Functional separable solutions of nonlinear convection–diffusion equations with variable coefficients *

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The paper presents a number of new functional separable solutions to nonlinear convection–diffusion equations of the form

$$c(x)u_t = [a(x)u_x]_x + [b(x) + p(x)f(u)]u_x,$$

where $f(u)$ is an arbitrary function. It shows that any three of the four variable coefficients $a(x)$, $b(x)$, $c(x)$, $p(x)$ of such equations can be chosen arbitrarily, and the remaining coefficient can be expressed through the others. Examples of specific equations and their exact solutions are given. The results obtained are generalized to more-complex nonlinear PDEs with variable coefficients. Also some functional separable solutions to nonlinear convection–diffusion equations with delay

$$u_t = u_{xx} + a(x)f(u, w)u_x, \qquad w = u(x, t - \tau),$$

where $\tau > 0$ is the delay time and $f(u, w)$ is an arbitrary function of two arguments, are obtained.

Keywords: nonlinear convection–diffusion equations, convection–diffusion equations with delay, equations with variable coefficients, exact solutions, functional separable solutions

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1 Introduction

1.1 A brief review of the literature

Transformations and exact solutions of various classes of nonlinear reaction–diffusion–convection equations

$$u_t = [f_1(u)u_x]_x + f_2(u)u_x + f_3(u) \tag{1}$$

and some other nonlinear equations that do not depend explicitly on the variables x, t, have been considered in many studies (see, for example, [1–18] and the literature cited therein). To construct exact solutions, the most frequently used methods were those based on the classical and nonclassical symmetry reductions [1–3, 5, 7, 9, 13, 14, 16–18], on generalized and functional separation of variables [6,8,10,13,15,16], and on differential constraints [6,12,13,15,16].

In the general case, equation (1) admits the traveling wave solution $u = U(kx - \lambda t)$ [2] and for $f_2(u) = f_3(u) = 0$, it has the self-similar solution $u = U(xt^{-1/2})$ [1]. In addition, other exact solutions to equations of the form (1), in which at least one of the functions $f_n(u)$ is arbitrary, are known [6,9,13,15,16].

A number of studies (e.g., see [11,16,19–23]) have been devoted to nonlinear reaction–diffusion equations with variable coefficients,

$$c(x)u_t = [a(x)f_1(u)u_x]_x + b(x)f_2(u). \tag{2}$$

In [16,24–29], symmetries and some exact solutions of nonlinear diffusion–convection equations with variable coefficients

$$c(x)u_t = [a(x)f_1(u)u_x]_x + b(x)f_2(u)u_x$$

were described.

Other related and more complex nonlinear evolution equations were considered in [16,30–36]. Exact solutions to a number of systems of coupled equations of the reaction–diffusion type are described in [16,37] (these books give an extensive list of publications on this topic); see also [38,39].

It is also noteworthy that lately much attention has been paid to studying hereditary systems, which are modeled by the nonlinear reaction–diffusion equations

$$u_t = au_{xx} + f(u, w), \qquad w = u(x, t - \tau),$$

where $\tau > 0$ is the delay time. Exact solutions of such and more complicated nonlinear equations (as well as systems of coupled equations with delay) were obtained in [40–54].

The present paper deals with exact solutions admitted by nonlinear convection–diffusion equations of a fairly general form (including some nonlinear delay PDEs) that depend on one or more arbitrary functions.

Remark 1. Importantly, exact solutions of nonlinear PDEs and delay PDEs that contain arbitrary functions and therefore have significant generality are of greatest practical interest for testing and evaluating the accuracy of various numerical and approximate analytical methods for solving corresponding initial-boundary value problems.

1.2 The concept of 'exact solution' for nonlinear PDEs

In what follows, the term 'exact solution' with regard to nonlinear partial differential equations is used in the following cases:

(i) the solution is expressible in terms of elementary functions;

(ii) the solution is expressible in closed form with definite or/and indefinite integrals;

(iii) the solution is expressible in terms of solutions to ordinary differential equations or systems of such equations.

Combinations of cases (i), (ii), and (iii) are also allowed.

Solutions of more-complex nonlinear partial differential equations with delay, which are expressed in terms of solutions of ordinary differential equations with delay, will also be attributed to exact solutions.

2 Construction of exact solutions of one-dimensional nonlinear convection–diffusion equations

2.1 Class of equations under consideration. Reduction of nonlinear convection–diffusion equations to ODEs

The paper deals with the nonlinear convection–diffusion equations with variable coefficients

$$c(x)u_t = [a(x)u_x]_x + [b(x) + p(x)f(u)]u_x, \tag{3}$$

where $f(u)$ is an arbitrary function. Some of the four functional coefficients $a = a(x) > 0$, $b = b(x)$, $c = c(x) > 0$, and $p = p(x)$ can be free, while the others can be expressed through them as a result of a subsequent analysis (the free coefficients can be chosen differently, as shown below).

Exact solutions to equation (3) will be sought in the form of a superposition of functions

$$u = U(z), \qquad z = \varphi(x,t). \tag{4}$$

Substituting (4) in (3) gives the functional-differential equation

$$a(x)\varphi_x^2 U''_{zz} + \big\{[a(x)\varphi_x]_x + b(x)\varphi_x - c(x)\varphi_t\big\}U'_z + p(x)\varphi_x f(U)U'_z = 0. \tag{5}$$

In the special case $U(z) = z$, equation (5) coincides with the original equation (3). At this stage no solutions are lost.

Let the coefficients of the equation satisfy the relations

$$p(x) = a(x)s(\varphi)\varphi_x, \tag{6}$$

$$c(x)\varphi_t = [a(x)\varphi_x]_x + b(x)\varphi_x + a(x)k(\varphi)\varphi_x^2, \tag{7}$$

where $s(\varphi)$ and $k(\varphi)$ are some functions ($s \not\equiv 0$). Then equation (5) reduces to the ordinary differential equation

$$U''_{zz} + [s(z)f(U) - k(z)]U'_z = 0. \tag{8}$$

Exact solutions of the nonlinear ordinary differential equation (8) for some functions $k(z)$, $s(z)$, $f(U)$ can be found in [55, 56].

In the special case $k(z) = k = \text{const}$ and $s(z) = s = \text{const}$, the general solution of equation (8) for any function $f(U)$ can be written in an implicit form

$$\int \frac{dU}{kU - sF(U) + C_1} = z + C_2, \tag{9}$$

where $F(U) = \int f(U)\,dU$, and C_1 and C_2 are arbitrary constants.

Equations (6)–(8) allow one to construct exact solutions for a wide class of nonlinear convection–diffusion equations of the form (3).

Remark 2. In equation (3), the number of functional coefficients dependent on the spatial coordinate can be reduced to two. In particular, transformations of an independent variable of the form $y = Y(x)$ allow us to reduce equation (3) to the canonical forms $u_t = [\bar{a}(y)u_y]_y + \bar{p}(y)f(u)u_y$ and $u_t = \bar{a}(y)u_{yy} + \bar{p}(y)f(u)u_y$. However, dealing with the equation in general form (3) is more convenient because it includes any canonical and noncanonical forms.

Remark 3. In equations (3), (5)–(7), the functions $a(x)$, $b(x)$, $c(x)$, and $p(x)$ can be replaced with functions of two variables $a(x,t)$, $b(x,t)$, $c(x,t)$, and $p(x,t)$ (see also Section 3.1).

2.2 A direct procedure for constructing exact solutions. Analysis and solutions of the determining system of equations in the general case

A direct procedure for constructing exact solutions of nonlinear equations of the form (3) suggests that the functions $a(x)$, $b(x)$, $c(x)$, and $f(u)$ are assumed given, and the functions $u = u(x,t)$ and $p = p(x)$ are the desired ones. In this case, with the functions $k(\varphi)$ and $s(\varphi)$ given in some way, one has first to find particular solutions $p(x)$ and $\varphi = \varphi(x,t)$ of equations (6) and (7) (the last equation can be linearized; see below). After this, with allowance for relation (6), a solution of equation (3) is determined by formula (4), where the function $U(z)$ is a solution of the ordinary differential equation (8).

In the general case, two equations (6) and (7) for given functions $a = a(x)$, $b = b(x)$, $c = c(x)$, $p = p(x)$, $k(\varphi)$, and $s(\varphi)$ form an overdetermined nonlinear system of coupled equations for one function φ (this system will be called the *determining system of equations*). The properties of equations (6) and (7) will be sequentially investigated.

The general solution of equation (6) is given by the formula

$$\int s(\varphi)\,d\varphi = \int \frac{p(x)}{a(x)}\,dx + \xi(t), \tag{10}$$

where $\xi(t)$ is an arbitrary function. Therefore, in the general case, the function φ

must have the form

$$\varphi = G(y), \qquad y = \xi(t) + \theta(x). \tag{11}$$

Note that solution (11) also admits another (however equivalent) representation of

$$\varphi = \bar{G}(\bar{y}), \qquad \bar{y} = \bar{\xi}(t)\bar{\theta}(x), \tag{12}$$

where $\bar{y} = e^y$, $\bar{\xi} = e^\xi$, and $\bar{\theta} = e^\theta$. Solutions of the form (11) and (12) often occur in mathematical physics and are called functional separable solutions [13, 16].

Nonlinear transformations

$$\varphi = F(\psi) \tag{13}$$

preserve the form of equations (6) and (7), while the functional coefficients $k(\varphi)$ and $s(\varphi)$ are changed by the rule:

$$k(\varphi) \Longrightarrow k(F(\psi))F'_\psi(\psi) + \frac{F''_{\psi\psi}(\psi)}{F'_\psi(\psi)}, \qquad s(\varphi) \Longrightarrow s(F(\psi))F'_\psi(\psi). \tag{14}$$

The degenerate case $k(\varphi) \equiv 0$ corresponds to the linear PDE with variable coefficients (7). For $k(\varphi) \not\equiv 0$, the nonlinear equation (7) can be reduced with the help of the substitution

$$\psi = C_1 \int K(\varphi)\, d\varphi + C_2, \qquad K(\varphi) = \exp\biggl[\int k(\varphi)\, d\varphi\biggr], \tag{15}$$

where C_1 and C_2 are arbitrary constants, to the linear equation

$$c(x)\psi_t = [a(x)\psi_x]_x + b(x)\psi_x. \tag{16}$$

In the special case $k(\psi) = k = \text{const}$, one can use the substitution

$$\varphi = k^{-1} \ln |\psi|, \tag{17}$$

which follows from (16).

Solutions of a linear equation with autonomous coefficients (16) can be constructed by the method of separation of variables. In particular, this equation has solutions with additive and multiplicative separation of variables:

$$\psi = \lambda t + \eta(x), \qquad [a(x)\eta'_x]'_x + b(x)\eta'_x - \lambda c(x) = 0; \tag{18}$$

$$\psi = \exp(\lambda t)\zeta(x), \quad [a(x)\zeta'_x]'_x + b(x)\zeta'_x - \lambda c(x)\zeta = 0, \tag{19}$$

where λ is an arbitrary constant. The equation for η in (18) is easily integrated through the substitution $w(x) = \eta_x$, and the solutions of the linear equation for ζ in (19) for various functions $a(x)$, $b(x)$, and $c(x)$ are given in [55,56]. Other exact solutions of equation (16) for certain functions $a(x)$, $b(x)$, and $c(x)$ can be found in [57].

Since transformations of the form (13) change only the functional coefficients $k(\varphi)$ and $s(\varphi)$ in equations (6) and (7), one can choose the function F, without loss of generality, so as to simplify one of these equations. Three possible ways of simplifying these equations are described below.

1°. For $s(\varphi) = 1$ and $k = k(\varphi)$, from formula (10) one finds

$$\varphi = \xi(t) + \theta(x), \tag{20}$$

which corresponds to $G(y) = y$ in (11). In this case, $p(x) = a(x)\theta'_x$.

2°. For $s(\varphi) = \varphi^{-1}$ and $k = k(\varphi)$, formula (10) gives

$$\varphi = \bar{\xi}(t)\bar{\theta}(x), \tag{21}$$

which corresponds to $\bar{G}(y) = y$ in (12).

3°. For $s = s(\varphi)$ and $k(\varphi) = 0$, equation (7) is a linear PDE of an autonomous form, the solutions of which are constructed by the method of separation of variables.

In what follows, the simplest representation of the solution in Item 1° will be used. Substituting (20) into equation (7) yields the functional-differential equation

$$c(x)\xi'_t = [a(x)\theta'_x]'_x + b(x)\theta'_x + a(x)(\theta'_x)^2 k(\varphi), \qquad \varphi = \xi(t) + \theta(x). \tag{22}$$

The intention is to find admissible forms of the function $k(\varphi)$ for which this equation can have solutions, using the differentiation method [13,16]. To this end, first, dividing by $c = c(x)$, allows us to represent equation (22) in the form

$$\xi'_t = Q(x) + R(x)k(\varphi), \qquad \varphi = \xi(t) + \theta(x), \tag{23}$$

where $Q(x) = [(a\theta'_x)'_x + b\theta'_x]/c$ and $R(x) = a(\theta'_x)^2/c$. Differentiating both parts of (23) with respect to t, we transform the obtained equation to the form $\xi''_{tt}/\xi'_t = R(x)k'_\varphi(\varphi)$. We logarithm both parts of this equation, and then again differentiate by t. After dividing by ξ'_t, we have $[\ln(\xi''_{tt}/\xi'_t)]'_t/\xi'_t = [\ln k'_\varphi(\varphi)]'_\varphi$. Differentiating further with respect to x, we obtain

$$[\ln k'_\varphi(\varphi)]''_{\varphi\varphi} = 0.$$

The solutions of this ordinary differential equation are determined by the formulas

$$k(\varphi) = k_1\varphi + k_2 \qquad \text{(degenerate solution)}, \tag{24}$$

$$k(\varphi) = k_1 e^{-k_2\varphi} + k_3 \qquad \text{(non-degenerate solution)}, \tag{25}$$

where k_1, k_2, and k_3 are arbitrary constants. Formulas (24) and (25) define all admissible functions $k(\varphi)$ for which the functional-differential equation (22) can have a solution.

Formulas (20), (24), and (25) will be used in the subsequent sections to construct exact solutions of nonlinear convection–diffusion equations with autonomous coefficients (3).

2.3 The construction of exact solutions for $k(\varphi) = k$ and $s(\varphi) = 1$

Direct method of constructing exact solutions. In the simplest case, $k(\varphi) = k = \text{const}$, which corresponds to the values $k_1 = 0$ and $k_2 = k$ in (24), substituting expression (20) into equation (22) gives $\xi(t) = t$ (the constant factor is chosen equal to unity). Therefore, the class of equations (3) in this case admits functional separable solutions of the form (4), where

$$\varphi(x,t) = t + \int g(x)\,dx. \tag{26}$$

Here, the function $g(x) = \theta'_x(x)$ can be prescribed by the researcher or determined in the subsequent analysis (depending on the goal; see below). Substituting (26) into equation (6) with $s(\varphi) = 1$ and equation (7) with $k(\varphi) = k$ yields

$$p(x) = a(x)g(x), \tag{27}$$

$$c(x) = [a(x)g(x)]'_x + b(x)g(x) + ka(x)g^2(x). \tag{28}$$

Relation (28) connects the first three functional coefficients of equation (3) and the function $g = g(x)$ in (26) (this relation is differential with respect to the functions a and g and algebraic with respect to the functions b and c), and relation (27) is algebraic and is used to determine the functional coefficient $p(x)$.

If the three functions $a(x)$, $b(x)$, and $c(x)$ are assumed to be given, then relation (28) with $k \neq 0$ is a Riccati equation for the function $g = g(x)$. Let us rewrite this equation in the standard form:

$$a(x)g'_x + ka(x)g^2 + [b(x) + a'_x(x)]g - c(x) = 0. \tag{29}$$

An extensive list of exact solutions of equation (29) for the functions $a(x)$, $b(x)$, and $c(x)$ of various forms can be found in [55, 56]. Two cases will be considered.

Degenerate case. For $k = 0$, the Riccati equation (29) degenerates into a linear equation whose general solution has the form

$$g(x) = \frac{1}{a(x)} E(x) \biggl[\int \frac{c(x)}{E(x)}\, dx + C_1 \biggr], \qquad E(x) = \exp\biggl[-\int \frac{b(x)}{a(x)}\, dx \biggr], \tag{30}$$

where C_1 is an arbitrary constant.

Example 1. In the case of constant coefficients $a = c = 1$ and $b = 0$, using formulas (30) with $C_1 = 0$, one finds $g(x) = x$. Substituting this function in (26) and (27) gives $\varphi(x,t) = t + \frac{1}{2}x^2$, $p(x) = x$. It follows that the nonlinear convection–diffusion equation

$$u_t = u_{xx} + x f(u) u_x \tag{31}$$

for an arbitrary function $f(u)$ admits a functional separable solution

$$u = U(z), \qquad z = t + \tfrac{1}{2}x^2. \tag{32}$$

Here, the function $U(z)$ is described by the autonomous ordinary differential equation

$$U''_{zz} + f(U) U'_z = 0 \tag{33}$$

(obtained by substituting $k = 0$ and $s = 1$ into (8)), whose general solution can be represented implicitly (9).

Example 2. Consider a more complicated situation when one of the coefficient of the equation depends in an arbitrary way on the spatial variable, $a = a(x)$, and the other two are constants, $b(x) = 0$ and $c(x) = 1$. Using formulas (30) with $C_1 = 0$, one finds that $g(x) = x/a(x)$. Substituting this function in (26) and (27) with $s(\varphi) = 1$ gives $\varphi(x,t) = t + \int \frac{x}{a(x)}\, dx$, $p(x) = x$. Therefore, the nonlinear convection–diffusion equation

$$u_t = [a(x) u_x]_x + x f(u) u_x, \tag{34}$$

dependent on two arbitrary functions $a(x)$ and $f(u)$, admits a functional separable solution

$$u = U(z), \qquad z = t + \int \frac{x}{a(x)}\, dx, \tag{35}$$

where the function $U(z)$ is described by the solvable autonomous ordinary differential equation (33).

Substituting $a(x) = x^n$ and $a(x) = e^{\lambda x}$ into (35) yields the nonlinear equations

$$u_t = (x^n u_x)_x + x f(u) u_x, \tag{36}$$

$$u_t = (e^{\lambda x} u_x)_x + x f(u) u_x, \tag{37}$$

which admit exact solutions for an arbitrary function $f(u)$.

Interestingly, the equation $u_t = (x u_x)_x + x f(u) u_x$, which is a special case of equation (36) with $n = 1$, admits a noninvariant traveling-wave solution $u = U(x + t)$.

Nondegenerate case. For $k = \text{const}$ ($k \neq 0$), the substitution

$$g = \frac{1}{k} \frac{y'_x}{y} \tag{38}$$

reduces equation (29) to the second-order linear differential equation

$$a(x) y''_{xx} + [b(x) + a'_x(x)] y'_x - k c(x) y = 0. \tag{39}$$

An extensive list of exact solutions of this equation for various forms of the functions $a(x)$, $b(x)$, and $c(x)$ can be found in [55, 56].

Example 3. In the case of constant coefficients $a = c = 1$ and $b = 0$, the general solution of equation (39) has the form

$$y = \begin{cases} C_1 \cosh(mx) + C_2 \sinh(mx) & \text{if } \ k = m^2 > 0, \\ C_1 \cos(mx) + C_2 \sin(mx) & \text{if } \ k = -m^2 < 0, \end{cases} \tag{40}$$

where C_1 and C_2 are arbitrary constants. By setting $C_1 = 1$, $C_2 = 0$, and $k = 1$ in (40) and using formula (38), we find that

$$g(x) = \tanh x.$$

Substituting this function into (26) and (27), we have

$$\varphi(x, t) = t + \ln \cosh x, \qquad p(x) = \tanh x.$$

It follows that the nonlinear convection–diffusion equation

$$u_t = u_{xx} + \tanh x f(u) u_x \tag{41}$$

for an arbitrary function $f(u)$ admits the functional separable solution

$$u = U(z), \qquad z = t + \ln\cosh x, \tag{42}$$

where the function $U(z)$ is described by the autonomous ordinary differential equation

$$U''_{zz} + [f(U) - 1]U'_z = 0. \tag{43}$$

This equation has the general solution (9), where $k = s = 1$.

Table 1 lists nonlinear equations $u_t = u_{xx} + p(x)f(u)u_x$, where $f(u)$ is an arbitrary function, that admit functional separable solutions of the form $u = U(z)$, $z = \varphi(x,t)$ (the function φ is determined to within an additive constant). For Equations 1, 2, 4–7, the function $\varphi(x,t)$ is the sum of functions of different arguments (26). A traveling wave solution (see Equation 1) corresponds to a degenerate solution of equation (29) of the form $g = \alpha = \text{const}$. Solutions of some equations of this type with more complicated functions $p(x)$ can be obtained using formulas (40) from Example 3. The solution to Equation 3 is self-similar (see Example 4).

No.	Function $p(x)$	Function $\varphi(x,t)$	Equation for the function $U=U(z)$
1	1	$t+\alpha x$	$\alpha^2 U''_{zz}+[\alpha f(U)-1]U'_z=0$
2	x	$t+\frac{1}{2}x^2$	$U''_{zz}+f(U)U'_z=0$
3	x^{-1}	$xt^{-1/2}$	$U''_{zz}+[\frac{1}{2}z+z^{-1}f(U)]U'_z=0$
4	$\tanh(\alpha x)$	$t+\alpha^{-2}\ln\cosh(\alpha x)$	$U''_{zz}+[\alpha f(U)-\alpha^2]U'_z=0$
5	$\coth(\alpha x)$	$t+\alpha^{-2}\ln\lvert\sinh(\alpha x)\rvert$	$U''_{zz}+[\alpha f(U)-\alpha^2]U'_z=0$
6	$\tan(\alpha x)$	$t-\alpha^{-2}\ln\lvert\cos(\alpha x)\rvert$	$U''_{zz}+[\alpha f(U)+\alpha^2]U'_z=0$
7	$\cot(\alpha x)$	$t-\alpha^{-2}\ln\lvert\sin(\alpha x)\rvert$	$U''_{zz}+[\alpha f(U)+\alpha^2]U'_z=0$

Таблица 1. Nonlinear equations $u_t=u_{xx}+p(x)f(u)u_x$ that admit exact solutions of the form $u=U(z)$, $z=\varphi(x,t)$. Here, $f(u)$ is an arbitrary function and α is an arbitrary constant ($\alpha\neq 0$).

Other ways of constructing exact solutions. We now consider other possibilities for constructing exact solutions of equations of the form (3) with $k(\varphi) = k$ and $s(\varphi) = 1$ without integrating the Riccati equation (29). To do this, we assume that $g(x)$ and any two of the three functions $a(x)$, $b(x)$, and $c(x)$ are given, and the remaining function will be found from (29). Table 2 lists the possible situations and provides formulas for determining the required function. The final form of the

No.	Functions assumed known	Function looked for
1	$a=a(x)$, $b=b(x)$, $g=g(x)$	$c(x)=ag'_x+kag^2+(b+a'_x)g$
2	$a=a(x)$, $c=c(x)$, $g=g(x)$	$b(x)=\frac{1}{g}(c-ag'_x)-a'_x-kag$
3	$b=b(x)$, $c=c(x)$, $g=g(x)$	$a(x)=\frac{1}{g}E\bigl[\int(c-bg)\frac{1}{E}\,dx+C_1\bigr]$, $E=\exp\bigl(-k\int g\,dx\bigr)$

Таблица 2. Different ways of specifying the functional coefficients of equation (3) with $p(x)=a(x)g(x)$. Here, k and C_1 are arbitrary constants.

nonlinear convection–diffusion equation is determined by substituting the function $p(x)=a(x)g(x)$ in (3).

Example 4. We use the third way described in Table 2 with $b=0$ and $c=1$ for an alternative representation of the equations and their exact solutions. There are two possible cases.

1. *Degenerate case for $k=0$.* From Table 2, row 3, we find $a(x)=xg^{-1}(x)$, $p(x)=x$, which leads to equation (34).

2. *Nondegenerate case for $k\neq 0$.* From Table 2, row 3, with $k\neq 0$ and $C_1=0$, we have $a(x)=g^{-1}E\int E^{-1}dx$. We introduce a new function $h=h(x)$ by putting $h=\int E^{-1}dx$. Differentiating this expression and taking into account the formula $E=\exp\bigl(-k\int g\,dx\bigr)$, we express the function g in terms of h. After simple manipulations, we finally get $g=k^{-1}h''_{xx}/h'_x$, $a=kh/h''_{xx}$, $p=h/h'_x$. It follows that the equation

$$u_t=[a(x)u_x]_x+p(x)f(u),\qquad a(x)=k\frac{h}{h''_{xx}},\quad p(x)=\frac{h}{h'_x},\tag{44}$$

where $f(u)$ and $h=h(x)$ are arbitrary functions, and $k\neq 0$ is an arbitrary constant, admits a generalized separable solution

$$u=U(z),\quad z=t+\frac{1}{k}\ln|h'_x|.$$

Here, the function $U(z)$ is determined from the solvable ordinary differential equation $U''_{zz}+[f(U)-k]U'_z=0$.

By setting, for example, $h=\sinh(\alpha x)$ and $k=\alpha^2$ in (44), we get Equation 4 from Table 1.

2.4 The direct construction of exact solutions for $k(\varphi)\neq\mathrm{const}$

1. Case $k(\varphi)=k_1\varphi$, $s(\varphi)=1$. For $k(\varphi)=k_1\varphi$, which corresponds to $k_2=0$ in (24), substituting expression (20) in equation (22), we obtain $\xi(t)=e^{\lambda t}$. Therefore, in this case the class of equations (3) admits functional separable solutions of the form (4), where

$$\varphi(x,t) = e^{\lambda t} + \theta(x). \tag{45}$$

Substituting (45) into relation (6) with $s(\varphi) = 1$ and equation (22) with $k(\varphi) = k_1\varphi$, we obtain

$$c(x) = \frac{k_1}{\lambda} a(x)(\theta'_x)^2, \qquad p(x) = a(x)\theta'_x. \tag{46}$$

In this case, the functions $a(x)$ and $b(x)$ remain arbitrary, and the function $\theta = \theta(x)$ is determined by solving the ordinary differential equation

$$[a(x)\theta'_x]'_x + b(x)\theta'_x + k_1 a(x)\theta(\theta'_x)^2 = 0. \tag{47}$$

The substitution $\eta = \int \exp\bigl(\frac{1}{2}k_1\theta^2\bigr)d\theta$ reduces equation (47) to the linear equation $[a(x)\eta'_x]'_x + b(x)\eta'_x = 0$, the general solution of which is expressed as $\eta = C_1 \int \frac{1}{a} \exp\bigl(-\int \frac{b}{a}\,dx\bigr)\,dx + C_2$.

2. Case $k(\varphi) = k_1 e^{-k_2\varphi} + k_3,\ s(\varphi) = 1$. For $k(\varphi) = k_1 e^{-k_2\varphi} + k_3$, which corresponds to using the dependence (25), substituting expression (20) into equation (22), we obtain $\xi(t) = k_2^{-1}\ln t$. In this case, the class of equations (3) admits functional separable solutions of the form (4), where

$$\varphi(x,t) = \frac{1}{k_2}\ln t + \theta(x). \tag{48}$$

Substituting (48) into relation (6) with $s(\varphi) = 1$ and equation (22) with $k(\varphi) = k_1 e^{-k_2\varphi} + k_3$, we get

$$c(x) = k_1 k_2 a(x) e^{-k_2\theta}(\theta'_x)^2, \qquad p(x) = a(x)\theta'_x. \tag{49}$$

The functions $a(x)$ and $b(x)$ remain arbitrary, and the function $\theta = \theta(x)$ is determined by solving the nonlinear ordinary differential equation

$$[a(x)\theta'_x]'_x + b(x)\theta'_x + k_3 a(x)(\theta'_x)^2 = 0. \tag{50}$$

This equation is easily integrated, since the substitution $\zeta(x) = \theta'_x$ leads to a Bernoulli equation. In particular, for $k_3 = 0$, the general solution of equation (50) is given by

$$\theta(x) = C_1 \int \frac{1}{a}\exp\biggl(-\int \frac{b}{a}\,dx\biggr)\,dx + C_2.$$

Example 5. Let

$$a(x) = 1, \quad b(x) = 0, \quad k_1 = -\tfrac{1}{2}, \quad k_2 = -2, \quad k_3 = 1. \tag{51}$$

In this case, equation (50) has a solution $\theta = \ln x$. Substituting this function into formulas (48) and (49), and taking (51) into account, we obtain

$$\varphi(x,t) = -\tfrac{1}{2}\ln t + \ln x, \quad p(x) = x^{-1}, \quad c(x) = 1. \tag{52}$$

Therefore, the equation

$$u_t = u_{xx} + x^{-1} f(u) u_x \tag{53}$$

admits the self-similar solution

$$u = U(z), \quad z = -\tfrac{1}{2}\ln t + \ln x \equiv \ln(xt^{-1/2}), \tag{54}$$

where the function $U(z)$ satisfies the ordinary differential equation

$$U''_{zz} + [\tfrac{1}{2}e^{2z} - 1 + f(U)]U'_z = 0. \tag{55}$$

Note that an alternative representation of similar solutions is often used in applications, which is based on the introduction of the self-similar variable $\bar{z} = e^z = xt^{-1/2}$ and reduction of equation (53) to the equation $U''_{zz} + [\frac{1}{2}z + z^{-1}f(U)]U'_z = 0$ (see Equation 3 in Table 1).

Example 6. Equations (49) and equation (50) are satisfied if we set

$$a(x) = 1, \ \ b(x) = 0, \ \ c(x) = e^{-x}, \ \ p(x) = 1, \ \ \theta(x) = x, \ \ k_1 = k_2 = 1, \ \ k_3 = 0.$$

Therefore, the equation $e^{-x}u_t = u_{xx} + f(u)u_x$ admits an exact solution of the form $u = U(z)$, where $z = x + \ln t$.

3 Some generalizations and modifications

3.1 More-complex one-dimensional nonlinear diffusion-type equations

Below is a useful theorem that allows one to construct exact solutions of more-complex nonlinear diffusion-type equations.

Theorem 1. Suppose $\varphi = \varphi(x,t)$ is a solution to the parabolic equation with quadratic nonlinearity

$$c(x,t)\varphi_t = [a(x,t)\varphi_x]_x + b(x,t)\varphi_x + ka(x,t)\varphi_x^2, \tag{56}$$

where k is an arbitrary constant. Then the nonlinear PDE

$$c(x,t)u_t = [a(x,t)u_x]_x + b(x,t)u_x + a(x,t)\varphi_x^2 F(\varphi, u, u_x/\varphi_x), \tag{57}$$

where $F(\varphi, u, w)$ is an arbitrary function of three arguments, admits a functional separable solution of the form

$$u = U(z), \qquad z = \varphi(x,t). \tag{58}$$

Here, the function $U(z)$ is determined by solving the ODE

$$U''_{zz} - kU'_z + F(z, U, U'_z) = 0. \tag{59}$$

Thus, exact solutions of equation (56) generate corresponding exact solutions of the nonlinear equation (57). Equation (56) will be called a *generating equation*. Note that this equation is invariant with respect to translation, $\varphi \Rightarrow \varphi + \text{const}$.

Theorem 1 is proved by direct verification by substituting function (58) into equation (57) while taking into account relation (56).

Remark 4. For $k = 0$, we get the linear PDE (56). For $k \neq 0$, substitution (17) takes the nonlinear equation (56) to the linear equation

$$c(x,t)\psi_t = [a(x,t)\psi_x]_x + b(x,t)\psi_x. \tag{60}$$

Thus, exact solutions of the linear equation (60) generate corresponding exact solutions of the nonlinear equation (57).

Exact solutions of equation (60) for certain functions $a(x,t)$, $b(x,t)$, and $c(x,t)$ can be found in [57].

Example 8. By setting

$$a(x,t) = a(x), \quad b(x,t) = 0, \quad c(x,t) = 1, \quad k = 0, \quad \varphi(x,t) = t + \int \frac{x\,dx}{a(x)},$$
$$F(\varphi, u, w) = f(u)w^2 + g(u)w + h(u)$$

in (56) and (57), we obtain the nonlinear PDE

$$u_t = [a(x)u_x]_x + a(x)f(u)u_x^2 + xg(u)u_x + \frac{x^2}{a(x)}h(u),$$

dependent on four arbitrary functions $a(x)$, $f(u)$, $g(u)$, $h(u)$, which has the exact solution

$$u = U(z), \qquad z = t + \int \frac{x\,dx}{a(x)}.$$

The function $U = U(z)$ is described by the nonlinear ODE

$$U''_{zz} + f(U)(U'_z)^2 + g(u)U'_z + h(u) = 0.$$

3.2 Nonlinear systems of coupled diffusion-type equations

Theorem 1 admits a generalization to systems of coupled equations.

Theorem 2. Suppose $\varphi = \varphi(x,t)$ is a solution to the parabolic equation with quadratic nonlinearity (56). Then the nonlinear system of coupled PDEs

$$\begin{aligned} c(x,t)u_t &= [a(x,t)u_x]_x + b(x,t)u_x + a(x,t)\varphi_x^2 F(\varphi,u,v,u_x/\varphi_x,v_x/\varphi_x),\\ c(x,t)v_t &= [a(x,t)v_x]_x + b(x,t)v_x + a(x,t)\varphi_x^2 G(\varphi,u,v,u_x/\varphi_x,v_x/\varphi_x), \end{aligned} \tag{61}$$

where $F(\varphi,u,v,w_1,w_2)$ and $G(\varphi,u,v,w_1,w_2)$ are arbitrary functions of five arguments, admits a functional separable solution of the form

$$u = U(z), \quad v = V(z), \quad z = \varphi(x,t). \tag{62}$$

The functions $U(z)$ and $V(z)$ are determined by solving the coupled ODEs

$$\begin{aligned} U''_{zz} - kU'_z + F(z,U,V,U'_z,V'_z) &= 0,\\ V''_{zz} - kV'_z + G(z,U,V,U'_z,V'_z) &= 0. \end{aligned} \tag{63}$$

Theorem 2 is proved by direct verification by substituting functions (62) into equations (61) while taking into account relation (56).

Example 9. By setting

$$\begin{aligned} &a(x,t) = c(x,t) = 1, \quad b(x,t) = 0, \qquad k = 0, \quad \varphi(x,t) = t + \tfrac{1}{2}x^2,\\ &F(\varphi,u,v,w_1,w_2) = f(u,v)w_2, \quad G(\varphi,u,v,w_1,w_2) = g(u,v)w_1 \end{aligned}$$

in (56) and (61), we obtain the nonlinear system of coupled PDEs

$$\begin{aligned} u_t &= u_{xx} + xf(u,v)v_x,\\ v_t &= v_{xx} + xg(u,v)u_x, \end{aligned}$$

involving arbitrary functions $f(u,v)$, and $g(u,v)$, which has the exact solution

$$u = U(z), \quad v = V(z), \quad z = t + \tfrac{1}{2}x^2.$$

The functions $U = U(z)$ and $V = V(z)$ are described by the nonlinear coupled ODEs

$$\begin{aligned} U''_{zz} + f(U,V)V'_z &= 0,\\ V''_{zz} + g(U,V)U'_z &= 0. \end{aligned}$$

3.3 Nonlinear diffusion equations with several spatial variables

Theorem 1 admits various multidimensional generalizations. An example is given below.

Theorem 3. Suppose $\varphi = \varphi(\mathbf{x}, t)$ is a solution to the parabolic equation with quadratic nonlinearity

$$c(\mathbf{x}, t)\varphi_t = \Delta\varphi + \mathbf{b}(\mathbf{x}, t) \cdot \nabla\varphi + k|\nabla\varphi|^2, \tag{64}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, Δ is the Laplace operator, ∇ is the gradient operator, and k is an arbitrary constant. Then the nonlinear PDE

$$c(\mathbf{x}, t)u_t = \Delta u + \mathbf{b}(\mathbf{x}, t) \cdot \nabla u + |\nabla\varphi|^2 F(\varphi, u, |\nabla u|/|\nabla\varphi|), \tag{65}$$

where $F(\varphi, u, w)$ is an arbitrary function of three arguments, admits a functional separable solution of the form

$$u = U(z), \qquad z = \varphi(\mathbf{x}, t). \tag{66}$$

The function $U(z)$ is determined by solving the ODE

$$U''_{zz} - kU'_z + F(z, U, |U'_z|) = 0. \tag{67}$$

Theorem 3 is proved by direct verification by substituting function (66) into equation (65) while taking into account relation (64).

Remark 5. For $k \neq 0$, substitution (17) takes the nonlinear generating equation (64) to the linear equation $c(\mathbf{x}, t)\psi_t = \Delta\psi + \mathbf{b}(\mathbf{x}, t) \cdot \nabla\psi$.

3.4 Nonlinear equations of convection–diffusion type with delay

The results obtained in Section 2 can also be generalized to the case of more complicated nonlinear convection–diffusion equations with delay of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)u_x + p(x)f(u, w)u_x, \qquad w = u(x, t - \tau), \tag{68}$$

where $\tau > 0$ is the delay time, $f(u, w)$ is an arbitrary function of two arguments.

Let us show how the solutions of the nonlinear convection–diffusion equation without delay (3), which are determined by formulas (4) and (26), can be used to construct exact solutions of the nonlinear equation with delay (68). Let equation (3) admit a functional separable solution of the form

$$u = U(z), \qquad z = t + \theta(x), \tag{69}$$

where the function $U(z)$ satisfies the ordinary differential equation (8) with $k(z) = k = \text{const}$ and $s(z) = 1$. Then the equation with delay (68) admits an exact solution of the form (69), where $U(z)$ satisfies the ordinary differential equation with delay

$$U''_{zz} + [f(U, W) - k]U'_z = 0, \qquad W = U(z - \tau). \tag{70}$$

Equations (31), (34), (36), (37), as well as Equations 4–7 from Table 1 (obtained for $k(z) = 0$ and $s(z) = 1$), have solutions of the form (69). Therefore, more-complex nonlinear convection–diffusion equations with delay, which are obtained from these equations by replacing the function $f(u)$ with $f(u, w)$, also admit exact solutions of the form (69).

Example 9. Nonlinear convection–diffusion equation with delay

$$u_t = u_{xx} + xf(u, w)u_x, \qquad w = u(x, t - \tau), \tag{71}$$

which is a generalization of equation (31), for an arbitrary function $f(u, w)$ admits a functional separable solution

$$u = U(z), \qquad z = t + \tfrac{1}{2}x^2, \tag{72}$$

where the function $U(z)$ is described by the delay ordinary differential equation

$$U''_{zz} + f(U, W)U'_z = 0, \qquad W = U(z - \tau). \tag{73}$$

Note that for $f(U, W) = g(W/U)$, equation (73) admits an exact solution of the form $U = Ce^{\lambda z}$, where C is an arbitrary constant, and λ is determined from the transcendental equation $\lambda + g(e^{-\tau\lambda}) = 0$.

3.5 Remarks on exact solutions of nonlinear wave-type equations

Nonlinear wave-type equations

$$c(x)u_{tt} = [a(x)u_x]_x + [b(x) + p(x)f(u)]u_x, \tag{74}$$

in which u_t is replaced with u_{tt}, can also have functional separable solutions of the form $u = U(z)$ with $z = \xi(t) + \theta(x)$. Some solutions of this type can be found in [16].

Below is an example of a new exact functional separable solution.

Example 10. Consider the equation

$$u_{tt} = [a(x)u_x]_x + [\lambda^2 a(x) - k^2]f(u)u_x, \tag{75}$$

where $a(x)$ and $f(u)$ are arbitrary functions, while k and λ are arbitrary constants. Equation (75) has the exact solutions

$$u = U(z), \qquad z = \lambda t + k \int \frac{dx}{a(x)}, \tag{76}$$

where the function $U = U(z)$ is described by an autonomous ordinary differential equation of the form (8),

$$U''_{zz} - kf(U)U'_z = 0. \tag{77}$$

Replacing k with $-k$ in (76) and (77) gives another solution to equation (75).

4 Brief conclusions

To summarize, the paper has presented a number of exact functional separable solutions to nonlinear convection–diffusion equations of the form

$$c(x)u_t = [a(x)u_x]_x + [b(x) + p(x)f(u)]u_x,$$

where $f(u)$ is an arbitrary function. Solutions were sought in the form $u = U(z)$ with $z = \varphi(x,t)$, where the functions $U(z)$ and $\varphi(x,t)$ are determined in the course of further analysis. It has been shown that any three of the four functional coefficients $a(x)$, $b(x)$, $c(x)$, $p(x)$ of the convection–diffusion equation can be chosen arbitrarily. Examples of specific equations and their exact solutions are given. Also some functional separable solutions of nonlinear convection–diffusion equations with delay

$$u_t = u_{xx} + a(x)f(u,w)u_x, \qquad w = u(x, t-\tau),$$

where $\tau > 0$ is the delay time and $f(u,w)$ is an arbitrary function of two arguments, have been obtained.

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References

1. L.V. Ovsiannikov, Group properties of nonlinear heat equations, Doklady Acad. Nauk USSR 125(3) (1959) 492–495 (in Russian).

2. V.A. Dorodnitsyn, On invariant solutions of the equation of non-linear heat conduction with a source, USSR Comput. Math. & Math. Phys. 22(6) (1982) 115–122.

3. V.A. Galaktionov, V.A. Dorodnitsyn, G.G. Yelenin, S.P. Kurdyumov, A.A. Samarskii, A quasilinear equation of heat conduction with a source: peaking, localization, symmetry, exact solutions, asymptotic behavior, structures, J. Soviet Math. 41(5) (1988) 1222–1292.

4. N.A. Kudryashov, On exact solutions of families of Fisher equations. Theor. Math. Phys. 94(2) (1993) 211–218.

5. P.A. Clarkson, E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Phys. D 70 (1994) 250–288.

6. V.A. Galaktionov, Quasilinear heat equations with first-order sign-invariants and new explicit solutions, Nonlinear Anal. Theor. Meth. Appl. 23 (1994) 1595–621.

7. N.H. Ibragimov (ed.), CRC Handbook of Lie Group Analysis of Differential Equations. Symmetries, Exact solutions and Conservation Laws, vol. 1, CRC Press, Boca Raton, 1994.

8. P.W. Doyle, P.J. Vassiliou, Separation of variables for the 1-dimensional non-linear diffusion equation, Int. J. Non-Linear Mech. 33(2) (1998) 315–326.

9. S. Hood, On direct, implicit reductions of a nonlinear diffusion equation with an arbitrary function - generalizations of Clarkson's and Kruskal's method, IMA J. Appl. Math. 64(3) (2000) 223–244.

10. P.G. Estevez, C. Qu, S. Zhang, Separation of variables of a generalized porous medium equation with nonlinear source, J. Math. Anal. Appl. 275 (2002) 44–59.

11. A.D. Polyanin, V.F. Zaitsev, Handbook of nonlinear equations of mathematical physics, Fizmatlit, Moscow, 2002 (in Russian).

12. O.V. Kaptsov, I.V. Verevkin, Differential constraints and exact solutions of nonlinear diffusion equations, J. Phys. A: Math. Gen. 36 (2003) 1401–1414.

13. A.D. Polyanin, V.F. Zaitsev, A.I. Zhurov, Solution methods for nonlinear equations of mathematical physics and mechanics, Fizmatlit, Moscow, 2005 (in Russian).

14. R.M. Cherniha, O. Pliukhin, New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations, J. Physics A: Math. Theor. 40(33) (2007) 10049–10070.

15. V.A. Galaktionov, S.R. Svirshchevskii, Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics, Chapman & Hall/CRC Press, Boca Raton, 2006.

16. A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, 2nd ed., CRC Press, Boca Raton, 2012 (see also 1st ed., 2004).

17. R.M. Cherniha, O. Pliukhin, New conditional symmetries and exact solutions of reaction-diffusion-convection equations with exponential nonlinearities, J. Math. Anal. Appl. 403 (2013) 23–37.

18. R. Cherniha, M. Serov, O. Pliukhin, Nonlinear Reaction-Diffusion-Convection Equations: Lie and Conditional Symmetry, Exact Solutions and Their Applications, Chapman & Hall/CRC Press, Boca Raton, 2018.

19. O.O. Vaneeva, A.G. Johnpillai, R.O. Popovych, C. Sophocleous, Extended group analysis of variable coefficient reaction-diffusion equations with power nonlinearities, J. Math. Anal. Appl. 330(2) (2007) 1363–1386.

20. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, Acta Appl. Math. 106(1) (2009) 1–46.

21. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Extended group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities, J. Math. Anal. Appl. 396 (2012) 225–242.

22. O. Vaneeva, A. Zhalij, Group classification of variable coefficient quasilinear reaction-diffusion equations, Publ. L'Institute Mathématique (Nouvelle série) 94(108) (2013) 81–90.

23. A.D. Polyanin, Functional separable solutions of nonlinear reactiondiffusion equations with variable coefficients, Appl. Math. Comput. 347 (2019) 282–292.

24. B.H. Bradshaw-Hajek, R.J. Moitsheki, Symmetry solutions for reaction-diffusion equations with spatially dependent diffusivity, Appl. Math. Comput. 254 (2015) 30–38.

25. M.L. Gandarias, J.L. Romero, J.M. Díaz, Nonclassical symmetry reductions of a porous medium equation with convection, J. Phys. A: Math. Gen. 32 (1999) 1461–1473.

26. R.O. Popovych, N.M. Ivanova, New results on group classification of nonlinear diffusion-convection equations, J. Physics A: Math. Gen. 37(30) (2004) 7547–7565.

27. N.M. Ivanova, C. Sophocleous, On the group classification of variable-coefficient nonlinear diffusion-convection equations, J. Comput. Appl. Math. 197(2) (2006) 322–344.

28. N.M. Ivanova, Exact solutions of diffusion-convection equations, Dynamics of PDE 5(2) (2008) 139–171.

29. O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Group analysis of variable coefficient diffusion-convection equations, I. Enhanced group classification, Lobachevskii J. Math. 31(2) (2010) 100–122.

30. P. Basarab-Horwath, V. Lahno, R. Zhdanov, The structure of Lie algebras and the classification problem for partial differential equations, Acta Appl. Math. 69 (2001) 43–94.

31. V.I. Lagno, S.V. Spichak, V.I. Stognii, Symmetry analysis of evolution type equations, Institute of Computer Sciences, Moscow–Izhevsk, 2004 (in Russian).

32. H. Jia, W.X.X. Zhao, Z. Li, Separation of variables and exact solutions to nonlinear diffusion equations with x-dependent convection and absorption, J. Math. Anal. Appl. 339 (2008) 982–995.

33. E. Pereira, E. Suazo, J. Trespalacios, Riccati–Ermakov systems and explicit solutions for variable coefficient reaction–diffusion equations, Appl. Math. Comput. 329 (2018) 278–296.

34. R. Cherniha, J.R. King, S. Kovalenko, Lie symmetry properties of nonlinear reaction-diffusion equations with gradient-dependent diffusivity, Commun. Nonlinear Sci. Numer. Simul. 36 (2016) 98–108.

35. I. Naeem, M.D. Khan, Symmetry classification of time-fractional diffusion equation, Commun. Nonlinear Sci. Numer. Simul. 42 (2017) 560–570.

36. M. Molati, H. Murakawa, Exact solutions of nonlinear diffusion-convection-reaction equation: A Lie symmetry analysis approach, Commun. Nonlinear Sci. Numer. Simul. 67 (2019) 253–263.

37. R. Cherniha, V. Davydovych, Nonlinear Reaction-Diffusion Systems: Conditional Symmetry, Exact Solutions and Their Applications in Biology, Springer, 2017.

38. S. Borazjani, A.J. Roberts, P. Bedrikovetsky, Splitting in systems of PDEs for two-phase multicomponent flow in porous media, Appl. Math. Lett. 53 (2016) 25–32.

39. N.A. Kudryashov, R.B. Rybka, A. G. Sboev, Analytical properties of the perturbed FitzHugh-Nagumo model, Appl. Math. Lett. 76 (2018) 142–147.

40. S.V. Meleshko, S. Moyo, On the complete group classification of the reaction-diffusion equation with a delay, J. Math. Anal. Appl. 338 (2008) 448–466.

41. A.D. Polyanin, A.I. Zhurov, Exact solutions of linear and non-linear differential-difference heat and diffusion equations with finite relaxation time, Int. J. Non-Linear Mech. 54 (2013) 115–126.

42. A.D. Polyanin, A.I. Zhurov, Exact separable solutions of delay reaction-diffusion equations and other nonlinear partial functional-differential equations, Commun. Nonlinear Sci. Numer. Simul. 19(3) (2014) 409–416.

43. A.D. Polyanin, A.I. Zhurov, Functional constraints method for constructing exact solutions to delay reaction-diffusion equations and more complex nonlinear equations, Commun. Nonlinear Sci. Numer. Simul. 19(3) (2014) 417–430.

44. A.D. Polyanin, A.I. Zhurov, New generalized and functional separable solutions to non-linear delay reaction-diffusion equations, Int. J. Non-Linear Mech. 59 (2014) 16–22.

45. A.D. Polyanin, A.I. Zhurov, Nonlinear delay reaction-diffusion equations with varying transfer coefficients: Exact methods and new solutions, Appl. Math. Letters 37 (2014) 43–48.

46. A.D. Polyanin, A.I. Zhurov, The functional constraints method: Application to non-linear delay reaction-diffusion equations with varying transfer coefficients, Int. J. Non-Linear Mech. 67 (2014) 267–277.

47. A.D. Polyanin, A.G. Sorokin, Nonlinear delay reaction-diffusion equations: Traveling-wave solutions in elementary functions, Appl. Math. Letters 46 (2015) 38–43.

48. Z. Zhao, W. Ge, Symmetry analysis of reaction diffusion equation with distributed delay, Commun. Nonlinear Sci. Numer. Simul. 24(1–3) (2015) 11–20.

49. A.D. Polyanin, Generalized traveling-wave solutions of nonlinear reaction–diffusion equations with delay and variable coefficients, Appl. Math. Letters 90 (2019) 49–53.

50. A.D. Polyanin, A.I. Zhurov, Generalized and functional separable solutions to nonlinear delay Klein–Gordon equations, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 2676–2689.

51. F-S. Long, S.V. Meleshko, On the complete group classification of the one-dimensional nonlinear Klein–Gordon equation with a delay, Math. Methods Appl. Sci. 39(12) (2016) 3255–3270.

52. F-S. Long, S.V. Meleshko, Symmetry analysis of the nonlinear two-dimensional Klein-Gordon equation with a time-varying delay. Math. Methods Appl. Sci. 40(13) (2017) 4658–4673.

53. A.D. Polyanin, A.I. Zhurov, Non-linear instability and exact solutions to some delay reaction-diffusion systems, Int. J. Non-Linear Mech. 62 (2014) 33–40.

54. A.D. Polyanin, A.I. Zhurov, The generating equations method: Constructing exact solutions to delay reaction-diffusion systems and other non-linear coupled delay PDEs, Int. J. Non-Linear Mech. 71 (2015) 104–115.

55. A.D. Polyanin, V.F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, 2003.

56. A.D. Polyanin, V.F. Zaitsev, Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems, CRC Press, Boca Raton, 2018.

57. A.D. Polyanin, V.E. Nazaikinskii, Handbook of Linear Partial Differential Equations for Engineers and Scientists, 2nd ed, CRC Press, Boca Raton, 2016.