Generalized traveling-wave solutions of nonlinear reaction-diffusion equations with delay and variable coefficients [1]

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Abstract

The paper presents a number of new exact solutions to nonlinear reaction-diffusion equations with delay of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)F(u, w), \qquad w = u(x, t - \tau),$$

where $\tau > 0$ is the delay time, and $F(u, w)$ is an arbitrary function of two arguments. Solutions are sought in the form of a generalized traveling-wave, $u = U(z)$ with $z = t + \theta(x)$. It is shown that one of the two functional coefficients $a(x)$ and $b(x)$ of the equation considered can be specified arbitrarily. Examples of delay reaction-diffusion equations and their solutions are given. New exact solutions of few other nonlinear delay PDEs are also obtained.

Keywords: reaction-diffusion equations with delay, nonlinear reaction-diffusions equations, equations with variable coefficients, exact solutions, generalized traveling-wave solutions, functional separable solutions

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1. Introduction

Symmetry, reductions and exact solutions of various classes of nonlinear reaction-diffusion equations

$$u_t = [g(u)u_x]_x + f(u)$$

and some related and more complex nonlinear equations and systems of equations that do not depend explicitly on the variables x, t, were considered in many works (see, for example, [1–13] and the literature cited therein).

A number of studies (e.g., see [9, 14–19]) is devoted to nonlinear reaction-diffusion equations with variable coefficients of an autonomous type

$$c(x)u_t = [a(x)g(u)u_x]_x + b(x)f(u).$$

Recently much attention has been paid to the study of hereditary systems, which are modeled by the nonlinear reaction-diffusion equations with delay

$$u_t = au_{xx} + F(u, w), \qquad w = u(x, t - \tau),$$

where $\tau > 0$ is the delay time and $a > 0$ is a constant. Exact solutions of such and more complicated related nonlinear equations were obtained in [20–27].

In what follows, the term exact solution with regard to nonlinear delay partial differential equations is used in the following cases:

(i) the solution is expressible in terms of elementary functions or in closed form with definite or/and indefinite integrals;

(ii) the solution is expressible in terms of solutions to ordinary differential or delay ordinary differential equations (or systems of such equations).

Combinations of cases (i) and (ii) are also allowed.

2. Class of equations under consideration. Reduction of nonlinear reaction-diffusion equations with delay to delay ODEs

We shall consider the nonlinear reaction-diffusion equations with delay

$$u_t = [a(x)u_x]_x + b(x)F(u, w), \qquad w = u(x, t - \tau), \tag{1}$$

where $\tau > 0$ is the delay time, $F(u, w)$ is an arbitrary function of two arguments, and $a(x) > 0$ and $b(x)$ are some functions.

We seek exact solutions of equation (1) in the form of a generalized traveling-wave,

$$u = U(z), \qquad z = t + \int g(x)\,dx. \tag{2}$$

Substituting (2) in (1), we obtain the functional-differential equation

$$a(x)g^2 U''_{zz} + \big\{[a(x)g]'_x - 1\big\}U'_z + b(x)F(U,W) = 0, \tag{3}$$

where $W = U(z-\tau)$ and $g = g(x)$.

Let the coefficients of the equation satisfy the relations

$$b(x) = a(x)g^2, \tag{4}$$

$$[a(x)g]'_x = -ka(x)g^2 + 1, \tag{5}$$

where k is a constant. Then equation (3) reduces to the delay ODE

$$U''_{zz} - kU'_z + F(U,W) = 0, \qquad W = U(z-\tau). \tag{6}$$

Remark 1. For $F(U,W) = Uf(W/U)$, the equation (6) admits an exact solution of the form $U = Ce^{\lambda z}$, where C is an arbitrary constant, and λ is determined from the transcendental equation $\lambda^2 - k\lambda + f(e^{-\tau\lambda}) = 0$ (different roots of the transcendental equation generate different exact solutions to the delay ODE in question).

Equations (4)–(6) allow us to construct exact solutions for a wide class of nonlinear reaction-diffusion equations of the form (1).

3. Direct way of construction exact solutions

If the function $a(x)$ is assumed to be given, then relation (5) is the Riccati ODE for the function $g = g(x)$. We consider two cases.

Degenerate case. For $k = 0$, the Riccati equation (5) degenerates in a linear equation whose general solution has the form

$$g(x) = \frac{x + C}{a(x)}, \tag{7}$$

where C is an arbitrary constant. Therefore, the nonlinear reaction-diffusion equation with delay

$$u_t = [a(x)u_x]_x + \frac{x^2}{a(x)}F(u,w), \qquad w = u(x, t-\tau), \tag{8}$$

depending on two arbitrary functions $a(x)$ and $F(u,w)$, admits the generalized traveling-wave solution

$$u = U(z), \qquad z = t + \int \frac{x}{a(x)}\,dx, \tag{9}$$

where the function $U(z)$ is described by a delay ordinary differential equation (6) with $k = 0$.

Example 1. Substituting the function $a(x) = x^n$ into (8), we obtain nonlinear delay equation

$$u_t = (x^n u_x)_x + x^{2-n} F(u,w), \tag{10}$$

where n is any number.

Nondegenerate case. For $k = \text{const}$ ($k \neq 0$), the substitution

$$g = \frac{1}{k}\frac{y'_x}{y} \tag{11}$$

reduces the Riccati equation (5) to the second-order linear ODE

$$a(x)y''_{xx} + a'_x(x)y'_x - ky = 0. \tag{12}$$

Exact solutions of this equation for some functions $a(x)$ can be found in [28].

Example 2. In the case of a constant coefficient $a(x) = 1$, the general solution of equation (12) has the form

$$y = \begin{cases} C_1\cosh(mx) + C_2\sinh(mx) & \text{if } \ k = m^2 > 0, \\ C_1\cos(mx) + C_2\sin(mx) & \text{if } \ k = -m^2 < 0, \end{cases} \tag{13}$$

where C_1 and C_2 are arbitrary constants, $m \neq 0$. Putting $C_1 = 1$, $C_2 = 0$, $k = 1$ in (13) and using the formula (11), we find

$$g(x) = \tanh x.$$

Substituting this function into (2) and (4), we have

$$z = t + \ln\cosh x, \qquad b(x) = \tanh^2 x.$$

It follows that the nonlinear reaction-diffusion equation with delay

$$u_t = u_{xx} + \tanh^2 x F(u,w), \qquad w = u(x, t-\tau), \tag{14}$$

for an arbitrary function $F(u,w)$, has the generalized traveling-wave solution

$$u = U(z), \qquad z = t + \ln\cosh x, \tag{15}$$

where the function $U(z)$ is described by the delay ordinary differential equation

$$U''_{zz} - U'_z + F(U,W) = 0, \qquad W = U(z-\tau). \tag{16}$$

In Table 1 shows the nonlinear equations $u_t = u_{xx} + b(x)F(u,w)$, where $F(u,w)$ is an arbitrary function that admit exact generalized traveling-wave solutions of the form $u = U(z)$, $z = t + \theta(x)$. A traveling wave solution (see equation No. 1) corresponds to a degenerate solution of equation (5) of the form $g = \text{const}$.

No.	Function $b(x)$	Function $\theta(x)$	Equation for the function $U = U(z)$
1	1	$t + \alpha x$	$\alpha^2 U''_{zz} - U'_z + F(U,W) = 0$
2	x^2	$t + \frac{1}{2}x^2$	$U''_{zz} + F(U,W) = 0$
3	$\tanh^2(\alpha x)$	$t + \alpha^{-2}\ln\cosh(\alpha x)$	$U''_{zz} - \alpha^2 U'_z + \alpha^2 F(U,W) = 0$
4	$\coth^2(\alpha x)$	$t + \alpha^{-2}\ln\lvert\sinh(\alpha x)\rvert$	$U''_{zz} - \alpha^2 U'_z + \alpha^2 F(U,W) = 0$
5	$\tan^2(\alpha x)$	$t - \alpha^{-2}\ln\lvert\cos(\alpha x)\rvert$	$U''_{zz} + \alpha^2 U'_z + \alpha^2 F(U,W) = 0$
6	$\cot^2(\alpha x)$	$t - \alpha^{-2}\ln\lvert\sin(\alpha x)\rvert$	$U''_{zz} + \alpha^2 U'_z + \alpha^2 F(U,W) = 0$

Table 1: Nonlinear equations $u_t = u_{xx} + b(x)F(u,w)$ that admit exact solutions of the form $u = U(z)$, $z = t + \theta(x)$. Here, $F(u,w)$ is an arbitrary function and α is an arbitrary constant ($\alpha \neq 0$).

Remark 2. Solutions Nos. 2-6 in Table 1 are also new for the simpler special case of reaction-diffusion equations without delay, which are determined by a kinetic function of the form $F(u,w) = f(u)$.

4. Alternative way of constructing exact solutions

We now consider another possibility of constructing exact solutions of equations of the form (1) without integrating the Riccati equation (5). To do this, we assume that the function $g(x)$ is given, and the function $a(x)$ will be found on the basis of (5). In this case, we have

$$a(x) = \frac{E(x)}{g(x)}\left[\int \frac{dx}{E(x)} + C\right], \qquad E = \exp\left[-k\int g(x)\,dx\right], \tag{17}$$

where C is an arbitrary constant. The final form of the nonlinear reaction-diffusion equation with delay (1) is given by (17) together with $b(x) = a(x)g^2(x)$, where $g(x)$ is an arbitrary function.

5. Exact solutions of other nonlinear PDEs with delay

1. Consider nonlinear equation of the reaction-diffusion type with delay

$$u_t = [a(x)u_x]_x + F(x, u - w), \quad w = u(x, t - \tau). \tag{18}$$

We seek exact solutions of this equation in the form (2). As a result, for the function $g = g(x)$ we get the first-order linear equation $(ag)'_x + f(x, \tau) - 1 = 0$, the general solution of which is

$$g(x) = \frac{1}{a(x)}\left[x - \int F(x, \tau)\,dx + C\right], \tag{19}$$

where C is an arbitrary constant.

Example 3. For $F(x, u) = b(x)f(u)$, equation (18), depending on three arbitrary functions $a(x)$, $b(x)$, $f(u)$, admits an exact solution (2) with $g(x) = \big[x - f(\tau)\int b(x)\,dx + C\big]/a(x)$.

2. Another nonlinear equation of reaction-diffusion type with delay

$$u_t = [a(x)u_x]_x + uF(x, w/u), \quad w = u(x, t - \tau), \tag{20}$$

has an exact solution in the form of a product of functions of different arguments $u = e^{\lambda t}\xi(x)$, where λ is an arbitrary constant and the function $\xi = \xi(x)$ is described by the linear ODE, $[a(x)\xi'_x]'_x + [F(x, e^{-\lambda\tau}) - \lambda]\xi = 0$.

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