Functional separable solutions of nonlinear reaction–diffusion equations with variable coefficients [1]

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Abstract

The paper presents a number of new functional separable solutions to nonlinear reaction–diffusion equations of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)u_x + p(x)f(u),$$

where $f(u)$ is an arbitrary function. It is shown that any three of the four variable coefficients $a(x)$, $b(x)$, $c(x)$, $p(x)$ of such equations can be chosen arbitrarily, and the remaining coefficient can be expressed through the others. Examples of specific equations and their exact solutions are given. The results obtained are generalized to more complex multidimensional nonlinear reaction–diffusion equations with variable coefficients. Also some functional separable solutions to nonlinear reaction–diffusion equations with delay

$$u_t = u_{xx} + a(x)f(u,w), \qquad w = u(x,t-\tau),$$

where $\tau > 0$ is the delay time and $f(u,w)$ is an arbitrary function of two arguments, are obtained.

It is important to note that the exact solutions of nonlinear PDEs and delay PDEs that contain arbitrary functions and therefore have sufficient generality are

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of the greatest practical interest for testing and evaluating the accuracy of various numerical and approximate analytical methods for solving corresponding initial-boundary value problems.

Keywords: nonlinear reaction–diffusion equations, reaction–diffusion equations with delay, equations with variable coefficients, exact solutions, functional separable solutions

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1. Introduction

1.1. A brief review of the literature

Transformations and exact solutions of various classes of nonlinear reaction–diffusion-convection equations (hereinafter briefly referred to as reaction–diffusion equations)

$$u_t = [f_1(u)u_x]_x + f_2(u)u_x + f_3(u) \tag{1}$$

and some other nonlinear equations that do not depend explicitly on the variables x, t, have been considered in many studies (see, for example, [1–19] and the literature cited therein). To construct exact solutions, the most frequently used methods were those based on the classical and nonclassical symmetry reductions [1–3, 5, 7, 10, 14, 15, 17–19], on generalized and functional separation of variables [6, 9, 11, 14, 16, 17], and on differential constraints [6, 13, 14, 16, 17].

In the general case, equation (1) admits the traveling wave solution $u = U(kx - \lambda t)$ [2] and for $f_2(u) = f_3(u) = 0$, it has the self-similar solution $u = U(xt^{-1/2})$ [1]. In addition to these cases, other exact solutions to equations of the form (1), in which at least one of the functions $f_n(u)$ is arbitrary, are known [6, 10, 14, 16, 17].

A number of studies (e.g., see [8, 17, 20–23]) have been devoted to nonlinear reaction–diffusion equations with variable coefficients,

$$c(x)u_t = [a(x)f_1(u)u_x]_x + b(x)f_2(u). \tag{2}$$

In particular, in [8, 12] (see also [17]) exact solutions to the following equations of the form (2) containing one or two arbitrary functions of the space variable were obtained:

$$u_t = [a(x)u^k u_x]_x + b(x)u^{k+1} \qquad [8, 12],$$
$$u_t = [a(x)u_x]_x + b_1 u \ln u + b_2 u \qquad [12],$$
$$u_t = a_1(u^{-4/3}u_x)_x + b(x)u^{-1/3} \qquad [8, 12],$$
$$u_t = [a(x)e^{\lambda u}u_x]_x + b(x)e^{\lambda u} \qquad [8, 12],$$

where $a(x)$ and $b(x)$ are arbitrary functions, and a_1, b_1, b_2, k, and λ are arbitrary constants.

In [6, 17], the exact (functional separable) solutions of four equations of the form

$$u_t = x^{-n}[x^n f_1(u)u_x]_x + f_2(u), \tag{3}$$

where the functions $f_1(u)$ and $f_2(u)$ are expressed in terms of an arbitrary function $\varphi(u)$ (n is any), were obtained. With the substitutions $z = (n+1)^{-n-1}x^{n+1}$ (for $n \neq -1$) and $z = -2\ln(x/2)$ (for $n = -1$), equation (3) can be reduced to an equation of the form (2) with a power or exponential dependence on the spatial coordinate:

$$\begin{aligned} &u_t = [z^k f_1(u)u_z]_z + f_2(u), \quad k = 2n/(n+1) \quad (n \neq -1);\\ &u_t = [e^z f_1(u)u_z]_z + f_2(u) \quad (n = -1). \end{aligned}$$

In [20–22], the methods of group analysis were used to analyze and construct exact solutions to equations of the form (2) with power and exponential nonlinearities.

In [23–29], symmetries and some exact solutions of nonlinear diffusion–convection equations with variable coefficients

$$c(x)u_t = [a(x)f_1(u)u_x]_x + b(x)f_2(u)u_x$$

were described.

Other related and more complex nonlinear evolution equations were considered in [17, 30–34]. Exact solutions to a number of systems of coupled equations of the reaction–diffusion type are described in [17, 35] (these books give an extensive list of publications on this topic), see also [36, 37].

It is also noteworthy that lately much attention has been paid to the study of hereditary systems, which are modeled by the nonlinear reaction–diffusion equations

$$u_t = au_{xx} + f(u,w), \quad w = u(x, t-\tau),$$

where $\tau > 0$ is the delay time. Exact solutions of such and more complicated nonlinear equations (as well as systems of coupled equations with delay) were obtained in [38–50].

The present paper deals with exact solutions allowed by nonlinear reaction–diffusion equations of a fairly general form (including nonlinear delay PDEs) that depend on one or more arbitrary functions.

1.2. The concept of 'exact solution' for nonlinear PDEs

In what follows, the term exact solution with regard to nonlinear partial differential equations is used in the following cases:

(i) the solution is expressible in terms of elementary functions;

(ii) the solution is expressible in closed form with definite or/and indefinite integrals;

(iii) the solution is expressible in terms of solutions to ordinary differential equations or systems of such equations.

Combinations of cases (i), (ii), and (iii) are also allowed.

Solutions of more complex nonlinear partial differential equations with delay, which are expressed in terms of solutions of ordinary differential equations with delay, will also be attributed to exact solutions.

2. Construction of exact solutions of one-dimensional nonlinear reaction–diffusion equations

2.1. Class of equations under consideration. Reduction of nonlinear reaction–diffusion equations to ODEs

The paper deals with the reaction–diffusion equations with a nonlinear source and variable coefficients

$$c(x)u_t = [a(x)u_x]_x + b(x)u_x + p(x)f(u), \tag{4}$$

where $f(u)$ is an arbitrary function. Some of the four functional coefficients $a = a(x) > 0$, $b = b(x)$, $c = c(x) > 0$, and $p = p(x)$ can be free, while the others can be expressed through them as a result of the subsequent analysis (the free coefficients can be chosen differently, see below). Without loss of generality, it will be assumed that $p > 0$ (for $p < 0$, the functions p and f must be redefined as $-p$ and $-f$).

Exact solutions to equation (4) will be sought in the form of a superposition of functions

$$u = U(z), \qquad z = \varphi(x, t). \tag{5}$$

Substituting (5) in (4) gives the functional-differential equation

$$a(x)\varphi_x^2 U''_{zz} + \bigl\{[a(x)\varphi_x]_x + b(x)\varphi_x - c(x)\varphi_t\bigr\}U'_z + p(x)f(U) = 0. \tag{6}$$

In the special case $U(z) = z$, equation (6) coincides with the original equation (4) (so at this stage no solutions are lost).

Let the coefficients of the equation satisfy the relations

$$p(x) = a(x)s(\varphi)\varphi_x^2, \tag{7}$$

$$c(x)\varphi_t = [a(x)\varphi_x]_x + b(x)\varphi_x + a(x)k(\varphi)\varphi_x^2, \tag{8}$$

where $s(\varphi)$ and $k(\varphi)$ are some functions ($s \not\equiv 0$). Then equation (6) reduces to the ordinary differential equation

$$U''_{zz} - k(z)U'_z + s(z)f(U) = 0. \tag{9}$$

Exact solutions of the nonlinear ordinary differential equation (9) for some functions $k(z)$, $s(z)$, $f(U)$ can be found in [51, 52].

In the special case $k(z) \equiv 0$, which corresponds to the linear equation (8), for $s(z) = 1$, the general solution of equation (9) for any function $f(U)$ can be written in an implicit form [51]:

$$\int \biggl[C_1 - 2\int f(U)\,dU\biggr]^{-1/2} dU = C_2 \pm z, \tag{10}$$

where C_1 and C_2 are arbitrary constants.

Equations (7)–(9) allow one to construct exact solutions for a wide class of nonlinear reaction–diffusion equations of the form (4).

Remark 1. Looking for solutions in the form (5) corresponds to a special form [17] of the Clarkson–Kruskal direct method for symmetry reductions [53, 54] (the use of the nonclassical method for symmetry reductions [54, 55] leads to more complex calculations).

Remark 2. In equation (4), without restriction of generality, two of four functional coefficients $a(x)$, $b(x)$, $c(x)$, $p(x)$ can be set equal to unity. In particular, if one divides both sides of the equation by c, and then changes from t, x to the new independent variables t, $y = \int \sqrt{c/a}\,dx$, then one obtains an equation in the canonical form $u_t = u_{yy} + b_1(y)u_y + p_1(y)f(u)$. It is not difficult to find a transformation t, $\bar{y} = \bar{y}(x)$, which reduces equation (4) to another canonical form $u_t = [a_2(\bar{y})u_{\bar{y}}]_{\bar{y}} + p_2(\bar{y})f(u)$. However, dealing with the equation in general form (4) is more convenient because it includes any canonical and noncanonical forms.

Remark 3. In equations (4), (6)–(8), the functions $a(x)$, $b(x)$, $c(x)$, and $p(x)$ can be replaced with functions of two variables $a(x,t)$, $b(x,t)$, $c(x,t)$, and $p(x,t)$.

2.2. A direct procedure for constructing exact solutions. Analysis and solutions of the determining system of equations in the general case

A direct procedure for constructing exact solutions of nonlinear equations of the form (4) suggests that the functions $a(x)$, $b(x)$, $c(x)$, and $f(u)$ are assumed given, and the functions $u = u(x,t)$ and $p = p(x)$ are the desired ones. In this case, having given in some way the functions $k(\varphi)$ and $s(\varphi)$, one has first to find particular solutions $p(x)$ and $\varphi = \varphi(x,t)$ of the equations (7) and (8) (the last equation can be linearized, see below). After this, with allowance for the relation (7), the solution of equation (4) is determined by formula (5), where the function $U(z)$ is a solution of the ordinary differential equation (9).

In the general case, two equations (7) and (8) for given functions $a = a(x)$, $b = b(x)$, $c = c(x)$, $p = p(x)$, $k(\varphi)$, and $s(\varphi)$ are an overdetermined nonlinear system of coupled equations for one function φ (this system will be called the *determining system of equations*). The properties of equations (7) and (8) will be sequentially investigated.

Equation (7) is transformed to an equation with separable variables $\sqrt{s(\varphi)}\,\varphi_x = \pm\sqrt{p(x)/a(x)}$. Its general solution is given by the formula

$$\int \sqrt{s(\varphi)}\,d\varphi = \pm \int \sqrt{p(x)/a(x)}\,dx + \xi(t), \tag{11}$$

where $\xi(t)$ is an arbitrary function. Therefore, in the general case, the function φ must have the form

$$\varphi = G(y), \qquad y = \xi(t) + \theta(x). \tag{12}$$

Note that solution (12) also admits another (but equivalent) representation of

$$\varphi = \bar{G}(\bar{y}), \qquad \bar{y} = \bar{\xi}(t)\bar{\theta}(x), \tag{13}$$

where $\bar{y} = e^y$, $\bar{\xi} = e^\xi$, and $\bar{\theta} = e^\theta$. Solutions of the form (12) and (13) often occur in mathematical physics and are called functional separable solutions [14, 17].

Nonlinear transformations

$$\varphi = F(\psi) \tag{14}$$

retain the form of equations (7) and (8), once the functional coefficients $k(\varphi)$ and $s(\varphi)$ are changed by the rule:

$$k(\varphi) \Longrightarrow k(F(\psi))F'_\psi(\psi) + \frac{F''_{\psi\psi}(\psi)}{F'_\psi(\psi)}, \qquad s(\varphi) \Longrightarrow s(F(\psi))[F'_\psi(\psi)]^2. \tag{15}$$

The degenerate case $k(\varphi) \equiv 0$ corresponds to the linear PDE with variable coefficients (8). For $k(\varphi) \not\equiv 0$, the nonlinear equation (8) can be reduced with the help of the substitution

$$\psi = C_1 \int K(\varphi)\, d\varphi + C_2, \qquad K(\varphi) = \exp\biggl[\int k(\varphi)\, d\varphi\biggr], \tag{16}$$

where C_1 and C_2 are arbitrary constants, to the linear equation

$$c(x)\psi_t = [a(x)\psi_x]_x + b(x)\psi_x. \tag{17}$$

In the special case $k(\psi) = k = \text{const}$, one can use the substitution

$$\varphi = k^{-1} \ln |\psi|, \tag{18}$$

which follows from (16).

Solutions of a linear equation with autonomous coefficients (17) can be constructed by the method of separation of variables. In particular, this equation has solutions with additive and multiplicative separation of variables:

$$\psi = \lambda t + \eta(x), \qquad [a(x)\eta'_x]'_x + b(x)\eta'_x - \lambda c(x) = 0; \tag{19}$$

$$\psi = \exp(\lambda t)\zeta(x), \qquad [a(x)\zeta'_x]'_x + b(x)\zeta'_x - \lambda c(x)\zeta = 0, \tag{20}$$

where λ is an arbitrary constant. The equation for η in (19) is easily integrated by the substitution $w(x) = \eta_x$, and the solutions of the linear equation for ζ in (20) for various functions $a(x)$, $b(x)$, and $c(x)$ are given in [51, 52]. Other exact solutions of equation (17) for certain functions $a(x)$, $b(x)$, and $c(x)$ can be found in [56].

Since transformations of the form (14) change only the functional coefficients $k(\varphi)$ and $s(\varphi)$ in equations (7) and (8), one can choose the function F, without loss of generality, so as to simplify one of these equations. Three possible ways of simplifying these equations are described below.

1°. For $s(\varphi) = 1$ and $k = k(\varphi)$, from formula (11) one finds

$$\varphi = \xi(t) + \theta(x), \tag{21}$$

which corresponds to $G(y) = y$ in (12). In this case, $p(x) = a(x)(\theta'_x)^2$.

2°. For $s(\varphi) = \varphi^{-1}$ and $k = k(\varphi)$, formula (11) gives

$$\varphi = \bar{\xi}(t)\bar{\theta}(x), \tag{22}$$

which corresponds to $\bar{G}(y) = y$ in (13).

3°. For $s = s(\varphi)$ and $k(\varphi) = 0$, equation (8) is a linear PDE of an autonomous form, the solutions of which are constructed by the method of separation of variables.

In what follows, the simplest representation of the solution in Item 1° will be used. Substituting (21) into equation (8) yields the functional-differential equation

$$c(x)\xi'_t = [a(x)\theta'_x]'_x + b(x)\theta'_x + a(x)(\theta'_x)^2 k(\varphi), \quad \varphi = \xi(t) + \theta(x). \tag{23}$$

The intention is to find the admissible forms of the function $k(\varphi)$ for which this equation can have solutions, using the differentiation method [14, 17]. To this end, first, dividing by c, allows us to represent equation (23) in the form

$$\xi'_t = Q(x) + R(x)k(\varphi), \quad \varphi = \xi(t) + \theta(x), \tag{24}$$

where $Q(x) = [(a\theta'_x)'_x + b\theta'_x]/c$ and $R(x) = a(\theta'_x)^2/c$. Differentiating both parts of (24) with respect to t, we transform the obtained equation to the form $\xi''_{tt}/\xi'_t = R(x)k'_\varphi(\varphi)$. We logarithm both parts of this equation, and then again differentiate by t. After dividing by ξ'_t, we have $[\ln(\xi''_{tt}/\xi'_t)]'_t/\xi'_t = [\ln k'_\varphi(\varphi)]'_\varphi$. Differentiating further with respect to x, we obtain

$$[\ln k'_\varphi(\varphi)]''_{\varphi\varphi} = 0.$$

The solutions of this ordinary differential equation are determined by the formulas

$$k(\varphi) = k_1\varphi + k_2 \qquad \text{(degenerate solution)}, \tag{25}$$

$$k(\varphi) = k_1 e^{-k_2\varphi} + k_3 \qquad \text{(non-degenerate solution)}, \tag{26}$$

where k_1, k_2, and k_3 are arbitrary constants. Formulas (25) and (26) define all admissible functions $k(\varphi)$ for which the functional-differential equation (23) can have a solution.

Formulas (21), (25), and (26) will be used in the following sections to construct exact solutions of nonlinear reaction–diffusion equations with autonomous coefficients (4).

2.3. The construction of exact solutions for $k(\varphi) = k$ and $s(\varphi) = 1$

Direct method of constructing exact solutions. In the simplest case, $k(\varphi) = k = \text{const}$, which corresponds to the values $k_1 = 0$ and $k_2 = k$ in (25), substituting expression (21) into equation (23) gives $\xi(t) = t$ (the constant factor is chosen equal to unity). Therefore, the class of equations (4) in this case admits exact solutions with a functional separation of variables of the form (5), where

$$\varphi(x,t) = t + \int g(x)\,dx. \tag{27}$$

Here, the function $g(x) = \theta'_x(x)$ can be given by the researcher or determined by further analysis (depending on the goal, see below). Substituting (27) into equation (7) for $s(\varphi) = 1$ and equation (8) for $k(\varphi) = k$ yields

$$p(x) = a(x)g^2(x), \tag{28}$$

$$c(x) = [a(x)g(x)]'_x + b(x)g(x) + ka(x)g^2(x). \tag{29}$$

Relation (29) connects the first three functional coefficients of equation (4) and the function $g = g(x)$ in (27) (this relation is differential with respect to functions a and g and algebraic with respect to functions b and c), and relation (28) is algebraic and is used to determine the functional coefficient $p(x)$.

If the three functions $a(x)$, $b(x)$, and $c(x)$ are assumed to be given, then relation (29) for $k \neq 0$ is the Riccati equation for the function $g = g(x)$. Let us rewrite this equation in the standard form:

$$a(x)g'_x + ka(x)g^2 + [b(x) + a'_x(x)]g - c(x) = 0. \tag{30}$$

An extensive list of exact solutions of equation (30) for the functions $a(x)$, $b(x)$, and $c(x)$ of various forms can be found in [51, 52]. Two cases will be considered.

Degenerate case. For $k = 0$, the Riccati equation (30) degenerates in a linear equation whose general solution has the form

$$g(x) = \frac{1}{a(x)}E(x)\biggl[\int \frac{c(x)}{E(x)}\,dx + C_1\biggr], \qquad E(x) = \exp\biggl[-\int \frac{b(x)}{a(x)}\,dx\biggr], \tag{31}$$

where C_1 is an arbitrary constant.

Example 1. In the case of constant coefficients of the equation $a = c = 1$, $b = 0$, using formulas (31) for $C_1 = 0$, one finds $g(x) = x$. Substituting this function in (27) and (28) for $s(\varphi) = 1$ gives $\varphi(x,t) = t + \frac{1}{2}x^2$, $p(x) = x^2$. It follows that the nonlinear reaction–diffusion equation

$$u_t = u_{xx} + x^2 f(u) \tag{32}$$

for an arbitrary function $f(u)$ admits a functional separable solution

$$u = U(z), \qquad z = t + \tfrac{1}{2}x^2. \tag{33}$$

Here, the function $U(z)$ is described by the autonomous ordinary differential equation

$$U''_{zz} + f(U) = 0 \tag{34}$$

(obtained by substituting the values $k = 0$ and $s = 1$ in (9)), whose general solution can be represented implicitly (10).

Example 2. Consider a more complicated situation when one of the coefficient of the equation depends in an arbitrary way on the spatial variable $a = a(x)$, and the other two are constants, $b(x) = 0$, $c(x) = 1$. Using formulas (31) with $C_1 = 0$, one finds that $g(x) = x/a(x)$. Substituting this function in (27) and (28), for $s(\varphi) = 1$ gives $\varphi(x,t) = t + \int \frac{x}{a(x)}\,dx$, $p(x) = x^2/a(x)$. Therefore, the nonlinear reaction–diffusion equation

$$u_t = [a(x)u_x]_x + \frac{x^2}{a(x)}f(u), \tag{35}$$

depending on two arbitrary functions $a(x)$ and $f(u)$, admits an exact solution with a functional separation of variables,

$$u = U(z), \qquad z = t + \int \frac{x}{a(x)}\,dx, \tag{36}$$

where the function $U(z)$ is described by a solvable autonomous ordinary differential equation (34).

Substituting $a(x) = x^n$, $a(x) = e^{\lambda x}$, $a(x) = xe^{\lambda x}$ into (35) yields the nonlinear equations

$$u_t = (x^n u_x)_x + x^{2-n}f(u), \tag{37}$$
$$u_t = (e^{\lambda x}u_x)_x + x^2 e^{-\lambda x}f(u), \tag{38}$$
$$u_t = (xe^{\lambda x}u_x)_x + xe^{-\lambda x}f(u), \tag{39}$$

which admit exact solutions for an arbitrary function $f(u)$.

It is interesting to note that the equation $u_t = (xu_x)_x + xf(u)$, which is a special case of equation (37) for $n = 1$, has a noninvariant solution of the traveling wave type, $u = U(x + t)$.

Nondegenerate case. For $k = \mathrm{const}$ ($k \neq 0$), the substitution

$$g = \frac{1}{k}\frac{y'_x}{y} \tag{40}$$

reduces equation (30) to the second-order linear differential equation

$$a(x)y''_{xx} + [b(x) + a'_x(x)]y'_x - kc(x)y = 0. \tag{41}$$

An extensive list of exact solutions of this equation for the functions $a(x)$, $b(x)$, and $c(x)$ of various forms can be found in [51, 52].

Example 3. In the case of constant coefficients $a = c = 1$, $b = 0$, the general solution of equation (41) has the form

$$y = \begin{cases} C_1 \cosh(mx) + C_2 \sinh(mx) & \text{if } \ k = m^2 > 0, \\ C_1 \cos(mx) + C_2 \sin(mx) & \text{if } \ k = -m^2 < 0, \end{cases} \tag{42}$$

where C_1 and C_2 are arbitrary constants. Putting $C_1 = 1$, $C_2 = 0$, $k = 1$ in (42) and using the formula (40), we find

$$g(x) = \tanh x.$$

Substituting this function into (27) and (28), we have

$$\varphi(x,t) = t + \ln\cosh x, \qquad p(x) = \tanh^2 x.$$

It follows that the nonlinear reaction–diffusion equation

$$u_t = u_{xx} + \tanh^2 x f(u) \tag{43}$$

for an arbitrary function $f(u)$ admits the functional separable solution

$$u = U(z), \qquad z = t + \ln\cosh x, \tag{44}$$

where the function $U(z)$ is described by the autonomous ordinary differential equation

$$U''_{zz} - U'_z + f(U) = 0. \tag{45}$$

The order of equation (45) can be reduced by one using the substitution $U'_z = \Phi(U)$, which leads to the Abel equation of the second kind in the canonical form. Exact solutions of equation (45) for some dependences of $f(U)$ are available in [51, 52].

In Table 1 shows the nonlinear equations $u_t = u_{xx} + p(x)f(u)$, where $f(u)$ is an arbitrary function, that admit exact solutions with a functional separation of variables of the form $u = U(z)$, $z = \varphi(x,t)$ (the function φ is determined to within an additive constant). For equations Nos. 1, 2, 4–7, the function $\varphi(x,t)$ is the sum of functions of different arguments (27). A traveling wave solution (see equation No. 1) corresponds to a degenerate solution of equation (30) of the form $g=\alpha=\text{const}$. Solutions of some equations of this type with more complicated functions $p(x)$ can be obtained by using formulas (42) from Example 3. The solution to the equation No. 3 is self-similar (see Example 4).

No.	Function $p(x)$	Function $\varphi(x,t)$	Equation for the function $U=U(z)$
1	1	$t+\alpha x$	$\alpha^2 U''_{zz} - U'_z + f(U) = 0$
2	x^2	$t+\frac{1}{2}x^2$	$U''_{zz} + f(U) = 0$
3	x^{-2}	$xt^{-1/2}$	$U''_{zz} + \frac{1}{2}zU'_z + z^{-2}f(U) = 0$
4	$\tanh^2(\alpha x)$	$t+\alpha^{-2}\ln\cosh(\alpha x)$	$U''_{zz} - \alpha^2 U'_z + \alpha^2 f(U) = 0$
5	$\coth^2(\alpha x)$	$t+\alpha^{-2}\ln\lvert\sinh(\alpha x)\rvert$	$U''_{zz} - \alpha^2 U'_z + \alpha^2 f(U) = 0$
6	$\tan^2(\alpha x)$	$t-\alpha^{-2}\ln\lvert\cos(\alpha x)\rvert$	$U''_{zz} + \alpha^2 U'_z + \alpha^2 f(U) = 0$
7	$\cot^2(\alpha x)$	$t-\alpha^{-2}\ln\lvert\sin(\alpha x)\rvert$	$U''_{zz} + \alpha^2 U'_z + \alpha^2 f(U) = 0$

Table 1: Nonlinear equations $u_t = u_{xx} + p(x)f(u)$ that admit exact solutions of the form $u = U(z)$, $z=\varphi(x,t)$. Here, $f(u)$ is an arbitrary function and α is an arbitrary constant ($\alpha \neq 0$).

Other ways of constructing exact solutions. We now consider other possibilities for constructing exact solutions of equations of the form (4) for $k(\varphi)=k$, $s(\varphi)=1$ without integrating the Riccati equation (30). To do this, we assume that $g(x)$ and any two of the three functions $a(x)$, $b(x)$, and $c(x)$ are given, and the remaining function will be found on the basis of (30). Table 2 describes the possible situations and provides formulas for determining the required function. The final form of the nonlinear reaction–diffusion equation is determined by substituting the function $p(x)=a(x)g^2(x)$ in (4).

No.	Functions that are known	Function we are looking for
1	$a=a(x)$, $b=b(x)$, $g=g(x)$	$c(x) = ag'_x + kag^2 + (b+a'_x)g$
2	$a=a(x)$, $c=c(x)$, $g=g(x)$	$b(x) = g^{-1}(c-ag'_x) - a'_x - kag$
3	$b=b(x)$, $c=c(x)$, $g=g(x)$	$a(x) = g^{-1}E\big[\int(c-bg)E^{-1}dx + C_1\big],\ E=\exp\big(-k\int g\,dx\big)$

Table 2: Different ways of specifying the functional coefficients of equation (4) for $p(x) = a(x)g^2(x)$. Here, k and C_1 are arbitrary constants and $g^{-1}=1/g$.

Example 4. We use the third way described in Table 2, with $b=0$, $c=1$ for an alternative representation of the equations and their exact solutions. There are two possible cases.

1. *Degenerate case for* $k = 0$. From Table 2, line No. 3, we find $a(x) = xg^{-1}(x)$, $p(x) = x^2/a(x)$, which leads to the equation (35).

2. *Nondegenerate case for* $k \neq 0$. From Table 2, line No. 3, with $k \neq 0$, $C_1 = 0$ we have $a(x) = g^{-1}E\int E^{-1}dx$. We introduce a new function $h = h(x)$ by putting $h = \int E^{-1}dx$. Differentiating this expression and taking into account the formula $E = \exp\bigl(-k\int g\,dx\bigr)$, we express the function g in terms of h. As a result of simple calculations, we finally get $g = k^{-1}h''_{xx}/h'_x$, $a = kh/h''_{xx}$, $p = k^{-1}hh''_{xx}/(h'_x)^2$. It follows that the equation

$$u_t = [a(x)u_x]_x + p(x)f(u), \quad a(x) = k\frac{h}{h''_{xx}}, \quad p(x) = \frac{1}{k}\frac{hh''_{xx}}{(h'_x)^2}, \tag{46}$$

where $f(u)$ and $h = h(x)$ are arbitrary functions, and $k \neq 0$ is an arbitrary constant, admits an exact solution with the generalized separation of variables

$$u = U(z), \quad z = t + \frac{1}{k}\ln|h'_x|.$$

Here, the function $U(z)$ is determined from the ordinary differential equation $U''_{zz} - kU'_z + f(U) = 0$.

Assuming, for example, in (46) $h = \sinh(\alpha x)$, $k = \alpha^2$, we obtain equation No. 4 from Table 1.

Substituting $h = \ln(\alpha x)$, $k = -1$ in (46), we arrive at the equation

$$u_t = [x^2\ln(\alpha x)u_x]_x + \ln(\alpha x)f(u), \tag{47}$$

which admits an exact solution of the form $u = U(z)$, where $z = t + \ln x$.

2.4. The direct construction of exact solutions for $k(\varphi) \neq \text{const}$

1. Case $k(\varphi) = k_1\varphi$, $s(\varphi) = 1$. For $k(\varphi) = k_1\varphi$, which corresponds to the value of $k_2 = 0$ in (25), substituting expression (21) in equation (23), we obtain $\xi(t) = e^{\lambda t}$. Therefore, in this case the class of equations (4) admits exact solutions with functional separation of variables of the form (5), where

$$\varphi(x,t) = e^{\lambda t} + \theta(x). \tag{48}$$

Substituting (48) into relation (7) for $s(\varphi) = 1$ and equation (23) for $k(\varphi) = k_1\varphi$ we obtain

$$c(x) = \frac{k_1}{\lambda}a(x)(\theta'_x)^2, \quad p(x) = a(x)(\theta'_x)^2. \tag{49}$$

In this case, the functions $a(x)$ and $b(x)$ remain arbitrary, and the function $\theta = \theta(x)$ is determined by solving the ordinary differential equation

$$[a(x)\theta'_x]'_x + b(x)\theta'_x + k_1 a(x)\theta(\theta'_x)^2 = 0. \tag{50}$$

The substitution $\eta = \int \exp\bigl(\frac{1}{2}k_1\theta^2\bigr)\,d\theta$ reduces equation (50) to the linear equation $[a(x)\eta'_x]'_x + b(x)\eta'_x = 0$, whose general solution is defined by the formula $\eta = C_1 \int \frac{1}{a}\exp\bigl(-\int \frac{b}{a}\,dx\bigr)\,dx + C_2$.

2. *Case* $k(\varphi) = k_1 e^{-k_2\varphi} + k_3$, $s(\varphi) = 1$. For $k(\varphi) = k_1 e^{-k_2\varphi} + k_3$, which corresponds to the use of the dependence (26), substituting expression (21) into equation (23), we obtain $\xi(t) = k_2^{-1}\ln t$. In this case, the class of equations (4) admits exact solutions with a functional separation of variables of the form (5), where

$$\varphi(x,t) = \frac{1}{k_2}\ln t + \theta(x). \tag{51}$$

Substituting (51) into relation (7) for $s(\varphi) = 1$ and equation (23) for $k(\varphi) = k_1 e^{-k_2\varphi} + k_3$, we get

$$p(x) = a(x)(\theta'_x)^2, \quad c(x) = k_1 k_2 a(x) e^{-k_2\theta}(\theta'_x)^2. \tag{52}$$

The functions $a(x)$ and $b(x)$ remain arbitrary, and the function $\theta = \theta(x)$ is determined by solving the nonlinear ordinary differential equation

$$[a(x)\theta'_x]'_x + b(x)\theta'_x + k_3 a(x)(\theta'_x)^2 = 0. \tag{53}$$

This equation is easily integrated, since the substitution $\zeta(x) = \theta'_x$ leads it to the Bernoulli equation. In particular, for $k_3 = 0$, the general solution of equation (53) is given by the formula

$$\theta(x) = C_1 \int \frac{1}{a}\exp\biggl(-\int \frac{b}{a}\,dx\biggr)\,dx + C_2.$$

Example 5. Let

$$a(x) = 1, \quad b(x) = 0, \quad k_1 = -\tfrac{1}{2}, \quad k_2 = -2, \quad k_3 = 1. \tag{54}$$

In this case, equation (53) has a solution $\theta = \ln x$. Substituting this function into formulas (51) and (52), and taking (54) into account, we obtain

$$\varphi(x,t) = -\tfrac{1}{2}\ln t + \ln x, \quad p(x) = x^{-2}, \quad c(x) = 1. \tag{55}$$

Therefore, the equation

$$u_t = u_{xx} + x^{-2} f(u) \tag{56}$$

admits the self-similar solution

$$u = U(z), \quad z = -\tfrac{1}{2}\ln t + \ln x \equiv \ln(xt^{-1/2}), \tag{57}$$

where the function $U(z)$ satisfies the ordinary differential equation

$$U''_{zz} + (\tfrac{1}{2}e^{2z} - 1)U'_z + f(U) = 0. \tag{58}$$

Note that in applications usually use an alternative representation of similar solutions, which is based on the introduction of the self-similar variable $\bar{z} = e^z = xt^{-1/2}$ and reduces equation (58) to the equation $U''_{\bar{z}\bar{z}} + \frac{1}{2}\bar{z}U'_{\bar{z}} + \bar{z}^{-2}U = 0$ (see equation No. 3 in the Table 1).

Example 6. Equations (52) and equation (53) are satisfied if we set

$$a(x) = 1, \ \ b(x) = 0, \ \ c(x) = e^{-x}, \ \ p(x) = 1, \ \ \theta(x) = x, \ \ k_1 = k_2 = 1, \ \ k_3 = 0.$$

Therefore, the equation $e^{-x}u_t = u_{xx} + f(u)$ admits an exact solution of the form $u = U(z)$, where $z = x + \ln t$.

3. Multidimensional equations and equations with delay

3.1. Nonlinear reaction–diffusion equations with several spatial variables

The results obtained in Sections 2.1 and 2.2 can be generalized to the case of a multidimensional reaction–diffusion equation with a nonlinear source

$$cu_t = \sum_{n=1}^{N} \frac{\partial}{\partial x_n}\left(a_n \frac{\partial u}{\partial x_n}\right) + \sum_{n=1}^{N} b_n \frac{\partial u}{\partial x_n} + pf(u), \tag{59}$$

whose coefficients depend on spatial coordinates and time: $a_n = a_n(\mathbf{x}, t)$, $b_n = b_n(\mathbf{x}, t)$, $c = c(\mathbf{x}, t)$, $p = p(\mathbf{x}, t)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $n = 1, \ldots, N$.

We seek exact solutions of equation (59) in the form of a superposition of functions

$$u = U(z), \quad z = \varphi(\mathbf{x}, t). \tag{60}$$

We require that the coefficients of equation (59) and the function φ be related by two relations

$$p = s(\varphi) \sum_{n=1}^{N} a_n \left(\frac{\partial \varphi}{\partial x_n} \right)^2, \tag{61}$$

$$c\varphi_t = \sum_{n=1}^{N} \frac{\partial}{\partial x_n} \left(a_n \frac{\partial \varphi}{\partial x_n} \right) + \sum_{n=1}^{N} b_n \frac{\partial \varphi}{\partial x_n} + k(\varphi) \sum_{n=1}^{N} a_n \left(\frac{\partial \varphi}{\partial x_n} \right)^2, \tag{62}$$

where $s(\varphi)$ and $k(\varphi)$ are certain functions ($s \not\equiv 0$). Then equation (59) reduces to the ordinary differential equation

$$U''_{zz} - k(z)U'_z + s(z)f(U) = 0. \tag{63}$$

The nonlinear transformation $\varphi = F(\psi)$ preserves the form of equations (61) and (62), and the functional coefficients $k(\varphi)$ and $s(\varphi)$ vary according to rule (15). The transformation (16) leads the nonlinear equation (62) to the linear equation

$$c\psi_t = \sum_{n=1}^{N} \frac{\partial}{\partial x_n} \left(a_n \frac{\partial \psi}{\partial x_n} \right) + \sum_{n=1}^{N} b_n \frac{\partial \psi}{\partial x_n}. \tag{64}$$

For the equation (59) with autonomous coefficients $a_n = a_n(\mathbf{x})$, $b_n = b_n(\mathbf{x})$, $c = c(\mathbf{x})$, and $p = p(\mathbf{x})$, the solution of equation (61) has the form $\varphi = G(y)$, where $y = \xi(t) + \theta(\mathbf{x})$. Without limiting generality, by putting $s(\varphi) = 1$, the exact solutions of equations (61) and (62) can be found as a function with additive separation of variables

$$\varphi = \xi(t) + \theta(\mathbf{x}). \tag{65}$$

Substituting (65) in equation (62), we obtain a functional differential equation that allows solutions only for coefficients $k(\varphi)$ of the form (25) and (26) (the analysis is carried out in the same way as it was done in Section 2.2).

Example 7. It is easy to verify that for $a_n = 1$, $b_n = 0$ ($n = 1, \dots, N$), $c = 1$, $k(\varphi) = 0$, $s(\varphi) = 1$, equations (61) and (62) can be satisfied if put $p = |\mathbf{x}|^2$, $\varphi = Nt + \frac{1}{2}|\mathbf{x}|^2$, where $|\mathbf{x}|^2 = x_1^2 + \dots + x_n^2$. Therefore, the N-dimensional nonlinear reaction–diffusion equation

$$u_t = \Delta u + |\mathbf{x}|^2 f(u) \tag{66}$$

(Δ is the Laplace operator), depending on an arbitrary function $f(u)$, admits an exact solution with a functional separation of the variables

$$u = U(z), \quad z = Nt + \tfrac{1}{2}|\mathbf{x}|^2. \tag{67}$$

Here, the function $U(z)$ is described by an autonomous ordinary differential equation $U''_{zz} + f(U) = 0$, whose general solution can be written in implicit form (10).

Remark 4. The functional coefficients a_n in equation (59) can have different signs. In particular, for $a_1 = -1$, $a_n > 0$ ($n = 2, \dots, N$), $c = 0$, equation (59) is an equation of hyperbolic type.

3.2. Nonlinear equations of reaction–diffusion type with delay

The results obtained in Section 2 can also be generalized to the case of more complicated nonlinear reaction–diffusion equations with a delay of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)u_x + p(x)f(u, w), \quad w = u(x, t - \tau), \tag{68}$$

where $\tau > 0$ is the delay time, $f(u, w)$ is an arbitrary function of two arguments.

Let us show how the solutions of the nonlinear reaction–diffusion equation without delay (4), which are determined by formulas (5) and (27), can be used to construct exact solutions of the nonlinear equation with delay (68). Let equation (4) admit a solution with a functional separation of the form

$$u = U(z), \quad z = t + \theta(x), \tag{69}$$

where the function $U(z)$ satisfies the ordinary differential equation (9). Then the equation with delay (68) admits an exact solution of the form (69), where the function $U(z)$ satisfies the ordinary differential equation with delay

$$U''_{zz} - k(z)U'_z + s(z)f(U, W) = 0, \quad W = U(z - \tau). \tag{70}$$

Equations (32), (35), (37)–(39), (47), as well as equations Nos. 4–7 of the Table 1 (obtained for $k(z) = 0$, $s(z) = 1$), have solutions of the form (69). Therefore, more complex nonlinear reaction–diffusion equations with delay, which are obtained from these equations by replacing the function $f(u)$ by the function $f(u, w)$, also admit exact solutions of the form (69).

Example 8. Nonlinear reaction–diffusion equation with delay

$$u_t = u_{xx} + x^2 f(u, w), \quad w = u(x, t - \tau), \tag{71}$$

which is a generalization of equation (32), for an arbitrary function $f(u,w)$ admits an exact solution with functional separation of variables

$$u = U(z), \qquad z = t + \tfrac{1}{2}x^2, \tag{72}$$

where the function $U(z)$ is described by the delay ordinary differential equation

$$U''_{zz} + f(U,W) = 0, \qquad W = U(z-\tau). \tag{73}$$

Note that for $f(U,W) = Ug(W/U)$, the equation (73) admits an exact the solution of the form $U = Ce^{\lambda z}$, where C is an arbitrary constant, and λ is determined from the transcendental equation $\lambda^2 + g(e^{-\tau\lambda}) = 0$.

4. Brief conclusions

To summarize, the paper has presented a number of exact functional separable solutions to nonlinear reaction–diffusion equations of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)u_x + p(x)f(u),$$

where $f(u)$ is an arbitrary function. Solutions were sought in the form $u = U(z)$ with $z = \varphi(x,t)$, where the functions $U(z)$ and $\varphi(x,t)$ are determined in the course of further analysis. It has been shown that any three of the four functional coefficients $a(x)$, $b(x)$, $c(x)$, $p(x)$ of the reaction–diffusion equation can be chosen arbitrarily. Examples of specific equations and their exact solutions are given. The results are to extend to multidimensional nonlinear reaction–diffusion equations with variable coefficients. Also some exact solutions with a functional separation of variables of nonlinear reaction–diffusion equations with delay

$$u_t = u_{xx} + a(x)f(u,w), \qquad w = u(x, t-\tau),$$

where $\tau > 0$ is the delay time and $f(u,w)$ is an arbitrary function of two arguments, have been obtained.

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