{1+|f(x,y)|}; \quad x(0) = x_0, \quad y(0) = y_0. \tag{29}$$

Note that we use here the absolute value sign to generalize the results, since the system (29) can also be used in the case $f < 0$ for numerical integration of the problems having solutions with a root singularity [2].

4°. We can also take the function

$$g = c_1 + \big(c_2 + |f|^s\big)^{1/s}$$

for $c_1 \geq 0$, $c_2 \geq 0$ ($|c_1| + |c_2| \neq 0$), and $s > 0$, which is a generalization of the functions in Items 2° and 3°.

5°. A very convenient problem for analysis can be obtained if we take

$$g = f/y \tag{30}$$

in (26). In this case, the second equation of the system is immediately integrated and, taking into account the initial condition, we get $y = y_0 e^{\xi}$. In addition, the variable x tends exponentially rapidly to a blow-up point x_* with increasing ξ. This transformation will be called the special exp-type transformation.

Remark 4. From Items 1° and 2° it follows that the method based on the hodograph transformation and the method of the arc-length transformation are particular cases of the method based on a nonlocal transformation of the general form (25).

Remark 5. The functions g in Items 1°–4° correspond to the value $k = 1$ in (27), and the function g in Item 5° gives $k = \infty$.

Remark 6. Nonlocal transformations of a special form were used in [29–32] to obtain exact solutions and to linearize some second-order ODEs.

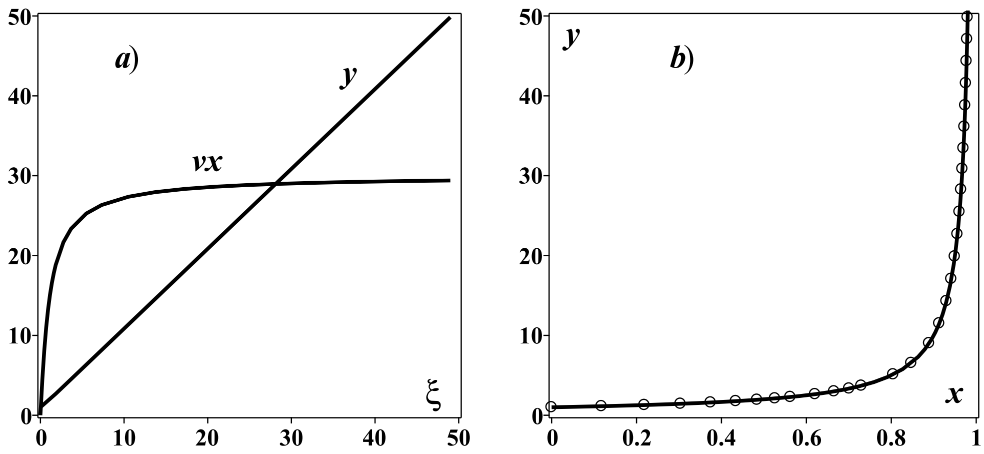

Figure 2: a)—the dependences $x = x(\xi)$ and $y = y(\xi)$, obtained by numerical solution of the problem (31) ($\nu = 30$); b)—exact solution (5) (solid line) and numerical solution of the problem (31) (circles).

3.2. Test problems and numerical solutions

Example 5. For the test Cauchy problem (4) with $f = y^2$, the equivalent problem for the system of equations (28) takes the form

$$x'_\xi = \frac{1}{\sqrt{1+y^4}}, \quad y'_\xi = \frac{y^2}{\sqrt{1+y^4}}; \qquad x(0)=0, \quad y(0)=1. \tag{31}$$

The second equation of this system is an equation with separable variables whose solution is not expressed in elementary functions.

The numerical solution of the Cauchy problem (31) in parametric form and its comparison with the exact solution (5) are shown in Fig. 2.

Example 6. For the test Cauchy problem (4), the equivalent the problem for the system of equations (29) admits an exact solution, which is expressed in terms of elementary functions in a parametric form as follows:

$$x = 1 + \tfrac{1}{2}\xi - \tfrac{1}{2}\sqrt{\xi^2+4}, \quad y = \tfrac{1}{2}\xi + \tfrac{1}{2}\sqrt{\xi^2+4} \qquad (\xi \ge 0). \tag{32}$$

This solution satisfies the initial conditions $x(0) = 0$ and $y(0) = 1$ and has no singularities. The function $x(\xi)$ is bounded, increases monotonically, and tends to its limiting value $x_* = \lim_{\xi\to\infty} x(\xi) = 1$. The function $y(\xi)$ increases monotonically and tends to infinity as $\xi \to \infty$. At large ξ we have $x \approx 1 - \xi^{-1}$ and $y \approx \xi + \xi^{-1}$.

The curves $x = x(\xi)$ and $y = y(\xi)$, determined by the exact solution (32) (and also the curves obtained by numerical integration of the corresponding system (29) with $f = y^2$, $x_0 = 0$, and $y_0 = 1$), are very close to the curves shown in Fig. 2 (they almost merge with them and therefore are not presented here).

Example 7. Consider the test problem (6)–(7), where $f = by^\gamma$, and take $g = f/y = by^{\gamma-1}$ (see Item 5° in Section 3.1). Substituting these functions into (26), we obtain the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{1}{by^{\gamma-1}}, \quad y'_\xi = y \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \end{gathered} \tag{33}$$

where $a > 0$, $b > 0$, and $\gamma > 1$. The exact solution of the problem (33) is written as follows:

$$x = \frac{1}{a^{\gamma-1}b(\gamma-1)}\big[1 - e^{-(\gamma-1)\xi}\big], \qquad y = ae^\xi. \tag{34}$$

It can be seen that the unknown function $x = x(\xi)$ tends exponentially to the asymptotic value $x_* = \dfrac{1}{a^{\gamma-1}b(\gamma-1)}$ as $\xi \to \infty$.

Let $b = 1$ and $\gamma = 2$. The numerical solutions of the problems (18) and (33), obtained by the Runge–Kutta method of the fourth-order of approximation, are shown in Fig. 3 for $a = 1$ and $a = 2$ and the same step of integration, equal to 0.2. We note that for this stepsize, the maximum difference between the exact solution (34) and the numerical solution of the system (33) is 0.0045% (and for stepsize 0.4, respectively, 0.061%).

It can be seen (see Figs. 1 and 3) that the numerical solutions are in a good agreement, but the rates of their approximation to the required asymptote $x = x_*$ are significantly different. For example, for the system (18), in order to obtain a good approximation to the asymptote, it is required to consider the interval $t \in [1, 2980]$, and for the system (33) it suffices to take $\xi \in [0, 4]$. Therefore, there is reason to believe that the method described in Item 5° (a special case of the transformation (25)) is more efficient than the method based on the differential transformation (see Section 2.1).

For comparison, similar calculations were also performed applying the method based on the hodograph transformation (see Section 3.1, Item 1°), and the method of the arc-length transformation (see Section 3.1, Item 2°). For both of these methods, in order to obtain a good approximation to the asymptote, it is required to consider the interval $\xi \in [0, 49]$. To control a numerical integration process, the calculations were carried out with the aid of the three most important and powerful mathematical software packages: Maple (2016), Mathematica (11), and MATLAB (2016a). It was found that the method based on the use of a special case of the system (26) with $g = f/y$ (see Item 5°) is essentially more efficient than the method based on the hodograph transformation and the method of the arc-length transformation.

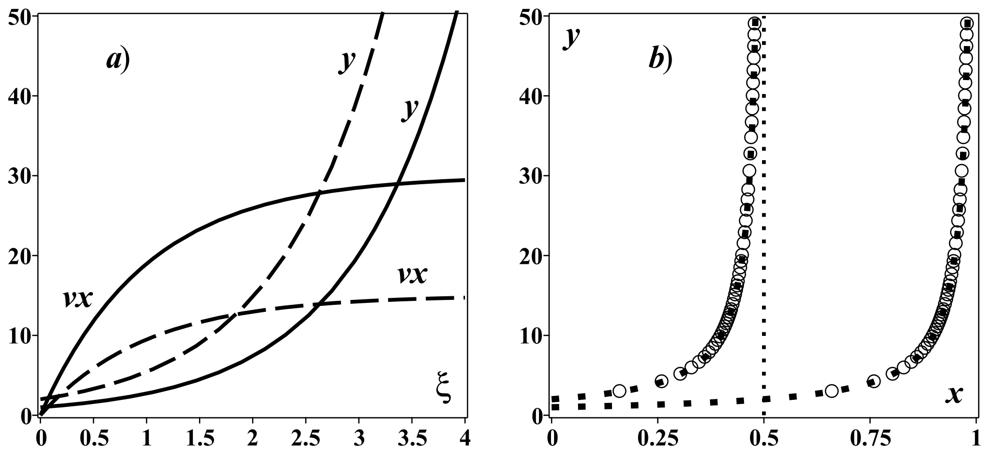

Figure 3: a)—the dependences $x = x(\xi)$ and $y = y(\xi)$, obtained by numerical solution of the problem (33) for $b = 1$, $\gamma = 2$ with $a = 1$ (solid lines) and $a = 2$ (dashed lines) ($\nu = 30$); b)—numerical solution of the problem (18) for $b = 1$, $\gamma = 2$ (circles) and numerical solution of the problem (33) for $b = 1$, $\gamma = 2$ (points); for left curves $a = 2$ and for right curves $a = 1$.

Example 8. We now consider the Cauchy problem (9) for $b = 1$, which is determined by the exponential $f = e^y$. Substituting the function $g = f/y = e^y/y$ (see Item 5° in Section 3.1) into the system (26), we obtain

$$x'_t = ye^{-y}, \quad y'_t = y;$$
$$x(0) = 0, \quad y(0) = a.$$

The exact solution of this problem in a parametric form is defined by the formulas

$$x = e^{-a} - \exp(-a\,e^t), \quad y = ae^t \qquad (t \geq 0),$$

which do not have singularities. The function $x = x(t)$ is bounded, monotonically increases with increasing t and very rapidly tends to the asymptote $x_* = \lim_{t\to\infty} x(t) = e^{-a}$, and the function $y = y(t)$ is unbounded and grows exponentially with respect to t.

3.3. Generalizations based on the use of differential constraints

Let us show that the method based on introducing a nonlocal variable (25) allows a further generalization.

We add to the equation (11) a first-order differential constraint of the form

$$\xi'_x = g(x, y, \xi) \tag{35}$$

and the initial condition $\xi(x = x_0) = \xi_0$.

The differential constraint (35) connects a new (nonlocal) independent variable ξ with the original variables x and $y = y(x)$ by means of a given differential equation. In a particular case, when the function g does not depend on ξ, the use of the differential constraint (35), after integrating it over x, leads to the nonlocal variable (25) for $\xi_0 = 0$ (therefore, the method based on the differential constraint generalizes the method based on introducing a nonlocal variable).

From (11) and (35) we obtain the following system of ordinary differential equations:

$$x'_\xi = \frac{1}{g(x, y, \xi)}, \qquad y'_\xi = \frac{f(x, y)}{g(x, y, \xi)}. \tag{36}$$

In a particular case,

$$g(x, y, \xi) = f_x + \xi f_y,$$

the system (36) coincides with the system (17), in which the variable t must be redenoted by ξ. If, in addition, we set $\xi_0 = f(x_0, y_0)$, then, up to renaming of variables, we also obtain the initial conditions (16). It follows that the method based on the differential constraint of general form (35) generalizes the method based on introducing a differential variable (see Section 2.1).

3.4. Comparison of efficiency of various transformations for numerical integration of first-order ODE blow-up problems

In Table 1, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (25) and differential constraints of the form (35) is presented by using the example of the test blow-up problem for the first-order ODE (4) with $f = y^2$. The comparison is based on the number of grid points needed to perform calculations with the same maximum error (approximately equal to 0.1, 0.01, and 0.005). In the last line of Table 1 for Example 4 we take $a = b = 1$ and $s = 2$.

It can be seen that for the first three transformations it is necessary to use a lot of grid points (the hodograph transformation is the least effective). This is due to the fact that in these cases x tends to the point x_* rather slowly for large ξ ($x_* - x \sim 1/\xi$, $y \sim \xi$). The last three transformations require a significantly less number of grid points; in these cases x tends exponentially fast to the point x_* for large ξ. In particular, the use of the exp-type transformation with $g = t/y$ and the nonlocal transformation with $g = \sqrt{1 + |f|}$ gives rather good results. The most effective analytical transformation is a modification of the method of differential transformations (see Example 4 in Section 2.3).

$\text{Error}_{\max}, \% = 0.1$				
Transformation or differential constraint	Function g or Example	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Hodograph, Item 1°	$g{=}f$	48.99	0.2300	213
Arc-length, Item 2°	$g{=}\sqrt{1{+}f^2}$	49.20	0.3000	164
Nonlocal, Item 3°	$g{=}1{+}\lvert f\rvert$	50.00	0.4000	125
Special exp-type, Item 5°	$g{=}f/y$	3.925	0.1570	25
Nonlocal	$g{=}\sqrt{1+\lvert f\rvert}$	4.170	0.1668	25
Differential constraint	$g{=}f/[y(1+2\xi)]$	1.543	0.0643	24
Modified differential	Example 4 with $\lambda=2$	3.910	0.2300	17
$\text{Error}_{\max}, \% = 0.01$				
Transformation or differential constraint	Function g or Example	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Hodograph, Item 1°	$g{=}f$	49.01	0.130	377
Arc-length, Item 2°	$g{=}\sqrt{1{+}f^2}$	49.30	0.170	290
Nonlocal, Item 3°	$g{=}1{+}\lvert f\rvert$	50.14	0.230	218
Special exp-type, Item 5°	$g{=}f/y$	3.960	0.090	44
Nonlocal	$g{=}\sqrt{1+\lvert f\rvert}$	4.136	0.094	44
Differential constraint	$g{=}f/[y(1+2\xi)]$	1.540	0.035	44
Modified differential	Example 4 with $\lambda=2$	3.900	0.130	30
$\text{Error}_{\max}, \% = 0.005$				
Transformation or differential constraint	Function g or Example	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Hodograph, Item 1°	$g{=}f$	49.035	0.1050	467
Arc-length, Item 2°	$g{=}\sqrt{1{+}f^2}$	49.266	0.1380	357
Nonlocal, Item 3°	$g{=}1{+}\lvert f\rvert$	50.135	0.1850	271
Special exp-type, Item 5°	$g{=}f/y$	3.9150	0.0725	54
Nonlocal	$g{=}\sqrt{1+\lvert f\rvert}$	4.1340	0.0780	53
Differential constraint	$g{=}f/[y(1+2\xi)]$	1.5420	0.0291	53
Modified differential	Example 4 with $\lambda=2$	3.9140	0.1030	38

Table 1: Various types of analytical transformations applied for numerical integration of the problem (4) for $f = y^2$ with a given accuracy (percent errors are 0.1, 0.01, and 0.005 for $\Lambda_{\rm m} \le 50$) and their basic parameters (maximum interval, stepsize, grid points number).

Remark 7. Up to now it has been assumed that the right-hand side of the equation (11) is positive. In cases where the right-hand side of equation (11) can change sign, for the numerical integration of system (26) one can use the regularizing functions $g = \sqrt{1+f^2}$, $g = 1 + |f|$, and $g = \sqrt{1+|f|}$.

4. Problems for first-order equations, the right-hand side of which has singularities or zeros

4.1. Blow-up problems for equations, the right-hand side of which has singularities in x

In this section we will analyze several blow-up problems for equations of the form (11), the right-hand side of which has a singularity at some $x = x_{\rm s}$, i.e. $\lim\limits_{x\to x_{\rm s}} f(x,y) = \infty$.

We assume that the right-hand side of equation (11) can be represented as a product of two functions

$$f(x,y) = f_{\rm b}(x,y) f_{\rm s}(x,y), \tag{37}$$

where the function $f_{\rm b}$ has the same properties as the function f in Section 2.1 (i.e. that the problem (11)–(12), where the function f is replaced by $f_{\rm b}$, has a blow-up solution).

Moreover, we will assume that the function $f_{\rm s}$ has an integrable or non-integrable singularity at $x = x_{\rm s}$, so that $\lim\limits_{x\to x_{\rm s}} f(x,y) = \infty$, and $f_{\rm s} > 0$ at $x_0 < x_{\rm s}$.

It is interesting to see how the two singularities of this problem will interact: the blow-up singularity and the coordinate singularity at $x = x_{\rm s}$.

For the sake of clarity, we give the following test problems and illustrative examples.

Example 9. Consider the two-parameter test Cauchy problem:

$$y'_x = \frac{y^2}{b-x}; \quad y(0) = a, \tag{38}$$

where $a > 0$ and $b > 0$. For this problem, we have $f_{\rm b} = y^2$ and $f_{\rm s} = 1/(b-x)$. The right-hand side of equation (38) has a pole of the first order at the point $x = x_{\rm s} = b$ (i.e., there exists a non-integrable singularity at this point); and the right-hand side of the equation becomes negative if $x > b$.

The exact solution of the problem (38) has the form

$$y = \left[\ln\Bigl(1 - \frac{x}{b}\Bigr) + \frac{1}{a}\right]^{-1}. \tag{39}$$

The singular point of this solution is determined by the formula

$$x_* = b\bigl(1 - e^{-1/a}\bigr) < b. \tag{40}$$

Here the blow-up singularity "overtakes" the non-integrable singularity of the equation at the point $x_\text{s} = b$. If $a \to \infty$, we have $x_* \to b$.

Example 10. Consider the test Cauchy problem with a stronger coordinate singularity:

$$y'_x = \frac{y^2}{(b-x)^2}; \qquad y(0) = a, \tag{41}$$

where $a > 0$ and $b > 0$. For this problem, we have $f_\text{b} = y^2$ and $f_\text{s} = 1/(b-x)^2$. The right-hand side of equation (41) has a pole of the second order at the point $x_\text{s} = b$ (i.e., there exists a non-integrable singularity at this point); and right-hand side of this equation is positive for all x.

The exact solution of the problem (41) is defined by the formula

$$y = \frac{ab}{a+b}\left[1 + \frac{ab}{b^2 - (a+b)x}\right]. \tag{42}$$

The blow-up point is determined as

$$x_* = \frac{b^2}{a+b} < b. \tag{43}$$

Here, as in Example 9, the blow-up singularity "overtakes" the non-integrable singularity of the equation at the point $x_\text{s} = b$.

Remark 8. A qualitatively similar picture will occur also for the problem (11)–(12) with $f = y^2/\sqrt{b-x}$, which has an integrable singularity at the point $x = b$. In this problem, the right-hand side of the equation is defined only on a part of the x-axis.

The solution property, described in Examples 9 and 10, has a general characteristic. Namely, let us assume that the right-hand side of equation (11) has the form (37), where the functions f_b and f_s have the properties described at the beginning of this section. Then the problem (11)–(12) has a blow-up solution, and the domain of definition of this solution is located to the left of the point x_s (i.e., $x_* < x_\text{s}$, where x_* is the blow-up point).

The methods described in Sections 2 and 3 can be applied for solving this type of problems with the coordinate singularity.

Example 11. For test problem (38) with $f = y^2/(b-x)$, we take (see Item 5° in Section 3.1),

$$g = \frac{f}{y} = \frac{y}{b-x}.$$

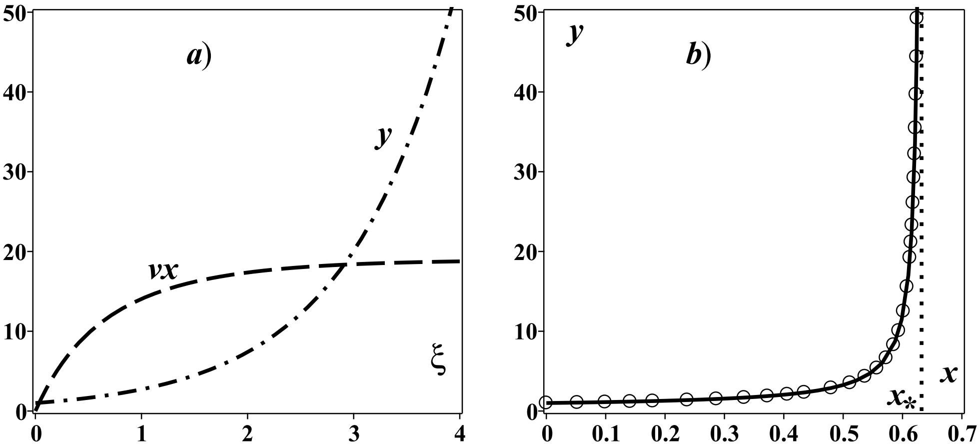

Figure 4: a)—numerical solutions of system (44), the dependences $x = x(\xi)$ and $y = y(\xi)$ (where $\nu = 30$); b)—exact solution (39), solid line, and the numerical solution of system (44) ($a = b = 1$ and $x_* = 0.6321$).

By substituting these functions in (26), we arrive at the Cauchy problem for a system of coupled ODEs

$$x'_\xi = \frac{b-x}{y}, \quad y'_\xi = y \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a. \tag{44}$$

The exact solution of this problem reads

$$x = b\left\{1 - \exp\left[-\frac{1}{a}(1 - e^{-\xi})\right]\right\}, \quad y = ae^{\xi}. \tag{45}$$

The numerical solution of the problem (44) with $a = b = 1$ is presented in Fig. 4*a*; the dependences $x = x(\xi)$ and $y = y(\xi)$ are obtained by the fourth-order Runge–Kutta method. Fig. 4*b* shows a comparison of the exact solution (39) of the Cauchy problem (38) for one equation with the numerical solution of the problem for the system of two equations (44).

Note that a form of the right-hand side of equation (11) with a coordinate singularity at the point x_s can mislead the researcher, inexperienced in blow-up problems. As a result, the researcher will start to refine a mesh (making it thinner) in the neighborhood of the point x_s (what should not to do).

4.2. Blow-up problems for equations, the right-hand side of which has zeros

In this section we will analyze blow-up problems for equations of the form (11), the right-hand side of which vanishes at some $x = x_z$, i.e. $f(x_z, y) = 0$.

Let us assume that the right-hand side of equation (11) can be represented as a product of two functions

$$f(x, y) = f_{\rm b}(x, y) f_{\rm z}(x, y), \tag{46}$$

where the function $f_{\rm b}$ has the same properties as the function f in Section 2.1 (i.e., the problem (11)–(12), where the function f is replaced by $f_{\rm b}$, has a blow-up solution). Moreover, we will assume that the function $f_{\rm z}$ vanishes at $x = x_{\rm z}$, so that $f(x_{\rm z}, y) = 0$, and $f_{\rm z} > 0$ at $x_0 < x_{\rm z}$.

Example 12. Consider the test two-parameter Cauchy problem

$$y'_x = y^2(b - x); \quad y(0) = a, \tag{47}$$

where $a > 0$ and $b > 0$. For this problem, we have $f_{\rm b} = y^2$ and $f_{\rm z} = b - x$. The right-hand side of equation (47) becomes zero at the point $x = x_{\rm z} = b$; and the right-hand side of the equation becomes negative if $x > b$.

It is interesting to see how two features of different types of such problem will interact: on the one hand, a possible blow-up singularity (which leads to an unlimited growth of the right-hand side of the equation), and on the other hand, vanishing of the right-hand side of the equation at $x = x_{\rm z}$.

The exact solution of the problem (47) has the form

$$y = \frac{a}{\frac{1}{2}ax^2 - abx + 1}. \tag{48}$$

The existence or absence of a blow-up singularity in this solution is determined by the existence or absence of real roots of the quadratic equation $\frac{1}{2}ax^2 - abx + 1 = 0$.

The elemental analysis shows that there are two qualitatively different cases:

(i) If $0 < b < \sqrt{2/a}$, there exists a smooth continuous solution of the problem for all $x \ge 0$. It is monotonically increasing on the interval $0 \le x < b$, reaches the maximum value $y_m = \dfrac{a}{1 - \frac{1}{2}ab^2}$, and decreases for $x > b$.

(ii) If $b \ge \sqrt{2/a}$, formula (48) defines a monotonically increasing blow-up solution with the singular point

$$x_* = b - \sqrt{b^2 - \frac{2}{a}}. \tag{49}$$

Therefore in this example the existence or absence of a blow-up solution is determined by a simple relation between the parameters a and b: if $b \ge \sqrt{2/a}$, then there exists a

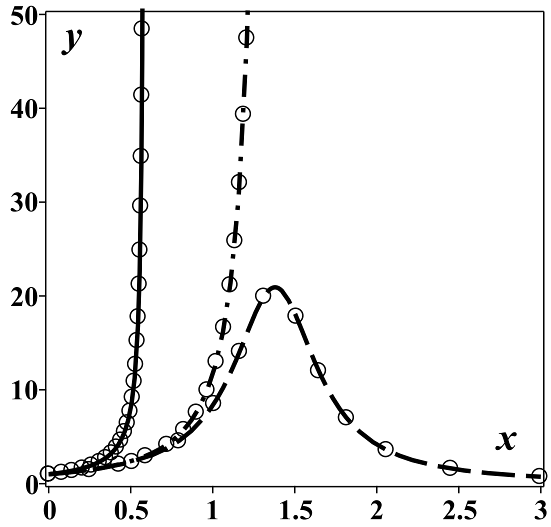

Figure 5: Exact solutions (48) of the Cauchy problem (47) for various values of the parameters: $a = 1$, $b = 1.38$ (dashed line); $a = 1$, $b = 2$ (solid line); and $a = 1$, $b = \sqrt{2}$ (dashed-dot line). The results of numerical solutions of this problem for the same values of the defining parameters for $g = \sqrt{1 + |f|}$ are denoted by circles.

blow-up solution, otherwise there is no a blow-up solution. The value $b = \sqrt{2/a}$ is a point of bifurcation of the two-parameter problem (47).

The exact solutions of the problem (47) obtained by formula (48) are presented in Fig. 5 for various values of the parameters: $a = 1$ and $b = 1$, $b = 2$, and $b = \sqrt{2}$ (the critical value at which there exists blow-up solutions).

For numerical integration of such problems, solutions of which can be qualitatively different in a wide range of changes of the determining parameters, one can use nonlocal transformations of the form (25) with regularizing functions of the form $g = \sqrt{1 + f^2}$, $g = 1 + |f|$, and $g = \sqrt{1 + |f|}$. The results of the corresponding numerical solutions of the problem (47) for $g = \sqrt{1 + |f|}$ are shown by circles in Fig. 5.

Remark 9. A formal replacing $b - x$ to $(b - x)^2$ in the problem (47) leads to a blow-up solution for any $a > 0$ and $b > 0$.

5. Problems for first-order equations. Two-sided estimates of the critical value

5.1. Autonomous equations. Analytical formula for the critical value

We consider the Cauchy problem for an autonomous equation of the general form

$$y'_x = f(y) \quad (x > 0), \qquad y(0) = a. \tag{50}$$

We assume that $a > 0$ and $f(y) > 0$ is a continuous function that is defined for all $y \geq a$. An exact solution of the Cauchy problem (50) for $x > 0$ can be represented implicitly as follows:

$$x = \int_a^y \frac{d\xi}{f(\xi)}. \tag{51}$$

This solution is a blow-up solution if and only if there exists a finite definite integral in (51) for $y = \infty$. In this case, the critical value x_* is calculated as follows:

$$x_* = \int_a^\infty \frac{d\xi}{f(\xi)}. \tag{52}$$

Let the conditions formulated after the problem (50) be satisfied.

A necessary criterion for the existence of a blow-up solution is:

$$\lim_{y\to\infty} \frac{f(y)}{y} = \infty.$$

Sufficient criterion of the existence of a blow-up solution. Let the conditions formulated above are also satisfied and the limiting ratio,

$$\lim_{y\to\infty} \frac{f(y)}{y^{1+\kappa}} = s, \qquad 0 < s \leq \infty, \tag{53}$$

takes place for some parameter $\kappa > 0$. Then the solution of the Cauchy problem (50) is a blow-up solution.

If $f(y)$ is a differentiable function, then instead of (53) we can propose an equivalent sufficient criterion of the existence of a blow-up solution:

$$\lim_{y\to\infty} \frac{f'_y(y)}{y^\kappa} = s_1, \qquad 0 < s_1 \leq \infty \qquad (\kappa > 0).$$

5.2. Non-autonomous equations. One-sided estimates

We consider the Cauchy problem for a first-order non-autonomous equation of the general form

$$y'_x = f(x, y) \quad (x > 0), \qquad y(0) = a. \tag{54}$$

We assume that $f(x, y)$ is a continuous function and the conditions

$$f(x, y) \geq g(y) > 0 \quad \text{for all} \quad y \geq a > 0, \qquad x \geq 0 \tag{55}$$

are satisfied. We also assume that there exists a finite integral

$$I_g = \int_a^\infty \frac{d\xi}{g(\xi)} < \infty. \tag{56}$$

Then the solution $y = y(x)$ of the Cauchy problem (54) is a blow-up solution, and the critical value x_* satisfies the inequality

$$x_* \leq I_g. \tag{57}$$

This estimate follows from the inequality (see, for example, the corresponding comparison theorems in [33, 34]):

$$y(x) \geq y_g(x), \tag{58}$$

where $y(x)$ is the solution of the Cauchy problem (54), and $y_g(x)$ is the solution of the auxiliary Cauchy problem

$$y'_x = g(y) \quad (x > 0), \qquad y(0) = a. \tag{59}$$

Example 13. We consider the Cauchy problem for the Abel equation of the first kind

$$y'_x = y^3 + h(x) \quad (x > 0); \qquad y(0) = 1. \tag{60}$$

If $h(x) \geq 0$ for $x \geq 0$, then the inequality is valid

$$f(x, y) \equiv y^3 + h(x) \geq g(y) \equiv y^3 > 0 \quad \text{for all} \quad y > 1.$$

Calculating the integral (56) with $g(y) = y^3$, we obtain

$$I_g = \int_1^\infty \frac{d\xi}{\xi^3} = \frac{1}{2} < \infty. \tag{61}$$

Therefore, the solution of the Cauchy problem (60) for $h(x) \geq 0$ is a blow-up solution, and $x_* \leq \frac{1}{2}$.

5.3. Non-autonomous equations. Two-sided estimates

We consider two cases in which the one-sided estimate (57) can be improved. We introduce the notations

$$I_1 = \int_{y_0}^{\infty} \frac{d\xi}{f(0,\xi)}, \qquad I_2 = \int_{y_0}^{\infty} \frac{d\xi}{f(I_1,\xi)}. \tag{62}$$

Case 1°*.* Let $f_x \geq 0$. Suppose that the integral I_1 in (62) exists and is finite. Suppose also that the conditions,

$$f(x,y) > 0, \quad f_x(x,y) \geq 0 \quad \text{for all} \quad 0 \leq x \leq I_1, \; y \geq y_0 > 0, \tag{63}$$

are satisfied. Then the integral I_2 exists and the inequalities are valid:

$$f(0,y) \leq f(x,y) \leq f(I_1,y) \quad \text{for} \quad 0 \leq x \leq I_1 \tag{64}$$

and

$$y_1(x) \leq y(x) \leq y_2(x) \quad \text{for} \quad 0 \leq x \leq I_2 \leq I_1. \tag{65}$$

Here $y(x)$ is the solution of the Cauchy problem (54), and $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding auxiliary Cauchy problems:

$$y'_x = f(0,y) \qquad (x > 0), \qquad y(0) = y_0; \tag{66}$$

$$y'_x = f(I_1,y) \quad (x > 0), \qquad y(0) = y_0. \tag{67}$$

The solutions $y_1(x)$ and $y_2(x)$ can be represented implicitly as follows:

$$x = \int_{y_0}^{y} \frac{d\xi}{f(0,\xi)}, \qquad x = \int_{y_0}^{y} \frac{d\xi}{f(I_1,\xi)}. \tag{68}$$

For the critical value x_*, the two-sided estimate

$$I_2 \leq x_* \leq I_1 \tag{69}$$

is valid.

Case 2°*.* Let $f_x \leq 0$. Suppose that the integrals I_1 and I_2 in (62) exist and are finite. Suppose also that the conditions,

$$f(x,y) > 0, \quad f_x(x,y) \leq 0 \quad \text{for all} \quad 0 \leq x \leq I_2, \; y \geq y_0 > 0, \tag{70}$$

are satisfied. Then the following inequalities are valid:

$$f(I_1, y) \le f(x, y) \le f(0, y) \quad \text{for} \quad 0 \le x \le I_2 \tag{71}$$

and

$$y_2(x) \le y(x) \le y_1(x) \quad \text{for} \quad 0 \le x \le I_1 \le I_2, \tag{72}$$

where $y(x)$ is the solution of the Cauchy problem (54), and $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding auxiliary Cauchy problems (66) and (67). The last two solutions can be represented implicitly (68). For the critical value x_*, the two-sided estimate

$$I_1 \le x_* \le I_2 \tag{73}$$

is valid.

Example 14. We consider the Cauchy problem for the Riccati equation

$$y'_x = y^2 + h(x) \quad (x > 0); \quad y(0) = a > 0. \tag{74}$$

Let us consider the two cases.

Case 1°. Let $h(x) \ge 0$ and $h'_x(x) \ge 0$. In this case, the first auxiliary Cauchy problem (66) is written as follows:

$$y'_x = y^2 + h(0) \quad (x > 0), \quad y(0) = a. \tag{75}$$

The exact solution of the problem (75) admits an implicit form of representation with the help of the first relation (68) for $y_0 = a$ and $f(0, y) = y^2 + h(0)$. After elementary calculations and transformations, this solution can be written in the explicit form

$$y = \sqrt{b}\,\frac{a\cos(\sqrt{b}\,x) + \sqrt{b}\sin(\sqrt{b}\,x)}{\sqrt{b}\cos(\sqrt{b}\,x) - a\sin(\sqrt{b}\,x)}, \quad b = h(0). \tag{76}$$

The singular point of this solution, I_1, which is the zero of the denominator and is equal to the improper first integral in (62) for $y_0 = a$ and $f(0, y) = y^2 + h(0)$, is defined by the formula

$$I_1 = \frac{1}{\sqrt{b}} \arctan\frac{\sqrt{b}}{a}, \quad b = h(0).$$

The solution of the second auxiliary Cauchy problem (67) is given by the formula (76), in which $h(0)$ must be replaced by $h(I_1)$. As a result, we obtain the two-sided estimate of the critical value x_*:

$$I_2 \le x_* \le I_1,$$
$$I_1 = \frac{1}{\sqrt{h(0)}} \arctan\frac{\sqrt{h(0)}}{a}, \quad I_2 = \frac{1}{\sqrt{h(I_1)}} \arctan\frac{\sqrt{h(I_1)}}{a}. \tag{77}$$

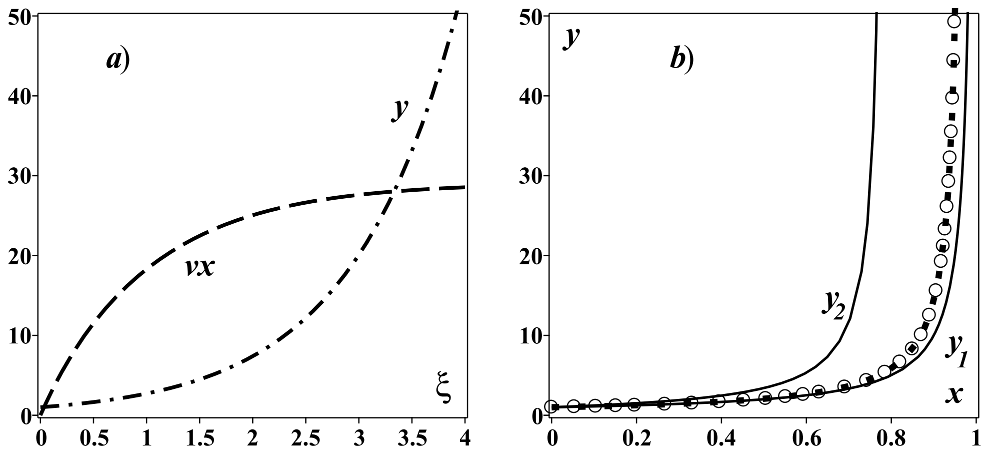

Figure 6: a)—the dependences $x = x(\xi)$ and $y = y(\xi)$ obtained numerically after the transformation of the Cauchy problem for one equation (74) for $a = 1$, $h(x) = x^2$ to the problem for the system of equations (26) for $f = y^2 + x^2$, $g = f/y$ ($\nu = 30$); b)—the exact solution of the problem (74) (points), the numerical solution of this problem (circles), and the majorizing functions $y_1(x)$ and $y_2(x)$ (solid lines).

In particular, setting $a = 1$, $h(x) = x^m$ and $m > 0$ in (74), we find that $I_1 = 1$ and $I_2 = \arctan 1$. Substituting these values into (77), we obtain the two-sided estimate $0.785 \le x_* \le 1$ for the critical value x_*.

In Fig. 6 we present the results of the numerical solution of the Cauchy problem (74) for $a = 1$ and $h(x) = x^2$ in parametric form, as well as a comparison of the numerical and exact solutions of this problem (the latter is expressed in terms of the Bessel functions and is omitted here), and also the majorizing functions $y_1 = 1/(1-x)$ and $y_2 = y_2(x)$, which are the solutions of the auxiliary Cauchy problems (66) and (67) (the solution of the Cauchy problem under consideration is located between these functions). The function $y_2(x)$ is determined by the formula (76), in which the parameter b must be replaced by $1/a$.

We note that if $h(x) = \text{const} > 0$, then the inequalities (77) give the exact result $x_* = I_1 = I_2$.

Case 2°. Let $h(x) \ge 0$ and $h'_x(x) \le 0$. In this case, the solution of the first auxiliary Cauchy problem (75) is also given by the formula (76), and the solution of the second auxiliary Cauchy problem is obtained from (76) by a formal replacement of $h(0)$ by $h(I_1)$. As a result, we obtain the two-sided estimate for the critical value x_*:

$$I_1 \le x_* \le I_2,$$

where the integrals I_1 and I_2 are determined by the formulas (77).

Remark 10. It should be noted that in Case 2° it does not matter how the function $f(x, y)$ and its derivative $f_x(x, y)$ behave for $x > I_2$; in particular, the right-hand side of the equation (54) can be negative for $x > I_2$.

Example 15. To illustrate what was said in Remark 10, we consider the test Cauchy problem

$$y'_x = (2 - x)y^2 \quad (x > 0); \qquad y(0) = 1, \tag{78}$$

which corresponds to *Case* 2°, where $f(x, y) < 0$ for $x > 2$.

Calculating the integrals (62), we have $I_1 = \frac{1}{2}$ and $I_2 = \frac{2}{3}$. Substituting these values into (73), we obtain the two-sided estimate for the singular point

$$\tfrac{1}{2} < x_* < \tfrac{2}{3}.$$

The exact solution of the problem (78) is given by the formula

$$y = \frac{2}{x^2 - 4x + 2}. \tag{79}$$

The zero of the denominator, equal to $x_* = 2 - \sqrt{2} \approx 0.5858$, determines the singular point of the solution (first-order pole).

6. Problems for second-order equations. Differential transformations

6.1. Solution method based on introducing a differential variable

The Cauchy problem for the second-order differential equation has the form

$$y''_{xx} = f(x, y, y'_x) \quad (x > x_0); \tag{80}$$

$$y(x_0) = y_0, \qquad y'_x(x_0) = y_1. \tag{81}$$

We note that the exact solutions of equations of the form (80), which can be used for the formulation of test problems with blow-up solutions, can be found in [34–36].

Let $f(x, y, u) > 0$ if $y > y_0 \geq 0$ and $u > y_1 \geq 0$, and the function f increases quite rapidly as $y \to \infty$ (for example, if f does not depend on y'_x, then $\lim\limits_{y\to\infty} f/y^{1+\varepsilon} = \infty$, where $\varepsilon > 0$).

First, as in Section 2.1, we represent the ODE (80) as an equivalent system of differential-algebraic equations

$$y'_x = t, \qquad y''_{xx} = f(x, y, t), \tag{82}$$

where $y = y(x)$ and $t = t(x)$ are the unknown functions.

Taking into account (82), we derive a standard system of ODEs for the functions $y = y(t)$ and $x = x(t)$. To do this, differentiating the first equation of the system (82) with respect to t, we obtain $(y'_x)'_t = 1$.

Taking into account the relations $y'_t = tx'_t$ (it follows from the first equation (82)) and $(y'_x)'_t = y''_{xx}/t'_x = x'_t y''_{xx}$, we have

$$x'_t y''_{xx} = 1. \tag{83}$$

Eliminating here the second derivative y''_{xx} by using the second equation (82), we arrive at the first-order equation

$$x'_t = \frac{1}{f(x, y, t)}. \tag{84}$$

Considering further the relation $y'_t = tx'_t$, we transform (84) to the form

$$y'_t = \frac{t}{f(x, y, t)}. \tag{85}$$

Equations (84) and (85) represent a system of coupled first-order differential equations for the unknown functions $x = x(t)$ and $y = y(t)$. The system (84)–(85) should be supplemented by the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = y_1, \tag{86}$$

which are derived from (81) and the first equation (82).

The Cauchy problem (84)–(86) has a solution without blow-up singularities and can be integrated by applying the standard fixed-step numerical methods (see, for example, [18–26]).

Remark 11. Systems of differential-algebraic equations (13) and (82) are particular cases of parametrically defined nonlinear differential equations, which are considered in [37, 38]. In [38], the general solutions of several parametrically defined ODEs were constructed via differential transformations based on introducing a new differential independent variable $t = y'_x$.

6.2. Test problems and numerical solutions

Example 16. We consider a test Cauchy problem for the second-order nonlinear ODE

$$y''_{xx} = b\gamma y^{\gamma-1} y'_x \quad (x > 0); \quad y(0) = a, \quad y'_x(0) = a^\gamma b, \tag{87}$$

which is obtained by differentiating equation (6). For $a > 0$, $b > 0$, and $\gamma > 1$, the exact solution of this problem is defined by the formula (8).

Introducing a new variable $t = y'_x$ in (87), we obtain the Cauchy problem, which exactly coincides with the problem (18). The exact solution of this problem is determined by the formulas (19).

Example 17. Let us now consider another Cauchy problem

$$y''_{xx} = b^2 \gamma y^{2\gamma-1} \qquad (x > 0); \quad y(0) = a, \quad y'_x(0) = a^\gamma b, \tag{88}$$

which is obtained by excluding the first derivative from the equations (6) and (87) (we recall that the second equation is a consequence of the first equation). The exact solution of the problem (88) is determined by the formula (8).

Introducing a new variable $t = y'_x$, we transform (88) to the Cauchy problem for the system of the first-order ODEs

$$\begin{gathered} x'_t = \frac{1}{b^2 \gamma y^{2\gamma-1}}, \quad y'_t = \frac{t}{b^2 \gamma y^{2\gamma-1}} \quad (t > t_0); \\ x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = a^\gamma b, \end{gathered} \tag{89}$$

which is a particular case of the problem (84)–(86) with $f = b^2 \gamma y^{2\gamma-1}$, $x_0 = 0$, and $y_0 = a$. The exact solution of the problem (89) is given by formulas (19).

Figure 7 shows a comparison of the exact solution (5) of the Cauchy problem for one equation (4) with the numerical solution of the related problem for the system of equations (89) for $a = b = 1$ and $\gamma = 2$, obtained by applying the Runge–Kutta method of the fourth-order of approximation.

The function $x(t)$ slowly tends to the asymptotic value x_*. Therefore to accelerate this process in the system (89) is useful additionally to make the exponential-type substitution (21).

For completeness of the picture, we also give an example of a blow-up problem whose solution has a logarithmic singularity.

Example 18. An exact solution of the Cauchy problem with exponential nonlinearity

$$y''_{xx} = e^{2y} \quad (x > 0); \qquad y(0) = 0, \quad y'_x(0) = 1, \tag{90}$$

has the form

$$y = \ln\Bigl(\frac{1}{1-x}\Bigr) = -\ln(1-x). \tag{91}$$

This solution has a logarithmic singularity at the point $x_* = 1$ and does not exist for $x > x_*$.

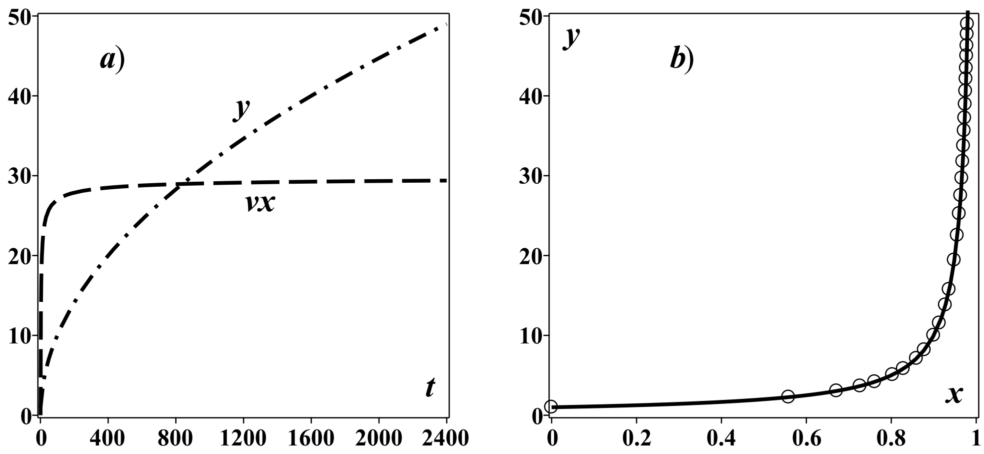

Figure 7: a)—the dependences $x = x(t)$ and $y = y(t)$ obtained by numerical solution of the problem (89) for $a = b = 1$ and $\gamma = 2$ ($\nu = 30$); b)—exact solution (5) (solid line) and numerical solution of problem (89) (circles).

Introducing the differential variable $t = y'_x$, we transform the problem (90) to the following Cauchy problem for a system of equations:

$$\begin{gathered} x'_t = e^{-2y}, \quad y'_t = te^{-2y} \quad (t > 1); \\ x(1) = 0, \quad y(1) = 0 \quad (t_0 = 1), \end{gathered} \tag{92}$$

which is a particular case of the system (84)–(85). The exact analytical solution of the problem (92) is determined by the formulas

$$x = 1 - \frac{1}{t}, \quad y = \ln t \qquad (t \geq 1),$$

which do not have singularities; the function $x = x(t)$ increases monotonically with $t > 1$ and tends to its limiting value $x_* = \lim\limits_{t\to\infty} x(t) = 1$, and the function $y = y(t)$ is unlimited and increases monotonically with respect to the logarithmic law.

The function $x(t)$ slowly tends to the asymptotic value x_*. Therefore to accelerate this process in the system (92) is useful additionally to make the exponential-type substitution (21).

7. Problems for second-order equations. Nonlocal transformations and differential constraints

7.1. Solution method based on introducing a nonlocal variable

First, we represent the equation (80) as the equivalent system of two equations

$$y'_x = t, \quad t'_x = f(x, y, t),$$

and then we introduce a nonlocal variable [27, 28] of the form

$$\xi = \int_{x_0}^{x} g(x, y, t)\, dx, \quad y = y(x), \quad t = t(x). \tag{93}$$

As a result, the Cauchy problem (80)–(81) is transformed to the following problem for the autonomous system of three equations:

$$\begin{gathered} x'_\xi = \frac{1}{g(x,y,t)}, \quad y'_\xi = \frac{t}{g(x,y,t)}, \quad t'_\xi = \frac{f(x,y,t)}{g(x,y,t)} \quad (\xi > 0); \\ x(0) = x_0, \quad y(0) = y_0, \quad t(0) = y_1. \end{gathered} \tag{94}$$

For a suitable choice of the regularizing function $g = g(x, y, t)$ (not very restrictive conditions of the type (27) must be imposed on it, see also Section 7.2), we obtain the Cauchy problem (94), the solution of which will not have blow-up singularities; therefore this problem can be integrated by applying the standard fixed-step numerical methods [18–26].

Remark 12. From the formula (93) for small increments of the argument Δx, we get $\Delta\xi = g(x, y, t)\Delta x$. It follows that the choice of a fixed stepsize for the new nonlocal variable $\Delta\xi = h$ is equivalent to using a variable stepsize for the original independent variable $\Delta x = h/g$.

Let us consider various possibilities for choosing the regularizing function g in the system (94).

1°. The special case $g = t$ is equivalent to the hodograph transformation with an additional shift of the dependent variable, which gives $\xi = y - y_0$.

2°. We can take $g = \bigl(c + |t|^s + |f|^s\bigr)^{1/s}$ for $c \ge 0$ and $s > 0$. The case $c = 1$ and $s = 2$ corresponds to the method of the arc-length transformation [4].

3°. Setting $g = f$, and then integrating the third equation (94), we obtain the problem (84)–(86) in which the variable $t = \xi + y_1$. Therefore the method based on the nonlocal transformation (93) generalizes the method based on the differential transformation, which is described in Section 6.1.

4°. We can take $g = t/y$ (or $g = kt/y$, where $k > 0$ is a numerical parameter that can be varied). In this case, the system (94) is much simplified, since the second equation is directly integrated, and taking into account the second initial condition, we obtain $y = y_0 e^{\xi}$. As a result, there remains a system of two equations for the determination of the functions $x = x(\xi)$ and $t = t(\xi)$. Taking into account the relation $t = y'_x$, we also have

$$\xi = \int_{x_0}^{x} \frac{t}{y}\, dx = \int_{x_0}^{x} \frac{y'_x}{y}\, dx = \ln \frac{y}{y_0}.$$

Therefore, the nonlocal transformation (93) with $g = t/y$ and the subsequent transition to the system (94) is equivalent to a point transformation $\xi = \ln(y/y_0)$, $z = x$, which is a combination of two more simple point transformations: 1) the transformation $\bar{x} = x$, $\bar{y} = \ln(y/y_0)$ and 2) the hodograph transformation $\xi = \bar{y}$, $z = \bar{x}$, where $z = z(\xi)$.

5°. Also, we can take $g = f/t$ (or $g = kf/t$, where $k > 0$ is a free numerical parameter). In this case, the system (94) is also much simplified, since the third equation is directly integrated, and taking into account the third initial condition, we obtain $t = y_1 e^{\xi}$. As a result, there remains a system of two equations for the determination of the functions $x = x(\xi)$ and $y = y(\xi)$. For the nonlocal transformation (93) with $g = f/t$, the new independent variable is expressed in terms of the derivative by the formula $\xi = \ln(y'_x/y_1)$ (that is, this transformation coincides with the modified differential transformation, see Section 2.3).

Remark 13. The transformations corresponding to the last two cases, 4° and 5°, will be called special exp-type transformations, they lead to the solutions, in which the variable x tends exponentially rapidly to a blow-up point x_*.

Remark 14. From Items 1°, 2°, and 3° it follows that the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation are particular cases of the nonlocal transformation of the general form (93), which leads to the Cauchy problem for the system of equations (94).

Remark 15. It is not necessary to calculate the integrals (93) (or (25)) when using nonlocal transformations.

7.2. Conditions that the regularizing functions must satisfy. Examples of regularizing functions

Since the transformed problem (94) must not have blow-up singularities, it is necessary that its solution $y = y(x)$, $t = t(x)$ satisfies the following condition:

$$\lim_{x \to x_*} I = \infty, \qquad I = \int_{x_0}^{x} g(x, y, t)\, dx. \tag{95}$$

We have

$$I = I_1 + I_2, \qquad I_1 = \int_{x_0}^{x_1} g(x, y, t)\, dx, \quad I_2 = \int_{x_1}^{x} g(x, y, t)\, dx, \tag{96}$$

where $x_1 > 0$ is a point sufficiently close to the blow-up point x_* ($x_1 < x_*$). The convergence or divergence of the integral I is determined by the convergence or divergence of the integral I_2 as $x \to x_*$.

1°. *Regularizing functions of the first type.* First, we consider regularizing functions of the form $g = g(|t|) > 0$, which in addition to the normalization condition $g(0) = 1$ satisfy the asymptotic condition of power growth for large $|t|$:

$$g(|t|) \to C|t|^\alpha \quad \text{as} \quad |t| \to \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0). \tag{97}$$

We use the asymptotics (2) and (97) to analyze the convergence or divergence of the integral I_2 in (96). We have

$$I_2 \simeq |A\beta|^\alpha C \int_{x_1}^{x} (x_* - x)^{-\alpha(\beta+1)}\, dx. \tag{98}$$

Hence it follows that the integral I_2 diverges if $\alpha(\beta + 1) \ge 1$ that is equivalent to the condition:

$$\alpha \ge \frac{1}{\beta + 1}. \tag{99}$$

For the most common singularity of the solution that has a first-order pole, which corresponds to the value $\beta = 1$, we should choose $\alpha \ge \frac{1}{2}$. Since $\beta > 0$, then $\alpha = 1$ is suitable for any blow-up solution of the power (and logarithmic) type.

The asymptotics, as $|t| \to \infty$, of regularizing functions of the form

$$g = (1 + k|t|^p)^q \qquad (k > 0,\ p > 0,\ q > 0) \tag{100}$$

is determined by the value $\alpha = pq$ in (97).

Let us consider two special types of the functions (100):

1. For the function (100) with $p = 2$ and $q = 1/2$ we have $\alpha = 1$ and the inequality (99) holds for any positive β.

2. For the function (100) with $p = 1$ and $q = 1/2$ we have $\alpha = 1/2$. The inequality (99) holds for a first-order pole, which is determined by the value $\beta = 1$, and also for all $\beta \geq 1$ (that is, for integer poles of any order).

2°. *Regularizing functions of the second type.* We now consider regularizing functions of the form $g = g(|f|) > 0$, where $f = f(x, y, t)$ is the right-hand side of the equation (80). In addition to the normalization condition $g(0) = 1$, let the regularizing function also satisfy the asymptotic condition of power growth for large $|f|$:

$$g(|f|) \to C|f|^\alpha \quad \text{as} \quad |f| \to \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0). \tag{101}$$

We use the asymptotics (2) and (101) to analyze the convergence or divergence of the integral I_2 in (96). Taking into account that $f = y''_{xx}$, we have

$$I_2 \simeq |A\beta(\beta + 1)|^\alpha C \int_{x_1}^{x} (x_* - x)^{-\alpha(\beta+2)}\, dx. \tag{102}$$

It follows that the integral I_2 diverges if $\alpha(\beta + 2) \geq 1$, or

$$\alpha \geq \frac{1}{\beta + 2}. \tag{103}$$

For the most common singularity of the solution that has a first-order pole, which corresponds to the value $\beta = 1$, we should choose $\alpha \geq 1/3$. Since $\beta > 0$, then $\alpha \geq 1/2$ is suitable for any blow-up solution of the power (and logarithmic) type.

The asymptotics, as $|f| \to \infty$, of regularizing functions of the form

$$g = (1 + k|f|^p)^q \qquad (k > 0,\ p > 0,\ q > 0) \tag{104}$$

is determined by the value $\alpha = pq$ in (101).

Let us consider two special types of the functions (104):

1. For the function (104) with $p = 1$ and $q = 1/2$ or $p = 1/2$ and $q = 1$ we have $\alpha = 1/2$ and the inequality (103) holds for any positive β.

2. For the function (104) with $p = 1$ and $q = 1/3$ or $p = 1/3$ and $q = 1$ we have $\alpha = 1/3$. The inequality (103) holds for a first-order pole, which is determined by the value $\beta = 1$, and also for all $\beta \geq 1$ (that is, for integer poles of any order).

Remark 16. A comparison of the exact solutions of a number of test problems and the corresponding numerical solutions obtained in this article by the method of nonlocal transformations shows that the most efficient regularizing functions are those that have the least admissible value of α, the exponent in the asymptotics (97) and (101), and are determined by the equality sign in (99) and (103).

3°. *Regularizing functions of the mixed type.* In a similar way, we can define the domain of divergence of regularizing functions of the mixed type $g = g(|t|, |f|)$, starting from the asymptotics of this function as $|t| + |f| \to \infty$. In particular, the function $g = (1+|t|+|f|)^{1/3}$ can be used if a solution has a singularity in the form of a pole of any integer order ($\beta = 1, 2, \ldots$).

7.3. Test problems and numerical solutions

Example 19. For the test problem (88), in which $f = b^2\gamma y^{2\gamma-1}$, we set $g = t/y$ (see Item 4° in Section 7.1). Substituting these functions into (94), we arrive at the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{y}{t}, \quad y'_\xi = y, \quad t'_\xi = \frac{b^2\gamma y^{2\gamma}}{t} \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \quad t(0) = a^\gamma b. \end{gathered} \tag{105}$$

The exact solution of this problem in parametric form is determined by the formulas

$$x = \frac{1}{a^{\gamma-1}b(\gamma-1)}\bigl[1 - e^{-(\gamma-1)\xi}\bigr], \quad y = ae^\xi, \quad t = a^\gamma b e^{\gamma\xi}. \tag{106}$$

It can be seen that the required function $x = x(\xi)$ exponentially tends to the asymptotic value $x_* = \dfrac{1}{a^{\gamma-1}b(\gamma-1)}$ as $\xi \to \infty$.

The numerical solutions of the problems (89) and (105) for $b = 1$, $\gamma = 2$, obtained by the Runge–Kutta method of the fourth-order of approximation, are shown in Fig. 8 for $a = 1$ and $a = 2$. For a fixed step of integration, equal to 0.2, the maximum difference between the exact solution (5) and the numerical solution of the related problem (105) is 0.0045%. For larger stepsizes, equal to 0.4 and 0.6, the maximum error in the numerical solutions is 0.061% and 0.24%, respectively. It can be seen that the numerical solutions of the problems (89) and (105) are in a good agreement, but the rates of their approximation to the required asymptote $x = x_*$ are significantly different. For example, for the

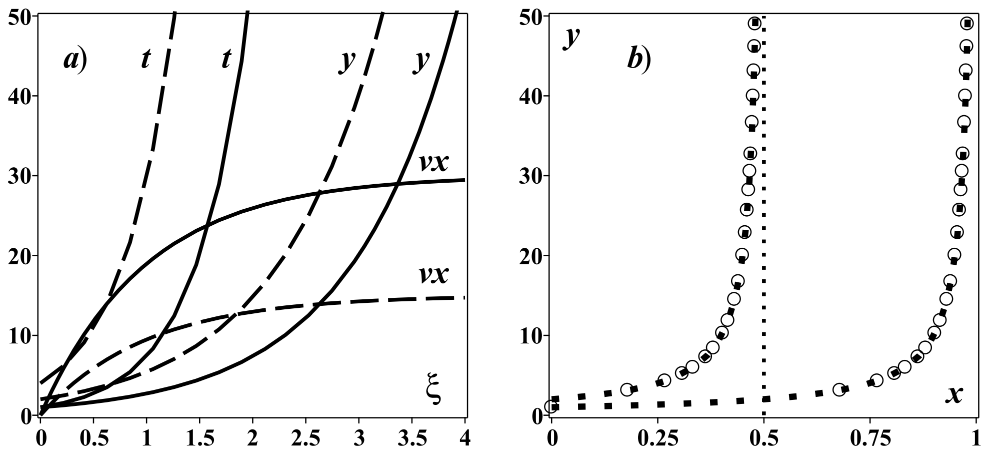

Figure 8: a)—the dependences $x = x(\xi)$, $y = y(\xi)$, $t = t(\xi)$, obtained by numerical solution of the problem (105) for $b = 1$, $\gamma = 2$ with $a = 1$ (solid lines) and $a = 2$ (dashed lines) ($\nu = 30$); b)—numerical solutions of the problems (89) for $b = 1$, $\gamma = 2$ (circles) and (105) (points); for left curves $a = 2$ and for right curves $a = 1$.

system (89), in order to obtain a good approximation to the asymptote, it is required to consider the interval $t \in [1, 2400]$, and for the system (105) it suffices to take $\xi \in [0, 4]$. Therefore, it should expect that the method based on the use of the system (94) with $g = t/y$ is much more efficient than the method based on the differential transformation.

For comparison, similar calculations were performed using Maple (2016), and applying the method based on the hodograph transformation (see Section 7.1, Item 1°) and the method of the arc-length transformation (see Section 7.1, Item 2° for $c = 1$ and $s = 2$). In order to obtain a good approximation to the asymptote, applying the method based on the hodograph transformation, it is required to consider the interval $\xi \in [0, 49]$, while using the method of the arc-length transformation leads to a significantly larger interval $\xi \in [0, 2500]$. To control a numerical integration process, the calculations were carried out with the aid of two other most important and powerful mathematical software packages: Mathematica (11), and MATLAB (2016a). It was found that the method based on the use of the system (105) with $g = t/y$ is essentially more efficient than the method based on the hodograph transformation and the method of the arc-length transformation.

Example 20. For the test problem (88), in which $f = b^2\gamma y^{2\gamma-1}$, we set $g = f/t$ (see Item 5° in Section 7.1). Substituting these functions into (94), we arrive at the Cauchy problem

$$x'_\xi = \frac{t}{b^2\gamma y^{2\gamma-1}}, \quad y'_\xi = \frac{t^2}{b^2\gamma y^{2\gamma-1}}, \quad t'_\xi = t \quad (\xi > 0);$$
$$x(0) = 0, \quad y(0) = a, \quad t(0) = a^\gamma b. \tag{107}$$

The exact solution of this problem in parametric form is

$$x = \frac{1}{a^{\gamma-1}b(\gamma-1)}\Big[1 - e^{-(\gamma-1)\xi/\gamma}\Big], \quad y = ae^{\xi/\gamma}, \quad t = a^\gamma b e^\xi. \tag{108}$$

The required value $x = x(\xi)$ tends exponentially to the asymptotic value $x_* = \dfrac{1}{a^{\gamma-1}b(\gamma-1)}$ as $\xi \to \infty$. However, in comparison with the method applied in Example 19, in this case the rate of approximation of the parametric solution to the asymptote is less (which is not important for application of the standard numerical methods for solving similar problems). Note that the solution (108) coincides with (106) if we redenote ξ by $\gamma\xi$.

7.4. Generalizations based on the use of differential constraints

The method of numerical integration of the Cauchy problems with blow-up solutions, which based on introducing a nonlocal variable, can be generalized if the relation (93) is replaced by the first-order differential constraint

$$\xi'_x = g(x, y, t, \xi) \tag{109}$$

with the initial condition $\xi(x = x_0) = \xi_0$.

If we set $\xi_0 = 0$, then the use of the differential constraint (109) leads to the problem (94), where the function $g(x, y, t)$ must be replaced by $g(x, y, t, \xi)$ in the equations.

Using differential constraints increases the possibilities for numerical analysis of blow-up problems.

In particular, if we choose a differential constraint of the form (109) with

$$g(x, y, t, \xi) = \frac{t}{\varphi(\xi)y + \psi(\xi)}, \tag{110}$$

where $\varphi(\xi)$ and $\psi(\xi)$ are given functions, then the second equation of the system (94) is reduced to the linear equation for $y = y(\xi)$, the solution of which is well known. As a result, the considered system, consisting of three equations, is simplified and reduced to two equations.

If we choose a differential constraint of the form (109) with

$$g(x,y,t,\xi) = \frac{f(x,y,t)}{\varphi(\xi)t + \psi(\xi)}, \tag{111}$$

then the third equation of the system (94) is reduced to the linear equation for $t = t(\xi)$. In this case, the system under consideration also is reduced to two equations.

Example 21. For the test Cauchy problem (88) with $b = 1$ and $\gamma = 2$, we take the differential constraint (109) with the function (111), where $f = 2y^3$, $\varphi(\xi) = 2(1+2\xi)$, and $\psi(\xi) = 0$. As a result, we arrive at the following problem for the ODE system:

$$\begin{gathered} x'_\xi = \frac{t(1+2\xi)}{y^3}, \quad y'_\xi = \frac{t^2(1+2\xi)}{y^3}, \quad t'_\xi = 2t(1+2\xi) \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \quad t(0) = a^2. \end{gathered} \tag{112}$$

The exact solution of the problem in parametric form is

$$x = \frac{1}{a}\bigl(1 - e^{-\xi-\xi^2}\bigr), \quad y = ae^{\xi+\xi^2}, \quad t = a^2 e^{2(\xi+\xi^2)}. \tag{113}$$

It can be seen that the required function $x = x(\xi)$ tends much faster to the asymptotic value $x_* = 1/a$ as $\xi \to \infty$ than in Examples 19 and 20.

7.5. Comparison of efficiency of various transformations for numerical integration of second-order blow-up ODE problems

In Table 2, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (93) and differential constraints of the form (109) is presented by using the example of the test blow-up problem for the second-order ODE (88) with $a = b = 1$ and $\gamma = 2$. The comparison is based on the number of grid points needed to make calculations with the same maximum error (approximately equal to 0.1 and 0.005).

It can be seen that the arc-length transformation is the least effective, since the use of this transformation is associated with a large number of grid points. In particular, when using the last four transformations, you need 150–200 times less of a number of grid points. The hodograph transformation has an intermediate (moderate) efficiency. The use of the exp-type transformation with $g = t/y$ gives rather good results.

The maximum absolute and relative errors of numerical integration of the problem (88) for $a = b = 1$ and $\gamma = 2$ by introducing a nonlocal variable (93)

$\text{Error}_{\max}, \% = 0.1$				
Transformation or differential constraint	Regularizing function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Arc-length	$g{=}\sqrt{1{+}t^2{+}f^2}$	2500.000	0.4150	6024
Nonlocal, Item 2°	$g{=}1{+}\|t\|{+}\|f\|$	2544.000	0.7550	3369
Hodograph	$g{=}t$	49.200	0.4510	109
Special exp-type, Item 5°	$g{=}f/t$	7.807	0.2110	37
Nonlocal	$g{=}(1{+}\|f\|)^{1/3}$	5.052	0.1486	34
Diff. constraint, p.c. of (111)	$g{=}f/[2t(1+2\xi)]$	1.550	0.0470	33
Nonlocal	$g{=}(1{+}\|t\|{+}\|f\|)^{1/3}$	5.217	0.1581	33
Nonlocal	$g{=}\sqrt{1{+}\|t\|}$	4.135	0.1334	31
Special exp-type, Item 4°	$g{=}t/y$	3.900	0.1300	30
Diff. constraint, p.c. of (110)	$g{=}t/[2(\xi+1)e^{2\xi+\xi^2}]$	1.218	0.0435	28
$\text{Error}_{\max}, \% = 0.005$				
Transformation or differential constraint	Regularizing function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Arc-length	$g{=}\sqrt{1{+}t^2{+}f^2}$	2500.000	0.2000	12500
Nonlocal, Item 2°	$g{=}1{+}\|t\|{+}\|f\|$	2544.000	0.3500	7268
Hodograph	$g{=}t$	49.000	0.1250	392
Special exp-type, Item 5°	$g{=}f/t$	7.821	0.0990	79
Diff. constraint, p.c. of (111)	$g{=}f/[2t(1+2\xi)]$	1.550	0.0210	74
Nonlocal	$g{=}(1{+}\|f\|)^{1/3}$	4.970	0.0700	71
Nonlocal	$g{=}(1{+}\|t\|{+}\|f\|)^{1/3}$	4.970	0.0738	70
Nonlocal	$g{=}\sqrt{1{+}\|t\|}$	4.134	0.0608	68
Special exp-type, Item 4°	$g{=}t/y$	3.900	0.0600	65
Diff. constraint, p.c. of (110)	$g{=}t/[2(\xi+1)e^{2\xi+\xi^2}]$	1.220	0.0200	61

Table 2: Various types of analytical transformations applied for numerical integration of the problem (88) for $a = b = 1$ and $\gamma = 2$ with a given accuracy (percent errors are 0.1 and 0.005 for $\Lambda_{\mathrm{m}} \le 50$) and their basic parameters (maximum interval, stepsize, grid points number). The abbreviation "p.c." stands for "particular case" and the notation $f = 2y^3$ is used.

with the regularizing function $g = (1 + |t|)^{1/2}$ for different values of stepsize h and Λ_{m} are given in Table 3. It can be seen that reducing the stepsize by one-half reduces the percent errors of numerical solutions by more than a factor of 23, and increasing Λ_{m} leads to an almost linear increasing of percent errors (increasing

$\Lambda_{\rm m}$ by a factor of 6 increases the percent errors by a factor of 6.15, for $h = 0.02$, or 4.07, for $h = 0.01$).

Stepsize $h = 0.02$				Stepsize $h = 0.01$			
$\Lambda_{\rm m}$	$\xi_{\max}$	$\lvert\text{error}\rvert_{\max}$	$\text{error}_{\max}$,%	$\Lambda_{\rm m}$	$\xi_{\max}$	$\lvert\text{error}\rvert_{\max}$	$\text{error}_{\max}$,%
50	4.14	0.000031237	0.000062344	50	4.14	0.000001980	0.000003951
100	4.84	0.000127968	0.000126838	100	4.84	0.000008111	0.000008039
150	5.24	0.000285740	0.000189849	150	5.24	0.000018111	0.000012033
200	5.54	0.000521560	0.000256717	200	5.53	0.000032401	0.000016108
300	5.94	0.001162662	0.000383607	300	5.93	0.000032401	0.000016108

Table 3: The maximum absolute and percent errors of numerical solutions of the problem (88) for $a = b = 1$ and $\gamma = 2$ by introducing a nonlocal variable (93) with the regularizing function $g = (1 + |t|)^{1/2}$ for various values of $\Lambda_{\rm m}$ and stepsize h.

So far, we have studied problems that have monotonic blow-up solutions. In the next section, we consider examples of more complex problems that have non-monotonic blow-up solutions.

7.6. Painlevé equations and their non-monotonic blow-up solutions

The Painlevé equations (whose solutions have movable singular points) and their properties play an important role not only in the theory of ODEs [35, 39–41], but also in the theory of nonlinear PDEs [39, 42].

It will be shown below that the first and the second Painlevé equations with suitable initial conditions have non-monotonic blow-up solutions. It is important to note that for problems having non-monotonic blow-up solutions, a method based on the hodograph transformation is inappropriate (since the inverse function is multivalued in such cases) and methods based on special exp-type transformations (see Remark 13) are also inappropriate. In the case of non-monotonic solutions, we must choose regularizing functions that satisfy the inequality $g > 0$.

Consider a Cauchy problem for the first Painlevé equation [35]

$$y''_{xx} = 6y^2 + x \quad (x > 0); \quad y(0) = a, \quad y'_x(0) = b. \tag{114}$$

For $a > 0$ and $b < 0$, the problem (114) has non-monotonic blow-up solutions.

Similarly, a Cauchy problem for the second Painlevé equation

$$y''_{xx} = 2y^3 + xy + c \quad (x > 0); \quad y(0) = a, \quad y'_x(0) = b, \tag{115}$$

for $a > 0$ and $b < 0$, can have non-monotonic blow-up solutions.

Numerical solution of the problem (114) for the first Painlevé equation with $a = 1$ and the three values of the parameter b ($b = 0$, $b = -10$, $b = -40$) and numerical solution of the problem (115) for the second Painlevé equation with $c = 0$, $a = 2$ and the three values of the parameter b ($b = 0$, $b = -2$, $b = -3$), which are obtained by integrating the transformed system with $g = (1 + |t| + |f|)^{1/3}$ by the Runge–Kutta method of the fourth-order approximation for the fixed stepsize $h = 0.01$ are shown by circles in Fig. 9*a* and Fig. 9*b* (the non-monotonic behavior of the solutions is presented in more detail on the corresponding lower figures). For example, if $b = -40$ and $b = -2$, the solutions of the first and second Painlevé equations have a non-monotonic character and exist in the finite regions $0 \le x < x_* = 1.0577704$ and $0 \le x < x_* = 0.8383873$, respectively. Reducing the stepsize by one-half, the maximum module of difference between the numerical solutions (with $h = 0.01$ and $h = 0.005$) for problem (114) with $b = -40$ is equal to 0.0000018 and for the problem (115) with $b = -2$ is equal to 0.0000101. It should be noted that $\min y = y(x_m) = -7.3590292$ with $x_m = 0.2829016$ for $b = -40$ and $\min y = y(x_m) = 1.8621645$ with $x_m = 0.1429965$ for $b = -2$.

8. Second-order autonomous equations. Solution of the Cauchy problem. Simple estimates

We consider the Cauchy problem for the second-order autonomous equation of the general form

$$y''_{xx} = f(y) \quad (x > 0), \qquad y(0) = a, \qquad y'_x(0) = b. \tag{116}$$

We assume that $a > 0$, $b \ge 0$ and $f(y) > 0$ is a continuous function that is defined for all $y \ge a$.

It is not difficult to show that, the equation (116) admits a first integral. As a result, with allowance for the initial conditions, we arrive at the Cauchy problem for the first-order autonomous equation

$$\begin{aligned} y'_x &= F(y) \quad (x > 0), \qquad y(0) = a; \\ F(y) &= \left[2 \int_a^y f(z)\, dz + b^2 \right]^{1/2}, \end{aligned} \tag{117}$$

which coincides with the problem (50), up to obvious modifications in notations. Therefore, we can use the results of Section 5.1.

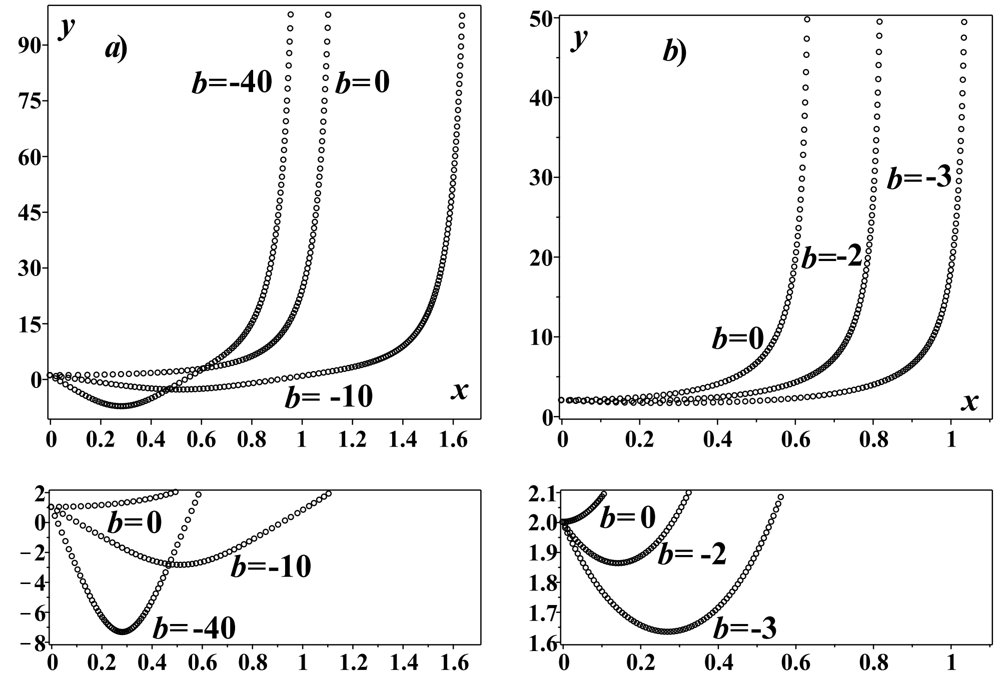

Figure 9: a)—numerical solutions of the problem (114) for the first Painlevé equation with $a = 1$, regularizing function $g = (1 + |t| + |f|)^{1/3}$, and $h = 0.01$; b)—numerical solutions of the problem (115) for the second Painlevé equation with $c = 0$ and $a = 2$, regularizing function $g = (1 + |t| + |f|)^{1/3}$, and $h = 0.01$.

The exact solution of the Cauchy problem (117) is determined by the formula (51), in which the function $f(y)$ should be replaced by $F(y)$. In blow-up problems, the critical value x_* is found by the formula (52), where the function $f(y)$ also must be replaced by $F(y)$.

Sufficient criterion of the existence of a blow-up solution. Suppose that for some $\kappa > 0$ we have the limiting relation

$$\lim_{y\to\infty} \frac{F(y)}{y^{1+\kappa}} = s, \qquad 0 < s \le \infty. \tag{118}$$

Then the solution of the Cauchy problem (116), when the above conditions are satisfied, is a blow-up solution.

The condition (118) is inconvenient, since it contains the function $F(y)$, which is rather complexly connected with the right-hand side $f(y)$ of the original equation (116). This condition can be simplified and transformed to a more convenient form:

$$\lim_{y\to\infty} \frac{f(y)}{y^{1+\kappa_1}} = s_1, \qquad 0 < s_1 \le \infty,$$

where κ_1 is a positive number.

By applying the sufficient criterion, we obtain the following useful result.

The Cauchy problem (116) for an autonomous equation with power nonlinearity, $f(y) = cy^\sigma$ ($c > 0$), has a blow-up solution if $\sigma > 1$.

9. Blow-up problems for systems of ODEs

9.1. Method based on nonlocal transformations

We consider the Cauchy problem for a system consisting of n first-order coupled ODEs of the general form

$$\frac{dy_m}{dx} = f_m(x, y_1, \dots, y_n), \qquad m = 1, \dots, n \quad (x > x_0), \tag{119}$$

with the initial conditions

$$y_m(x_0) = y_{m0}, \qquad m = 1,\, 2,\, \dots,\, n. \tag{120}$$

In blow-up problems, the right-hand side of at least one of the equations (119) (after substituting the solution into it) tends to infinity as $x \to x_*$, where the value x_* is unknown in advance.

In the general case, the functions f_m may have different signs. Further, we assume that $\sum_{m=1}^{n} |f_m| > 0$.

We associate the system (119) with the equivalent system of equations consisting of $(n+1)$ equations

$$\frac{dx}{d\xi} = \frac{1}{g(x, y_1, \dots, y_n)}, \quad \frac{dy_m}{d\xi} = \frac{f_m(x, y_1, \dots, y_n)}{g(x, y_1, \dots, y_n)}, \quad m{=}1, \dots, n \ \ (\xi{>}0) \qquad (121)$$

with the initial conditions

$$x(0) = x_0, \qquad y_m(0) = y_{m0}, \qquad m = 1,\, 2,\, \dots,\, n. \qquad (122)$$

Here ξ is a nonlocal variable defined by the formula

$$\xi = \int_{x_0}^{x} g(x, y_1, \dots, y_n)\, dx, \qquad y_m = y_m(x), \qquad m = 1, \dots, n \quad (\xi \geq 0). \quad (123)$$

In (121), it is assumed that $g > 0$ if $\sum_{m=1}^{n} |y_m| > 0$. Below we will describe some possible ways of choosing the function $g = g(x, y_1, \dots, y_n)$.

9.2. Special cases of nonlocal transformations

Let us consider some possible ways of choosing the function g in the system (121).

1°. We can take

$$g = \Big[c_0 + \sum_{m=1}^{n} c_m |f_m(x, y_1, \dots, y_n)|^s\Big]^{1/s}, \ \ c_0 > 0, \ \ c_m > 0, \ \ s > 0. \quad (124)$$

In particular, if we set $c_0 = c_m = s = 1$ ($m = 1, \dots, n$) in (124), then the system (121) takes the form

$$\frac{dx}{d\xi} = \frac{1}{1 + \sum_{m=1}^{n} |f_m(x, y_1, \dots, y_n)|}, \quad \frac{dy_m}{d\xi} = \frac{f_m(x, y_1, \dots, y_n)}{1 + \sum_{m=1}^{n} |f_m(x, y_1, \dots, y_n)|}, \quad (125)$$

where $m = 1, \dots, n$.

Unlike the right-hand sides of the original system (119), the right-hand sides of the system (121) with (124) have no singularities since all the derivatives are bounded, $|(y_m)'_\xi| \le 1$ ($m = 1, \dots, n$); we recall that for blow-up solutions at least one of the derivatives $(y_m)'_x$ tends to infinity as $x \to x_*$.

The numerical solution of the problem (121)–(122) with (124) can be obtained, for example, applying the Runge–Kutta method or other standard numerical methods, see above.

2°. For the system (121) with

$$g = \Big[1 + \sum_{m=1}^{n} f_m^2(x, y_1, \dots, y_n)\Big]^{1/2}, \tag{126}$$

we get the method of the arc-length transformation [4] (the function (126) is a particular case of (124) with $c_0 = c_m = 1$ and $s = 2$). Therefore for blow-up problems, the method based on introducing the nonlocal variable (123) is more general than the method of the arc-length transformation.

3°. In the general case, it is not known in advance whether the solution of the Cauchy problem (119)–(120) is a solution with usual properties, or is a blow-up solution. Therefore, in the first stage, the problem (119)–(120) can be solved by any standard fixed-step numerical method, for example, by the Runge–Kutta method. If one of the components, for example, y_k, begins to grow very rapidly (and increases faster than exponential and faster than the other components), then a hypothesis arises that the corresponding solution is a blow-up solution. Numerical confirmation of this hypothesis is a rapid growth of the ratio $|f_k/y_k|$ with increasing of the integration region with respect to x. In this case, it is reasonable to choose the function g in (121), for example, as follows:

$$g = \frac{1}{y_k} f_k(x, y_1, \dots, y_n). \tag{127}$$

As a result, the $(k+1)$-th equation of the system (121) is easily integrated and, taking into account the corresponding initial condition (122), we arrive at the dependence

$$y_k = y_{k0} e^{\xi}. \tag{128}$$

Substituting the relations (127) and (128) into the remaining equations of the system (121), we obtain the Cauchy problem

$$\frac{dx}{d\xi} = \frac{y_k}{f_k(x, y_1, \dots, y_n)}, \qquad \frac{dy_m}{d\xi} = \frac{y_k f_m(x, y_1, \dots, y_n)}{f_k(x, y_1, \dots, y_n)}, \qquad m = 1, \dots, n; \quad m \neq k \quad (\xi > 0), \tag{129}$$

with the initial conditions (122).

In the right-hand sides of the system (129), the function y_k should be replaced by the right-hand side of the formula (128).

The numerical solution of the system (129) with (128) and the initial conditions (122) can be obtained, for example, by applying the Runge–Kutta method or the other standard numerical methods with a sufficiently large stepsize in ξ.

Example 22. We consider the test Cauchy problem for the system of three equations

$$\frac{dy_1}{dx} = -y_1 y_2, \quad \frac{dy_2}{dx} = y_2^4 y_3, \quad \frac{dy_3}{dx} = -2y_1; \qquad y_1(0) = y_2(0) = y_3(0) = 1. \tag{130}$$

The exact solution of this problem has the form

$$y_1 = 1 - x, \quad y_2 = \frac{1}{1 - x}, \quad y_3 = (1 - x)^2. \tag{131}$$

A trial numerical integration of the problem (130) by the Runge–Kutta method shows that the component y_2 grows faster (in magnitude) that the other components. Using the formulas (127) and (128), we obtain that $g = y_2^3 y_3$ and $y_2 = e^{\xi}$. Substituting these functions into (129), and taking into account that $f_1 = -y_1 y_2$ and $f_3 = -2y_1$, we arrive at the equivalent Cauchy problem

$$\frac{dx}{d\xi} = \frac{e^{-3\xi}}{y_3}, \quad \frac{dy_1}{d\xi} = -\frac{e^{-2\xi} y_1}{y_3}, \quad \frac{dy_3}{d\xi} = -\frac{2e^{-3\xi} y_1}{y_3}; \qquad x(0) = 0, \quad y_1(0) = y_3(0) = 1. \tag{132}$$

Unlike the original problem (130), the problem (132) does not have blow-up singularities. Its exact solution is written in parametric form as follows:

$$x = 1 - e^{-\xi}, \quad y_1 = e^{-\xi}, \quad y_2 = e^{\xi}, \quad y_3 = e^{-2\xi}. \tag{133}$$

The numerical solution of the problem (132) is shown in Fig. 10. We do not present here the exact dependences (133), since they almost coincide (up to the maximum error 0.025%) with the results of the numerical solution.

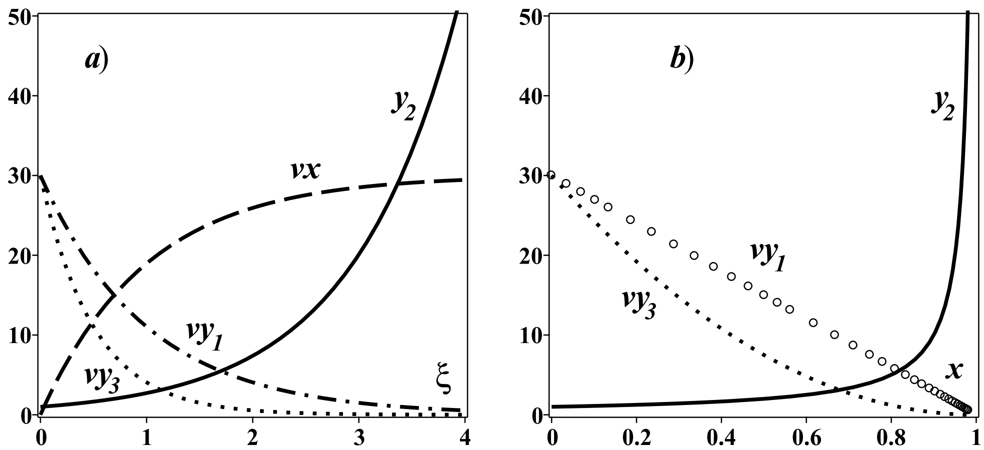

Figure 10: *a*)—the dependences $x = x(\xi)$, $y_1 = y_1(\xi)$, $y_3 = y_3(\xi)$, obtained by numerical solution of the problem (132) ($\nu = 30$), and $y_2 = e^{\xi}$; *b*)—numerical solution of the problem (132): $y_1(x)$ (circles), $y_2(x)$ (solid line), and $y_3(x)$ (points).

Remark 17. In the methods described in Items 1° and 2°, the rate of approximation of the function $x = x(\xi)$ to the asymptote, that determines the singular point x_*, will be power-law behavior with respect to ξ, while the method presented in Item 3°, yields the exponential rate of approximation of the singular point.

Remark 18. For systems of equations (119) of polynomial type, the most growing component y_k can be determined by substituting the approximate functions $y_1 = \alpha_1 z^{-\beta_1}$, $\dots$, $y_n = \alpha_n z^{-\beta_n}$ with $z = x_* - x$ into the equations. Then, from the analysis of the obtained linear algebraic relations (in the right-hand sides of these relations only the principal terms of the expansion in z must be taken into account), the largest exponent $\beta_k = \max[\beta_1, \dots, \beta_n]$ is found, where $\beta_k > 0$. The component y_k is used in formula (127) for the function g.

Example 23. Consider the problem (130). The solution in the neighborhood of the singular point is sought in the form

$$y_1 = \alpha_1 (x_* - x)^{-\beta_1}, \quad y_2 = \alpha_2 (x_* - x)^{-\beta_2}, \quad y_3 = \alpha_3 (x_* - x)^{-\beta_3}. \qquad (134)$$

Substituting the expressions (134) into (130), we obtain a simple system of linear algebraic equations for the exponents β_m ($m = 1, 2, 3$). The solution of the system is

$$\beta_1 = -1, \quad \beta_2 = 1, \quad \beta_3 = -2.$$

The maximum exponent is β_2. Therefore, the component y_2 should be used for the function g in formula (127).

Remark 19. If the two components, y_k and y_j, simultaneously have a blow-up behavior (with the same or different rate of approaching to infinity as $x \to x_*$), then we also can choose, for example, $g = c + |f_k| + |f_j|$ or $g = c_1 + \sqrt{c_2 + f_k^2 + f_j^2}$ in (121). Here c, c_1, and c_2 are some non-negative constants.

Remark 20. The technique developed in Section 9 can also be used in Cauchy problems for partial differential equations (PDEs) with blow-up solutions, if to apply the numerical methods in which the PDEs are approximated by ODE systems (for example, in projection methods and the method of lines [43, 44]).

9.3. Method based on differential constraints

For the Cauchy problems that are described by the systems of ODEs and have blow-up solutions, the method of numerical integration, based on introducing a nonlocal variable, can be generalized if, instead of the relation (123) to take the first-order differential constraint

$$\xi'_x = g(x, y_1, \dots, y_n, \xi) \tag{135}$$

with the initial condition $\xi(x = x_0) = \xi_0$.

If we set $\xi_0 = 0$, then the use of the differential constraint (135) leads to the problem (121), where the function $g(x, y_1, \dots, y_n)$ must be replaced by the function $g(x, y_1, \dots, y_n, \xi)$ in the equations.

10. Blow-up problems for higher-order ODEs

10.1. Reduction of higher-order ODEs to a system of first-order ODEs

Consider the Cauchy problem for the n th-order ODE:

$$\begin{gathered} y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}) \quad (x > x_0); \\ y(x_0) = y_0, \quad y'_x(x_0) = y_0^{(1)}, \quad \dots, \quad y_x^{(n-1)}(x_0) = y_0^{(n-1)}, \end{gathered} \tag{136}$$

where $y_x^{(k)} = d^k y/dx^k$ ($k = 3, \dots, n$).

The Cauchy problem for one n th-order ODE (136) is equivalent to the Cauchy problem for a system of n coupled first-order equations of the special form

$$\begin{gathered} y'_1 = y_2, \quad y'_2 = y_3, \quad \dots, \quad y'_{n-1} = y_n, \quad y'_n = f(x, y_1, y_2, \dots, y_n); \\ y_1(x_0) = y_0, \quad y_2(x_0) = y_0^{(1)}, \quad \dots, \quad y_n(x_0) = y_0^{(n-1)}, \end{gathered} \tag{137}$$

where the prime denotes the derivative with respect to x and $y_1 \equiv y$.

The problem (137) is a particular case of the Cauchy problem (119)–(120) and the general methods described in Sections 9.1 and 9.2 are applicable to it. As before, we introduce a new nonlocal independent variable ξ by means of an integral

$$\xi = \int_{x_0}^{x} g(x, y_1, y_2, \dots, y_n)\, dt, \qquad y_k = y_k(x) \equiv y_x^{(k)}, \tag{138}$$

where $g = g(x, y_1, y_2, \dots, y_n)$ is a regularizing function that can be varied. Then, by using (138), we pass from x to a new independent variable ξ in (137). As a result, the Cauchy problem for one equation of the n th order (136) is transformed to the following problem for a system consisting of $(n+1)$ st equation of the first order:

$$\begin{gathered} x' = \frac{1}{g}, \quad y_1' = \frac{y_2}{g}, \quad y_2' = \frac{y_3}{g}, \quad \dots, \quad y_{n-1}' = \frac{y_n}{g}, \quad y_n' = \frac{f}{g}; \\ x(0) = t_0, \quad y_1(0) = y_0, \quad y_2(0) = y_0^{(1)}, \quad \dots, \quad y_n(0) = y_0^{(n-1)}, \end{gathered} \tag{139}$$

where $f = f(x, y_1, y_2, \dots, y_n)$, $g = g(x, y_1, y_2, \dots, y_n)$, and the prime denotes the derivative with respect to ξ.

10.2. Examples of regularizing functions for n th-order equations

From (1), we obtain an approximate formula for the derivative of an arbitrary order in a neighborhood of the singular point t_*:

$$y_x^{(n)} \simeq A_n (x_* - x)^{-\beta-n}, \qquad A_n = A\beta(\beta+1)\cdots(\beta+n-1). \tag{140}$$

We consider regularizing functions of the form $g = g(|f|) > 0$, where f is the right-hand side of the equation (136). Suppose, in addition to the normalization condition $g(0) = 1$, that the function g satisfies the asymptotic condition of power growth for large $|f|$:

$$g(|f|) \to C|f|^{\alpha} \quad \text{as} \quad |f| \to \infty \quad \text{with} \quad \alpha > 0 \quad (C > 0). \tag{141}$$

The asymptotics (140) and (141) are needed to determine the conditions for the convergence or divergence of the integral in the right-hand side of (138). Using the same reasoning as in Section 7.2, it can be shown that the integral diverges if the following condition is satisfied:

$$\alpha \ge \frac{1}{\beta + n}. \tag{142}$$

Since $\beta > 0$, then $\alpha = 1/n$ is suitable for any blow-up power-type solution. For the most common singularity of the solution that has a first-order pole, which corresponds to the value $\beta = 1$, we can choose $\alpha = 1/(n+1)$ (this value of α is also suitable for a pole of any integer order).

The asymptotics, as $|f| \to \infty$, of regularizing functions of the form

$$g = (1 + k|f|^p)^q \qquad (k > 0,\ p > 0,\ q > 0) \tag{143}$$

is determined by the value $\alpha = pq$ in (141).

Let us consider two special types of suitable functions of the form (143):

1. For the function (143) with $p = 1$ and $q = 1/n$ or $p = 2$ and $q = 1/(2n)$ we have $\alpha = 1/n$ and the inequality (142) holds for any positive β.
2. For the function (143) with $p = 1$ and $q = 1/(n+1)$ or $p = 1/(n+1)$ and $q = 1$ we have $\alpha = 1/(n+1)$. The inequality (142) holds for a first-order pole, which is determined by the value $\beta = 1$, and also for all $\beta \geq 1$ (that is, for integer poles of any order).

10.3. Blow-up problems for third-order ODEs

Let us consider the Cauchy problem for the nonlinear third-order ODE of the general form

$$y'''_{xxx} = f(x, y, y', y''_{xx}) \quad (x > 0); \quad y(0) = y_0,\ y'_x(0) = y_1,\ y''_{xx}(0) = y_2. \tag{144}$$

The problem for one third-order ODE (144) is equivalent to the following problem for the system of three coupled first-order equations:

$$\begin{gathered} y'_x = t, \quad t'_x = w, \quad w'_x = f(x, y, t, w) \quad (x > 0); \\ y(0) = y_0, \quad t(0) = y_1, \quad w(0) = y_2. \end{gathered} \tag{145}$$

The introduction of the nonlocal variable (123) transforms the system (145) to the form

$$\begin{gathered} x'_\xi = \frac{1}{g}, \quad y'_\xi = \frac{t}{g}, \quad t'_\xi = \frac{w}{g}, \quad w'_\xi = \frac{f}{g} \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = y_0, \quad t(0) = y_1, \quad w(0) = y_2, \end{gathered} \tag{146}$$

where $f = f(x, y, t, w)$ and $g = g(x, y, t, w)$.

Let us consider various possibilities for choosing the function g in the system (146).

1°. We can take $g = \big(c_1 + c_2|t|^s + c_3|w|^s + c_4|f|^s\big)^{1/s}$ for $c_m > 0$ and $s > 0$. The case $c_1 = c_2 = c_3 = c_4 = 1$ and $s = 2$ corresponds to the method of the arc-length transformation [4].

2°. We can take $g = t/y$ (or $g = kt/y$, where $k > 0$ is a constant). In this case, the system (146) is simplified, since the second equation is directly integrated, and taking into account the second initial condition, we obtain $y = y_0 e^{\xi}$.

3°. We can take $g = w/t$ (or $g = kw/t$ with $k > 0$). In this case, the system (146) is simplified, since the third equation is directly integrated, and we obtain $t = y_1 e^{\xi}$. Taking into account the relations (145), we also have

$$\xi = \int_{x_0}^{x} \frac{w}{t}\,dx = \int_{x_0}^{x} \frac{y''_{xx}}{y'_x}\,dx = \ln \frac{y'_x}{y_1}.$$

Thus, this nonlocal transformation coincides with the modified differential transformation, which was considered in Section 2.3.

4°. We can take $g = f/w$ (or $g = kf/w$ with $k > 0$). In this case, the system (146) is also simplified, since the fourth equation is directly integrated, and we obtain $w = y_2 e^{\xi}$.

5°. Also, we can take $g{=}(1 + |f|)^{1/3}$ and $g{=}(1 + |f|)^{1/4}$ (see Section 10.2 for $n = 3$).

The transformations corresponding to the last three cases, 2°, 3°, and 4°, will be called the special exp-type transformations, they lead to the solutions, in which the variable x tends exponentially rapidly to a blow-up point x_*.

Example 24. We consider in more detail the test Cauchy problem of the form

$$y'''_{xxx} = 6y^4 \quad (x > 0); \quad y(0) = y'_x(0) = 1, \quad y''_{xx}(0) = 2. \tag{147}$$

The exact solution of this problem is determined by the formula (5).

In Table 4, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (93) is presented by using the example of the test blow-up problem for the third-order ODE (147). The comparison is based on the number of grid points needed to make calculations with the same maximum error (e.g., equal to 0.1 and 0.01).

It can be seen that the arc-length transformation is the least effective, since the use of this transformation is associated with a large number of grid points. In particular, when

$\text{Error}_{\max}, \% = 0.1$				
Transformation	Regularizing function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Arc-length, Item 1°	$g=\sqrt{1+t^2+w^2+f^2}$	249600.000	0.78000	320000
Nonlocal, Item 1°	$g=1+\|t\|+\|w\|+\|f\|$	252000.000	1.40000	180000
Hodograph	$g=t$	49.010	0.16900	290
Nonlocal, Item 5°	$g=(1+\|f\|)^{1/3}$	14.654	0.21550	68
Special exp-type, Item 4°	$g=f/w$	11.741	0.24980	47
Nonlocal, Item 5°	$g=(1+\|f\|)^{1/4}$	6.132	0.14600	42
Special exp-type, Item 2°	$g=t/y$	3.912	0.09780	40
Special exp-type, Item 3°	$g=w/t$	7.828	0.20600	38
$\text{Error}_{\max}, \% = 0.01$				
Transformation	Regularizing function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Arc-length, Item 1°	$g=\sqrt{1+t^2+w^2+f^2}$	252000.000	0.45000	560000
Nonlocal, Item 1°	$g=1+\|t\|+\|w\|+\|f\|$	253580.000	0.81800	310000
Hodograph	$g=t$	49.020	0.09500	516
Nonlocal, Item 5°	$g=(1+\|f\|)^{1/3}$	14.669	0.11830	124
Special exp-type, Item 4°	$g=f/w$	11.738	0.13810	85
Nonlocal, Item 5°	$g=(1+\|f\|)^{1/4}$	6.160	0.08000	77
Special exp-type, Item 2°	$g=t/y$	3.920	0.05600	70
Special exp-type, Item 3°	$g=w/t$	7.827	0.12230	64

Table 4: Various types of analytical transformations applied for numerical integration of the problem (147) with a given accuracy (percent errors are 0.1 and 0.01 for $\Lambda_{\rm m} \le 50$) and their basic parameters (maximum interval, stepsize, grid points number).

using the last three transformations, you need 6580–8750 times less of a number of grid points. The hodograph transformation has an intermediate (moderate) efficiency. The use of the last two special exp-type transformations with $g = t/y$ and $g = w/t$ gives rather good results. Note that an even smaller number of grid points can be obtained by using suitable differential constraints.

11. Elementary approaches allowing one to find the form of new variables

We now describe an elementary approach, based on simple semi-geometric considerations, which allows us to find the form of new variables that transform the original blow-up problem to a problem, more suitable for numerical integration, that does not have blow-up singularities.

11.1. Combination of point transformation and hodograph transformation

Let us consider the approximate relation (1) as an equation connecting the variables x and y. Solving it with respect to x (for concreteness, we assume that $A > 0$), we obtain

$$x = x_* - B_1 y^{-1/\beta}, \qquad B_1 = A^{1/\beta}. \tag{148}$$

It is seen that x tends to the blow-up point x_* slowly enough as $y \to \infty$ (by the power law $\sim y^{-1/\beta}$). If we make the substitution $y = e^{\xi}$, then the rate of approximation to the desired asymptotic value x_* will become exponential with respect to the new variable ξ (i.e., will increase significantly). It is convenient to represent the described procedure in the form of a transformation

$$\xi = \ln y, \qquad z = x, \tag{149}$$

where $z = z(\xi)$ is the new unknown function. As a result, we arrive at the dependence $z = x_* - B_1 e^{-\xi/\beta}$, which can be written in the parametric form

$$x = x_* - B_1 e^{-\xi/\beta}, \qquad y = e^{\xi}. \tag{150}$$

The transformation (149) is a combination of two simple point transformations: 1) the non-linear transformation $\bar{x} = x$, $\bar{y} = \ln y$ and 2) the hodograph transformation $\xi = \bar{y}$, $z = \bar{x}$. The transformation (149) is equivalent to the transformation (25) if $g = f/y$ (see Section 3.1, Item 5°) and to the transformation (93) if $g = t/y$ (see Section 7.1, Item 5°).

11.2. Combination of transformation, based on a differential variable, and point transformation

Differentiating the asymptotics (1), we have the following relations:

$$y'_x = A\beta(x - x_*)^{-\beta-1}, \qquad y = A\left(\frac{y'_x}{A\beta}\right)^{\frac{\beta}{\beta+1}} \tag{151}$$

(the second relation is obtained from the first one after elimination of x by means of (1)).

Excluding y from (148) with the help of the second relation (151), we obtain

$$x = x_* - B_2 (y'_x)^{-\frac{1}{\beta+1}}, \quad B_2 = (A\beta)^{\frac{1}{\beta+1}}. \tag{152}$$

It is seen that x tends to the blow-up point x_* slowly enough as $y'_x \to \infty$ (in accordance with the power law $\sim (y'_x)^{-1/(\beta+1)}$).

If we make the substitution $y'_x = e^{\xi}$, then the rate of approximation to the desired asymptotic value x_* will become exponential with respect to the new variable ξ (i.e., will increase significantly). The described procedure can be represented as a modified differential transformation

$$\xi = \ln y'_x, \quad x = x(\xi), \quad y = y(\xi), \tag{153}$$

which is based on a combination of the differential transformation $t = y'_x$ (see Sections 2.1 and 6.1) and the point transformation $\xi = \ln t$.

The transformation (153) determines the asymptotics of the solution (1) in a neighborhood of the blow-up singularity in the parametric form

$$x = x_* - B_2 e^{-\frac{1}{\beta+1}\xi}, \quad y = AB_2^{-\beta} e^{\frac{\beta}{\beta+1}\xi}. \tag{154}$$

In Section 2.3, apart from other considerations, it was described how one can obtain a transformation of the type (153).

11.3. Relation allowing one to control the calculation process

We now derive a useful formula that makes it possible to control the calculation process.

Taking into account the relations (1) and (151), we differentiate the relation y/y'_x. After elementary transformations, we obtain

$$\frac{1}{\beta} = \frac{y y''_{xx}}{(y'_x)^2} - 1 = \frac{y}{y'_\xi}\left(\frac{y''_{\xi\xi}}{y'_\xi} - \frac{x''_{\xi\xi}}{x'_\xi}\right) - 1, \tag{155}$$

where $x = x(\xi)$, $y = y(\xi)$ is the representation of the solution in the parametric form.

For blow-up problems with a power singularity, the constant β must be greater than zero. Therefore, for numerical representation of solutions in the parametric form, for large values of ξ the right-hand side of (155) must tend to a positive constant (asymptote), which allows us to control the calculation process.

12. Brief conclusions

Three new methods of numerical integration of Cauchy problems for nonlinear ODEs of the first and second-order, which have a blow-up solution, are described. These methods are based on differential and nonlocal transformations, and also on differential constraints, that lead to the equivalent problems for systems of equations, whose solutions are represented in parametric form and have no singularities.

It is shown that:

(i) the method based on a nonlocal transformation of the general form includes themselves, as particular cases, the method based on the hodograph transformation, the method of the arc-length transformation, and the methods based on the differential and modified differential transformations;

(ii) the methods based on the exp-type and modified differential transformations are much more efficient than the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation;

(iii) the method based on the differential constraints is the most general of the proposed methods.

In the Cauchy problems described by the first-order equations, two-sided theoretical estimates are established for the critical value of the independent variable $x = x_*$, when an unlimited growth of the solution occurs as approaching it.

It is shown that the method based on a nonlocal transformation of the general form as well as the method based on the differential constraints admit generalizations to the nth-order ordinary differential equations and systems of coupled differential equations.

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