Construction of functional separable solutions in implicit form for non-linear Klein–Gordon type equations with variable coefficients*

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Abstract

The paper deals with non-linear Klein–Gordon type equations

$$c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u).$$

The direct method for constructing functional separable solutions in implicit form to non-linear PDEs is used. This effective method is based on the representation of solutions in the form

$$\int h(u)\,du = \xi(x)\omega(t) + \eta(x),$$

where the functions $h(u)$, $\xi(x)$, $\eta(x)$, and $\omega(t)$ are determined further by analyzing the resulting functional-differential equations. Examples of specific Klein–Gordon type equations and their exact solutions are given. The main attention is paid to non-linear equations of a fairly general form, which contain several arbitrary functions dependent on the unknown u and/or the spatial variable x (it is important to note that exact solutions of non-linear PDEs, that contain arbitrary functions and therefore have significant generality, are of great practical interest

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* This is a preprint of an article that will be published in *Int. J. Non-Linear Mechanics*, 2019, Vol. 114, pp. 29–40.

for testing various numerical and approximate analytical methods for solving corresponding initial-boundary value problems). Many new generalized traveling-wave solutions and functional separable solutions (in closed form) are described. Solutions of several Klein–Gordon equations with delay are also given.

Keywords: non-linear Klein–Gordon equations, non-linear PDEs with variable coefficients, exact solutions in implicit form, generalized traveling-wave solutions, functional separable solutions

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1. Introduction

Non-linear Klein–Gordon type equations arises in relativistic quantum mechanics and field theory [1, 2]. These equations model various physical phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles in a condensed medium [2, 3].

Transformations, symmetries, and exact solutions of various classes of non-linear Klein–Gordon type equations

$$u_{tt} = [f(u)u_x]_x + g(u) \tag{1}$$

have been considered in many studies (see, for example, [4–20] and the literature cited therein). To construct exact solutions, the most frequently used methods were those based on the classical and nonclassical symmetry reductions [4, 6, 7, 9, 13, 19, 20] and on generalized and functional separation of variables [8, 10, 12, 15, 17, 18, 20].

In the general case, equation (1) admits the traveling wave solution $u = U(kx - \lambda t)$. For $g(u) = 0$, it has the self-similar solution $u = U(x/t)$ [4], and also more complex exact solutions that can be presented in implicit form [20]:

$$x - t\sqrt{f(u)} = \varphi_1(u),$$
$$x + t\sqrt{f(u)} = \varphi_2(u),$$

where $\varphi_1(u)$ and $\varphi_2(u)$ are arbitrary functions (degenerate cases $\varphi_1 = 0$ and $\varphi_2 = 0$ correspond to self-similar solutions of the special forms). It is important to note that for $g(u) = 0$ equation (1) can be linearized [14, 16, 17, 19]; in this case, for an arbitrary function $f(u)$ some exact solutions presented in parametric form are given in [16, 17, 20]. In addition to these cases, the exact solutions of the equation of the form (1) are also known, in which two functions $f(u)$ and $g(u)$ are expressed in terms of one arbitrary function $h(u)$ [20].

In [11, 16, 17, 20–23] non-linear Klein–Gordon type equations with variable coefficients,

$$c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u), \tag{2}$$

were considered. Table 1 lists the forms of exact solutions to some equations of this type with one arbitrary function (in the solution in line 1, $\sinh(\frac{1}{2}\lambda t)$ can also be taken instead of $\cosh(\frac{1}{2}\lambda t)$).

No.	Equation	Form of solution or remark	References
1	$u_{tt}=u_{xx}+e^{\lambda x}g(u)$	$u=U(z)$, $z=e^{\frac{1}{2}\lambda x}\cosh(\frac{1}{2}\lambda t)$	[16, 17]
2	$u_{tt}=(x^k u_x)_x+g(u),\ k\neq 2$	$u=U(z)$, $z=4x^{2-k}-(2-k)^2t^2$	[11, 16, 17]
3	$u_{tt}=(x^k u_x)_x+x^{k-2}g(u),\ k\neq 2$	$u=U(z)$, $z=x^{(k-2)/2}t$	[21]
4	$u_{tt}=(x^2 u_x)_x+g(u)$	$u=U(z)$, $z=\lambda t+\ln x$	[11, 16, 17]
5	$u_{tt}=(e^{\lambda x}u_x)_x+g(u)$	$u=U(z)$, $z=4e^{-\lambda x}-\lambda^2t^2$	[11, 16, 17]
6	$u_{tt}=(u^k u_x)_x+b(x)u^{k+1}$	$u=\varphi(x)\psi(t)$	[17]
7	$u_{tt}=(e^{\lambda u}u_x)_x+b(x)e^{\lambda u}$	$u=\varphi(x)+\psi(t)$	[17]
8	$u_{tt}=(u^{-4/3}u_x)_x+b(x)u^{-1/3}$	Reduces to $v_{tt}=(v^{-4/3}v_z)_z$	[11, 16, 17]
9	$u_{tt}=[a(x)u^k u_x]_x$	$u=\varphi(x)\psi(t)$	[11, 16, 17]
10	$u_{tt}=[a(x)u_x]_x+cu\ln u+ku$	$u=\varphi(x)\psi(t)$	[16, 17]

Table 1: Non-linear Klein–Gordon type equations with variable coefficients and their exact solutions. Here, $a(x)$, $b(x)$, and $g(u)$ are arbitrary functions, and c, k, and λ are free parameters.

Below are four equations that generalize the equations in rows 3, 4, 6, 7, 9 of Table 1, which contain two arbitrary functions and admit exact solutions:

$u_{tt} = [a(x)u^k u_x]_x + b(x)u^{k+1}$, $\quad u = \varphi(x)\psi(t)$ (it generalizes Eqs 6 and 9);
$u_{tt} = [a(x)e^{\lambda u} u_x]_x + b(x)e^{\lambda u}$, $\quad u = \varphi(x) + \psi(t)$ (it generalizes Eq. 7);
$u_{tt} = [x^k f(u)u_x]_x + x^{k-2}g(u)$, $\quad u = U(x^{(k-2)/2}t)$ ($k \neq 2$, it generalizes Eq. 3);
$u_{tt} = [x^2 f(u)u_x]_x + g(u)$, $\quad u = U(z),\ z = \lambda t + \ln x$ (it generalizes Eq. 4).

Note also that the non-linear equation

$$u_{tt} = [e^{\lambda x} f(u)u_x]_x + e^{\lambda x} g(u),$$

has an exact (invariant) solution of the form $u = U(z)$, $z = e^{\lambda x/2}t$.

In [11, 16, 17, 24–26], symmetries and some exact solutions of non-linear telegraph type equations,

$$c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u)u_x,$$

were described (in [24, 25] the case $a(x) = b(x) = 1$ was considered).

Other related and more complex non-linear hyperbolic equations were studied, for example, in [27–32].

Remark 1. Some exact solutions of non-linear delay Klein–Gordon equations of the form

$$u_{tt} = au_{xx} + F(u, w), \quad w = u(x, t - \tau),$$

where $\tau > 0$ is the delay time, were obtained in [33, 34].

Remark 2. Symmetries, reductions and exact solutions of non-linear reaction-diffusion equations (which can be obtained from (2) by the formal replacement of u_{tt} by u_t) and some more-complex non-linear parabolic equations were considered, for example, in [11, 16, 17, 20, 35–58].

The present paper deals with new exact solutions (in implicit form) admitted by non-linear Klein–Gordon type equations of a fairly general form (2) that depend on one or more arbitrary functions. It is important to note that exact solutions of mathematical physics equations, which contain arbitrary functions and therefore have a significant generality, are of great practical interest for evaluating the accuracy of various numerical and approximate analytical methods for solving corresponding initial-boundary value problems.

2. Nonlinear Klein–Gordon type equations. Direct method for constructing functional separable solutions in implicit form

2.1. Preliminary remarks. Examples of solutions defined implicitly

The method proposed below for constructing implicitly given exact functional separable solutions is based on the generalization of traveling-wave solutions of some non-linear PDEs. Prior to describing the method, let us first give two simple examples that illustrate the existence of solutions defined implicitly.

Example 1. Let us look at the non-linear wave equation

$$u_{tt} = [f(u)u_x]_x, \tag{3}$$

which contains an arbitrary function $f(u)$. This equation does not depend explicitly on x and t and admits the traveling-wave solution

$$u = u(z), \qquad z = \lambda t + \kappa x, \tag{4}$$

where κ and λ are arbitrary constants. Substituting (4) in (3), we obtain an ODE of the form $\lambda^2 u''_{zz} = \kappa^2 [f(u) u'_z]'_z$. Integrating, we find its solution in implicit form

$$\int [\kappa^2 f(u) - \lambda^2]\, du = C_1(\lambda t + \kappa x) + C_2, \tag{5}$$

where C_1 and C_2 are arbitrary constants. On the right-hand side of (5), z has been replaced by the original variables using (4).

Example 2. The non-linear Klein–Gordon equation

$$u_{tt} = u_{xx} + g(u), \tag{6}$$

where $g(u)$ is an arbitrary function, also admits traveling-wave solutions $u = u(z)$, $z = \lambda t + \kappa x$ (with $\lambda \neq \pm\kappa$), which can be represented in the implicit form

$$\int \left[C_1 + \frac{2}{\lambda^2 - \kappa^2} G(u)\right]^{-1/2} du = C_2 \pm (\lambda t + \kappa x), \qquad G(u) = \int g(u)\, du. \tag{7}$$

Examples 1 and 2 show that non-linear equations (3) and (6) have traveling-wave solutions that can be represented in implicit form. It is important that in the general case of arbitrary $f(u)$ and $g(u)$, these solutions cannot be represented explicitly. More complex examples of non-linear PDEs that have solutions in implicit form can be found, for example, in [20].

Section 2.3 will describe a method for constructing exact solutions to non-linear equations of mathematical physics, based on the generalization of solutions (5) and (7).

2.2. Class of non-linear Klein–Gordon type equations with variable coefficients in question

We will consider non-linear Klein–Gordon type equations with variable coefficients

$$c(x) u_{tt} = [a(x) f(u) u_x]_x + b(x) g(u). \tag{8}$$

Further, we will assume that $a = a(x) > 0$, and $c = c(x)$ can have any sign (for $c < 0$ equation (8) can be interpreted as a non-linear stationary anisotropic heat equation with source, where $t = y$ is the spatial coordinate).

Note that in the case of $a(x) = c(x) = x^n$, equation (8) describes non-linear wave processes with radial symmetry in two-dimensional (with $n = 1$) and three-dimensional (with $n = 2$) cases (x is the radial coordinate).

In this paper, the main attention is focused on equations (8) of a fairly general form, which depend on one or two arbitrary functions.

2.3. Direct method for constructing functional separable solutions in implicit form

Exact solutions of Klein–Gordon type equations (8) or other non-linear PDEs that explicitly depend on x but do not depend on t,

$$G(x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{9}$$

are sought in the implicit form [58]

$$\int h(u)\, du = \xi(x)\omega(t) + \eta(x), \tag{10}$$

where the functions $h(u)$, $\xi(x)$, $\eta(x)$, and $\omega(t)$ are determined in the subsequent analysis. The representation of the solution in the form (10) is based on a substantial generalization of solution (5), which is carried out as follows:

$$[\kappa^2 f(u) - \lambda^2] \implies h(u), \quad C_1\lambda \implies \xi(x), \quad t \implies \omega(t), \quad C_1\kappa x + C_2 \implies \eta(x).$$

The arguments of the functions $a = a(x)$, $b = b(x)$, $c = c(x)$, $f = f(u)$, $g = g(u)$, $h = h(u)$, $\xi = \xi(x)$, $\eta = \eta(x)$, and $\omega = \omega(t)$, which appear in equation (8) and solution (10), will often be omitted. It is further assumed that $\omega \neq \text{const}$.

We describe the procedure for constructing exact solutions in implicit form. First, using (10), one calculates the partial derivatives u_x, u_t, u_{xx}, ..., which are expressed in terms of functions h, ξ, η, and ω and their derivatives. Then, these partial derivatives must be substituted into equation (9) followed by eliminating the variable t with the help of (10). As a result (with a suitable choice of ω as shown below), one arrives at a bilinear functional-differential equation,

$$\sum_{j=1}^{N} \varphi_j[x]\psi_j[u] = 0. \tag{11}$$

Here, $\varphi_j[x] \equiv \varphi_j(x, \xi, \eta, \xi'_x, \eta'_x, \dots)$ and $\psi_j[u] \equiv \psi_j(u, h, h'_u, \dots)$ are differential forms (in some cases, functional coefficients) that depend respectively on only x and u. The following statement is true.

The splitting principle (first indicated by Birkhoff [59]). Functional-differential equations of the form (11) can have solutions only if the forms $\psi_j[u]$ ($j = 1, \dots, N$) are connected by linear relations (see, for example, [20, 58, 60]):

$$\sum_{j=1}^{m_i} k_{ij}\psi_j[u] = 0, \qquad i = 1, \dots, n, \tag{12}$$

where k_{ij} are some constants, $1 \le m_i \le N-1$, and $1 \le n \le N-1$. It is necessary to consider also degenerate cases when, in addition to the linear relations, some individual differential forms $\psi_j[u]$ vanish.

The splitting principle is also true for the forms $\varphi_j[x]$.

The splitting principle will be used in Sections 3 and 4 to construct exact solutions of some functional-differential equations of the form (11), which arise in seeking solutions of the corresponding non-linear Klein–Gordon type equations (8). Note that, in the generic case, different linear relations of the form (12) correspond to different solutions of the PDE under consideration.

Remark 3. In general, equation (8) is invariant under transformations $t = -\bar{t}$ and $t = \tilde{t} + t_0$, where t_0 is an arbitrary constant. Therefore, in all the exact solutions below, t can be replaced by $\pm t + t_0$.

The high efficiency of the described direct method for constructing functional separable solutions in implicit form was clearly demonstrated in [58], where more than 20 functional separable solutions of nonlinear reaction-diffusion equations with variable coefficients, which depended on arbitrary functions, were obtained.

2.4. Derivation of functional-differential equation

We are looking for exact solutions of Klein–Gordon type equations (8) in implicit form (10). Differentiating (10) with respect to t and x, we get

$$u_t = \frac{\xi\omega'_t}{h}, \quad u_{tt} = \frac{\xi\omega''_{tt}}{h} - \xi^2(\omega'_t)^2\frac{h'_u}{h^3}, \quad u_x = \frac{\xi'_x\omega + \eta'_x}{h};$$
$$(afu_x)_x = [(a\xi'_x)'_x\omega + (a\eta'_x)'_x]\frac{f}{h} + a(\xi'_x\omega + \eta'_x)^2\frac{1}{h}\Big(\frac{f}{h}\Big)'_u.$$

Substituting these expressions into (8) yields the functional-differential equation

$$\omega''_{tt} - \xi\frac{h'_u}{h^2}(\omega'_t)^2 = \Phi_1(x,u)\omega^2 + \Phi_2(x,u)\omega + \Phi_3(x,u), \tag{13}$$

where the functions Φ_n are not explicitly dependent on t and are defined by the formulas

$$\begin{aligned}
\Phi_1(x,u) &= \frac{a(\xi'_x)^2}{c\xi}\Big(\frac{f}{h}\Big)'_u,\\
\Phi_2(x,u) &= \frac{1}{c\xi}\Big[(a\xi'_x)'_x f + 2a\xi'_x\eta'_x\Big(\frac{f}{h}\Big)'_u\Big],\\
\Phi_3(x,u) &= \frac{1}{c\xi}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big].
\end{aligned} \tag{14}$$

Equation (13)–(14) depends on three variables t, x, u, which are connected by one additional relation (10); it contains unknown functions (and their derivatives) that have different arguments. This equation is more complex than equations of the form (11).

The functional-differential equation (13)–(14) is significantly simplified in two cases: (i) $\xi'_x = 0$ and (ii) $(f/h)'_u = 0$. These cases are discussed further in Sections 3 and 4.

3. Exact solutions in the case $\boldsymbol{\xi} = 1$

3.1. Generalized traveling-wave solutions for $\omega(t) = kt$

For $\xi'_x = 0$, without loss of generality, we can set $\xi = 1$. On substituting $\xi = 1$ into (14), we get $\Phi_1(x,u) = \Phi_2(x,u) = 0$. As a result, equation (13) is reduced to the form

$$\omega''_{tt} - \frac{h'_u}{h^2}(\omega'_t)^2 = \frac{1}{c}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big]. \tag{15}$$

It is seen that when $\omega(t) = kt$ with $k = \text{const}$, the variables in (15) are separated. The situation under consideration corresponds to generalized traveling-wave solutions given in implicit form

$$\int h(u)\,du = kt + \int \theta(x)\,dx. \tag{16}$$

Here, the integrands $h(u)$ and $\theta(x) = \eta'_x(x)$ will be determined in the subsequent analysis from the functional-differential equation

$$k^2 c\frac{h'_u}{h^3} + (a\theta)'_x\frac{f}{h} + a\theta^2\frac{1}{h}\Big(\frac{f}{h}\Big)'_u + bg = 0. \tag{17}$$

Equation (17) is a bilinear functional-differential equation of the form (11) with $N = 4$.

Solution 1. First, let us look at the degenerate case where the differential form $(f/h)'_u$ in (17) is zero. In this case, according to the splitting principle (it will no longer be referred in what follows), equation (17) has solutions under the following conditions

$$h = f, \quad g = A + Bf^{-3} f'_u, \quad (a\theta)'_x + Ab = 0, \quad Bb + k^2 c = 0, \tag{18}$$

where A and B are arbitrary constants. From the relations (18) with $c(x) = 1$ and $B = k = 1$ it follows that the equation

$$u_{tt} = [a(x)f(u)u_x]_x - A - \frac{f'_u(u)}{f^3(u)}, \tag{19}$$

which contains two arbitrary functions $a(x)$ and $f(u)$, has the generalized traveling-wave exact solution

$$\int f(u)\,du = t + A\int \frac{x\,dx}{a(x)} + C_1 \int \frac{dx}{a(x)} + C_2, \tag{20}$$

where C_1 and C_2 are arbitrary constants. In the special case $a(x) = 1$, equation (19) and its solution (20) become the equation and solution obtained in [17].

Remark 4. It is easy to verify that the equation

$$u_{tt} = [a(x)f(u)u_x]_x + b(x) - \frac{f'_u(u)}{f^3(u)},$$

which contains three arbitrary functions $a(x)$, $b(x)$, and $f(u)$ and generalizes the equation (19), has the exact solution in implicit form

$$\int f(u)\,du = t - \int \frac{1}{a(x)}\left(\int b(x)\,dx\right)dx + C_1 \int \frac{dx}{a(x)} + C_2.$$

Solution 2. Equation (17) holds if we set

$$f = A\frac{h'_u}{h^2}, \quad g = -\frac{1}{h}\left(\frac{f}{h}\right)'_u, \quad A(a\theta)'_x + k^2 c = 0, \quad b = a\theta^2, \tag{21}$$

where A is an arbitrary constant.

Using the formulas (21) with $c(x) = 1$, one can obtain the non-linear Klein–Gordon type equation

$$u_{tt} = [a(x)F(u)u_x]_x - \frac{x^2}{a(x)} G(u), \tag{22}$$

where $a(x)$ is an arbitrary function. The functions $F(u)$ and $G(u)$ are expressed through the arbitrary function $h = h(u)$ as

$$F(u) = A\frac{h'_u}{h^2}, \quad G(u) = \frac{k^4}{Ah}\left(\frac{h'_u}{h^3}\right)'_u. \tag{23}$$

Equation (22) under condition (23) admits the exact solution

$$\int h(u)\,du = kt - \frac{k^2}{A}\int \frac{x\,dx}{a(x)} + C. \tag{24}$$

Example 3. By setting $h = u^{-n-1}$ and $A = -1/(n+1)$ in (22)–(24), we can get the Klein–Gordon type equation with power non-linearities

$$u_{tt} = [a(x)u^n u_x]_x - \frac{x^2}{a(x)}u^{3n+1}, \qquad n \neq -1,\ -1/2. \tag{25}$$

Exact solutions of this equation are determined by formula (24), where $\int h(u)\,du = -u^{-n}/n$ and $k = \pm[(n+1)^2(2n+1)]^{-1/4}$.

Example 4. By setting $h = e^{-\lambda u}$ and $A = -1/\lambda$ in (22)–(24), we can get the Klein–Gordon type equation with exponential non-linearities

$$u_{tt} = [a(x)e^{\lambda u}u_x]_x - \frac{x^2}{a(x)}e^{3\lambda u}, \qquad \lambda \neq 0. \tag{26}$$

Exact solutions of this equation are determined by formula (24), where $\int h(u)\,du = -e^{-\lambda u}/\lambda$ and $k = \pm(2\lambda^3)^{-1/4}$.

Equations (25) and (26) include an arbitrary function $a = a(x)$.

Solution 3. Equation (17) can be satisfied by setting

$$Af = \left(\frac{f}{h}\right)'_u, \quad g = -k^2\frac{h'_u}{h^3}, \quad (a\theta)'_x + Aa\theta^2 = 0, \quad b = c, \tag{27}$$

where A is an arbitrary constant. Taking $c(x) = 1$ in (27), we obtain the non-linear Klein–Gordon type equation

$$u_{tt} = [a(x)f(u)u_x]_x + g(u), \tag{28}$$

where $a(x)$ is an arbitrary function, and the functions $f(u)$ and $g(u)$ are expressed through the arbitrary function $h = h(u)$ as follows:

$$f(u) = Bh\exp\left(A\int h\,du\right), \quad g(u) = -k^2\frac{h'_u}{h^3}, \tag{29}$$

where B is an arbitrary constant. Equation (28)–(29) admits the exact solution

$$\int h(u)\,du = kt + \frac{1}{A}\ln\biggl(C_1\int\frac{dx}{a(x)} + C_2\biggr). \tag{30}$$

Formulas (29) and (30) include two arbitrary functions $a = a(x)$ and $h = h(u)$ and five arbitrary constants A, B, C_1, C_2, and k.

Example 5. By setting $h = 1/u$, $A = m+1$, and $B = 1$ in (28) and (29), we get the Klein–Gordon type equation with power non-linearity

$$u_{tt} = [a(x)u^m u_x]_x + k^2 u, \qquad m \neq -1, \tag{31}$$

which has the exact solution

$$u = e^{kt}\biggl(C_1\int\frac{dx}{a(x)} + C_2\biggr)^{\frac{1}{m+1}}.$$

Remark 5. It is easy to verify that equation (31) also admits a more general solution,

$$u = (A_1e^{kt} + B_1e^{-kt})\biggl(C_1\int\frac{dx}{a(x)} + C_2\biggr)^{\frac{1}{m+1}},$$

where A_1, B_1, C_1, and C_2 are arbitrary constants.

Remark 6. The equation

$$u_{tt} = [a(x)u^m u_x]_x - k^2 u, \qquad m \neq -1, \tag{32}$$

which differs from the equation (31) by the sign of the last term, has the exact solution

$$u = [A_1\cos(kt) + B_1\sin(kt)]\biggl(C_1\int\frac{dx}{a(x)} + C_2\biggr)^{\frac{1}{m+1}}.$$

Remark 7. The equation

$$u_{tt} = [a(x)e^{\lambda u}u_x]_x + k, \tag{33}$$

has the exact solution

$$u = \frac{1}{2}kt^2 + C_3t + \frac{1}{\lambda}\ln\biggl(C_1\int\frac{dx}{a(x)} + C_2\biggr).$$

Solution 4. Equation (17) holds if we set

$$\frac{h'_u}{h^2} + A\Big(\frac{f}{h}\Big)'_u = 0, \quad g = -\frac{f}{h}, \quad a\theta^2 = Ak^2 c, \quad (a\theta)'_x = b, \tag{34}$$

where A is an arbitrary constant. With $c(x) = 1$ in (34), we obtain the non-linear Klein–Gordon type equation

$$u_{tt} = [a(x)F(u)u_x]_x \mp \frac{a'_x(x)}{\sqrt{a(x)}}G(u), \tag{35}$$

where the functions $f(u)$ and $g(u)$ are expressed through the arbitrary function $h = h(u)$ as

$$F(u) = Bh + \frac{1}{A}, \quad G(u) = \frac{1}{2}k\sqrt{A}\Big(B + \frac{1}{Ah}\Big). \tag{36}$$

Equations (35)–(36) have the exact solutions

$$\int h(u)\,du = kt \pm k\sqrt{A}\int \frac{dx}{\sqrt{a(x)}} + C. \tag{37}$$

Formulas (36) and (37) include arbitrary functions $a(x)$ and $h(u)$ and arbitrary constants A, B, C, and k.

Remark 8. From formulas (36) one can express G through F in the form

$$G = \frac{kA^{3/2}BF}{2(AF-1)}.$$

Example 6. By setting $A = 1$, $B = 0$, and $k = 2$ in (35)–(37), we get $F = 1$ and $G = 1/h$. Taking this into account, replacing in (37) first h with G, and then denoting $G = \mp g$, we obtain the equation

$$u_{tt} = [a(x)u_x]_x + \frac{a'_x(x)}{\sqrt{a(x)}}g(u), \tag{38}$$

which include two arbitrary functions $a(x)$ and $g(u)$ and admits the following exact solutions:

$$\int \frac{du}{g(u)} = \pm 2t - 2\int \frac{dx}{\sqrt{a(x)}} + C. \tag{39}$$

Substituting in (38) $a(x) = x^{2\beta}$ and $a(x) = e^{2\beta x}$, we obtain two equations

$$
\begin{aligned}
u_{tt} &= (x^{2\beta}u_x)_x + x^{\beta-1}\bar{g}(u), \qquad \bar{g}(u) = 2\beta g(u);\\
u_{tt} &= (e^{2\beta x}u_x)_x + e^{\beta x}\bar{g}(u), \qquad \bar{g}(u) = 2\beta g(u);
\end{aligned}
$$

their exact solutions can be determined by the formula (39).

Solution 5. Equation (17) holds if we set

$$
g = A_1\frac{f}{h} + A_2\frac{h'_u}{h^3}, \quad \frac{1}{h}\Big(\frac{f}{h}\Big)'_u = A_3\frac{f}{h} + A_4\frac{h'_u}{h^3}, \tag{40}
$$

$$
(a\theta)'_x + A_3 a\theta^2 + A_1 b = 0, \quad A_4 a\theta^2 + k^2 c + A_2 b = 0, \tag{41}
$$

where A_1, A_2, A_3, and A_4 are arbitrary constants. From equations (40), we obtain the following representations of the functions f and g in terms of h:

$$
\begin{aligned}
f &= hE\bigg(A_4\int\frac{h'_u\,du}{h^2E} + C_1\bigg), \quad E = \exp\bigg(A_3\int h\,du\bigg),\\
g &= A_1C_1E + A_1A_4E\int\frac{h'_u\,du}{h^2E} + A_2\frac{h'_u}{h^3},
\end{aligned} \tag{42}
$$

where C_1 is an arbitrary constant.

Equations (41) for given functions $a = a(x)$ and $c = c(x)$ allow us to find two other functions, $b(x)$ and $\theta(x)$. Eliminating $b(x)$ from equations (41), we obtain a first-order ODE with quadratic non-linearity in $\theta(x)$,

$$
A_2 a\theta'_x + (A_2A_3 - A_1A_4)a\theta^2 + A_2 a'_x\theta - A_1k^2c = 0, \tag{43}
$$

which is a Riccati equation [61]. With the substitution

$$
\theta = \lambda\frac{\psi'_x}{\psi}, \quad \lambda = \frac{A_2}{A_2A_3 - A_1A_4} \quad (A_2A_3 - A_1A_4 \neq 0), \tag{44}
$$

it is reduced to the linear second-order ODE

$$
A_2\lambda(a\psi'_x)'_x - A_1k^2c\psi = 0. \tag{45}
$$

Exact solutions of equation (45) for some functions $a = a(x)$ and $c = c(x)$ can be found in [61].

Using the last relation in (41), we can express the functional coefficient b through θ:

$$
b = -\frac{1}{A_2}(A_4a\theta^2 + k^2c). \tag{46}
$$

Example 7. For $a(x) = c(x) = 1$, the general solution of equation (45) is given by

$$\psi = \begin{cases} C_2 \cosh(mx) + C_3 \sinh(mx) & \text{if } A_1(A_2A_3 - A_1A_4) > 0, \\ C_2 \cos(mx) + C_3 \sin(mx) & \text{if } A_1(A_2A_3 - A_1A_4) < 0, \end{cases} \tag{47}$$

where C_2 and C_3 are arbitrary constants, and $m = \sqrt{k^2|A_1|/|A_2\lambda|}$. In particular, substituting $A_1 = A_2 = A_4 = 1$, $A_3 = 2$, $C_2 = 1$, $C_3 = 0$, and $k = \pm 1$ in formulas (44), (46), and (47), we obtain $m = \lambda = 1$, $\psi = \cosh x$, $\theta = \tanh x$, and $b = -(1 + \tanh^2 x)$.

Solution 6. Equation (17) also has solutions under the following conditions:

$$(a\theta)'_x = Ac, \quad a\theta^2 = Bc, \quad b = c, \quad k^2 \frac{h'_u}{h^3} + A\frac{f}{h} + B\frac{1}{h}\Big(\frac{f}{h}\Big)'_u + g = 0, \tag{48}$$

where A and B are arbitrary constants.

1. Substituting in (48) $a(x) = b(x) = c(x) = 1$, $\theta(x) = \kappa$, $A = 0$, $B = \kappa^2$, and $k = \lambda$ we obtain the traveling wave solution of the form (4), which is lowered here.

2. By setting $c(x) = 1$ and $A = B = 1$ in the first three equations (48), we find

$$a(x) = x^2, \quad b(x) = 1, \quad \theta(x) = 1/x. \tag{49}$$

As a result, we get the equation

$$u_{tt} = [x^2 f(u) u_x]_x + g(u), \tag{50}$$

where

$$g(u) = -k^2 \frac{h'_u(u)}{h^3(u)} - \frac{f(u)}{h(u)} - \frac{1}{h(u)}\frac{d}{du}\left[\frac{f(u)}{h(u)}\right], \tag{51}$$

which has the exact solution in implicit form

$$\int h(u)\, du = kt + \ln x. \tag{52}$$

Note that equation (50)–(51) includes two arbitrary functions $f = f(u)$ and $h = h(u)$.

Remark 9. The invariant solution (52) of equation (50) can be found in the usual form $u = U(z)$, where $z = kt + \ln x$ (in this case, the relation (51) between the functions g and h is not used). The function $U(z)$ is determined from the ODE:

$$k^2 U''_{zz} = [f(U)U'_z]'_z + f(U)U'_z + g(U).$$

Remark 10. The non-linear Klein–Gordon type equation with delay (τ being the delay time)

$$u_{tt} = [x^2 f(u,w)u_x]_x + g(u,w), \quad w = u(x, t-\tau),$$

which is more general than (50), also has an exact solution of the form $u = U(z)$, where $z = kt + \ln x$.

3.2. Functional separable solutions for $\omega(t) = k\ln t$

Substituting $\xi = 1$ and $\omega(t) = k \ln t$ in (10), we look for solutions in the form

$$\int h(u)\,du = k\ln t + \eta(x). \tag{53}$$

Eliminating t from (15), with $\omega = k\ln t$, and (53), we obtain the functional-differential equation

$$\begin{aligned} &(a\eta'_x)'_x f + a(\eta'_x)^2\left(\frac{f}{h}\right)'_u + bgh + kce^{2\eta/k}\left(1 + k\frac{h'_u}{h^2}\right)e^{-2H/k} = 0, \\ &H = \int h(u)\,du. \end{aligned} \tag{54}$$

Solution 7. First, let us look at the degenerate case where the differential form $(f/h)'_u$ vanishes. In this case, equation (54) has solutions under the following conditions

$$\begin{aligned} &h = f, \quad g = A + \frac{B}{f}\left(1 + k\frac{f'_u}{f^2}\right)e^{-2F/k}, \\ &(a\eta'_x)'_x + Ab = 0, \quad Bb + kce^{2\eta/k} = 0, \end{aligned} \tag{55}$$

where A and B are arbitrary constants, and $F = \int f(u)\,du$. From relations (55) with $B = k$, it follows that the equation

$$c(x)u_{tt} = [a(x)f(u)u_x]_x - c(x)e^{2\eta(x)/k}\left[A + \frac{k}{f(u)}\left(1 + k\frac{f'_u(u)}{f^2(u)}\right)e^{-2F(u)/k}\right], \tag{56}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, and the function $\eta = \eta(x)$ is the solution of the second-order non-linear ODE

$$[a(x)\eta'_x]'_x - Ac(x)e^{2\eta/k} = 0, \tag{57}$$

has the functional separable solution

$$\int f(u)\,du = k\ln t + \eta(x). \tag{58}$$

Note that for $a(x) = c(x) = 1$ and $A = -k$ a solution of equation (57) is defined by the formula $\eta = -k\ln\cosh x$.

Solution 8. Equation (54) can be satisfied if we take

$$\begin{aligned} &Af = \left(\frac{f}{h}\right)'_u, \quad g = \frac{1}{h}\left(1 + k\frac{h'_u}{h^2}\right)e^{-2H/k}, \\ &(a\eta'_x)'_x + Aa(\eta'_x)^2 = 0, \quad b = -kce^{2\eta/k}, \end{aligned} \tag{59}$$

where A is an arbitrary constant. Substituting $c = 1$ into (59), we obtain the non-linear Klein–Gordon type equation

$$u_{tt} = [a(x)f(u)u_x]_x - k\left(C_1\int\frac{dx}{a(x)} + C_2\right)^{2/(Ak)} g(u), \tag{60}$$

where $a(x)$ is an arbitrary function, and the functions $f(u)$ and $g(u)$ are expressed as follows through the arbitrary function $h = h(u)$:

$$f(u) = Bh\exp\left(A\int h\,du\right), \quad g(u) = \frac{1}{h}\left(1 + k\frac{h'_u}{h^2}\right)\exp\left(-\frac{2}{k}\int h\,du\right). \tag{61}$$

Equation (60)–(61) has the exact solution

$$\int h(u)\,du = k\ln t + \frac{1}{A}\ln\left(C_1\int\frac{dx}{a(x)} + C_2\right). \tag{62}$$

Formulas (61) and (62) include arbitrary functions $a(x)$ and $h(u)$ and arbitrary constants A, B, C_1, C_2, and k.

Example 8. If $a(x) = 1$, $A = 2/(kn)$, $C_1 = k^{-1/n}$, and $C_2 = 0$, equation (60) becomes

$$u_{tt} = [f(u)u_x]_x + x^n g(u),$$

where the functions $f(u)$ and $g(u)$ are given by formulas (61). This equation has the exact solution $\int h(u)\,du = k(\ln t + \frac{1}{2}n\ln x - \frac{1}{2}\ln k)$.

Solution 9. Equation (54) holds if we set

$$\begin{aligned} &\Big(\frac{f}{h}\Big)'_u = -A\bigg(1 + k\frac{h'_u}{h^2}\bigg)e^{-2H/k}, \quad g = -\frac{f}{h}, \\ &Aa(\eta'_x)^2 = kce^{2\eta/k}, \quad (a\eta'_x)'_x = b, \end{aligned} \tag{63}$$

where A is an arbitrary constant. As a result, we obtain the non-linear Klein–Gordon type equation

$$c(x)u_{tt} = [a(x)f(u)u_x]_x + b(x)g(u), \tag{64}$$

with the functions $f(u)$ and $g(u)$ expressed in terms of the arbitrary function $h = h(u)$ as follows:

$$\begin{aligned} &f(u) = -h(u)\bigg[A\int\bigg(1 + k\frac{h'_u}{h^2}\bigg)e^{-2H/k}du + B\bigg], \\ &g(u) = A\int\bigg(1 + k\frac{h'_u}{h^2}\bigg)e^{-2H/k}du + B, \quad H = \int h(u)\,du, \end{aligned} \tag{65}$$

and the function $b(x)$ is determined by the formulas (two variants are possible)

$$b(x) = [a(x)\eta'_x]'_x, \qquad \eta = -k\ln\bigg[C_1 \pm \frac{1}{\sqrt{Ak}}\int\sqrt{\frac{c(x)}{a(x)}}dx\bigg]. \tag{66}$$

Equations (64)–(66) have the exact solutions

$$\int h(u)\,du = k\ln t - k\ln\bigg[C_1 \pm \frac{1}{\sqrt{Ak}}\int\sqrt{\frac{c(x)}{a(x)}}dx\bigg]. \tag{67}$$

Formulas (65)–(67) include arbitrary functions $a(x)$, $c(x)$, and $h(u)$ and arbitrary constants A, B, and C.

Solution 10. Equation (54) can be satisfied by setting

$$\begin{aligned} &f = A\bigg(1 + k\frac{h'_u}{h^2}\bigg)e^{-2H/k}, \quad g = \frac{1}{h}\Big(\frac{f}{h}\Big)'_u, \\ &A(a\eta'_x)'_x + kce^{2\eta/k} = 0, \quad b = -a(\eta'_x)^2, \end{aligned} \tag{68}$$

where A is an arbitrary constant.

Suppose that in the last two equations in (68), the variable functions $c = c(x)$ and $\eta = \eta(x)$ are given. Then the functional coefficients $a = a(x)$ and $b = b(x)$ are determined by the formulas

$$a(x) = -\frac{k}{A\eta'_x}\biggl(\int ce^{2\eta/k}dx + B\biggr), \quad b(x) = \frac{k\eta'_x}{A}\biggl(\int ce^{2\eta/k}dx + B\biggr),$$

where B is an arbitrary constant.

Solution 11. Equation (54) holds if we set

$$\begin{aligned} &(a\eta'_x)'_x = Ace^{2\eta/k}, \quad a(\eta'_x)^2 = Bce^{2\eta/k}, \quad b = ce^{2\eta/k}, \\ &Af + B\Bigl(\frac{f}{h}\Bigr)'_u + gh + k\biggl(1 + k\frac{h'_u}{h^2}\biggr)e^{-2H/k} = 0, \end{aligned} \tag{69}$$

where A and B are arbitrary constants and $H = \int h(u)\,du$.

Substituting $c(x) = 1$, $A = 2/k$, and $B = 1$ into the first three equations in (69), we find that

$$a(x) = b(x) = e^{\lambda x}, \quad \eta(x) = x, \quad \lambda = \frac{2}{k}. \tag{70}$$

As a result, we obtain the equation

$$u_{tt} = [e^{\lambda x} f(u)u_x]_x + e^{\lambda x} g(u), \tag{71}$$

where

$$g(u) = -\frac{2}{\lambda h(u)}\biggl(1 + \frac{2}{\lambda}\frac{h'_u}{h^2}\biggr)\exp\biggl[-\lambda\int h(u)\,du\biggr] - \lambda\frac{f(u)}{h(u)} - \frac{1}{h(u)}\frac{d}{du}\biggl[\frac{f(u)}{h(u)}\biggr], \tag{72}$$

which has the exact solution in implicit form

$$\int h(u)\,du = x + \frac{2}{\lambda}\ln t. \tag{73}$$

Note that equation (71)–(72) includes two arbitrary functions $f = f(u)$ and $h = h(u)$.

Remark 11. The invariant solution (73) of equation (71) can be found in the usual form $u = U(z)$, where $z = x + (2/\lambda)\ln t$ (in this case, the relation (72) between the functions g and h is not used). The function $U(z)$ is determined by the ODE

$$\frac{4}{\lambda^2}U''_{zz} - \frac{2}{\lambda}U'_z = [e^{\lambda z} f(U)U'_z]'_z + e^{\lambda z} g(U).$$

Solution 12. By setting $c(x)=1$, $A=(k+2)/k$, and $B=1$ in the first three equations of (69), we find that

$$a(x)=x^n,\quad b(x)=x^{n-2},\quad \eta(x)=\ln x,\quad n=2+\frac{2}{k}. \tag{74}$$

This leads to the Klein–Gordon type equation

$$u_{tt}=[x^n f(u)u_x]_x+x^{n-2}g(u), \tag{75}$$

where $n\neq 2$ and

$$\begin{aligned} g(u)=&-\frac{2}{(n-2)h(u)}\left(1+\frac{2}{n-2}\frac{h'_u(u)}{h^2(u)}\right)\exp\left[-(n-2)\int h(u)\,du\right]\\ &-(n-1)\frac{f(u)}{h(u)}-\frac{1}{h(u)}\frac{d}{du}\left[\frac{f(u)}{h(u)}\right], \end{aligned} \tag{76}$$

which has the exact solution in implicit form

$$\int h(u)\,du=\ln x+\frac{2}{n-2}\ln t. \tag{77}$$

Remark 12. The self-similar solution (77) of equation (75) can be found in the usual form $u=U(z)$, where $z=xt^{2/(n-2)}$ (in this case, relation (76) between the functions g and h is not used). The function $U(z)$ is determined by the ODE

$$\frac{4}{(n-2)^2}z(zU'_z)'_z-\frac{2}{n-2}zU'_z=[z^n f(U)U'_z]'_z+z^{n-2}g(U).$$

Solution 13. Equation (54) holds if we set

$$\begin{aligned} &\left(\frac{f}{h}\right)'_u=Af,\quad \left(1+k\frac{h'_u}{h^2}\right)e^{-2H/k}=Bf,\quad gh=f,\\ &(a\eta'_x)'_x+Aa(\eta'_x)^2+b+Bkce^{2\eta/k}=0, \end{aligned} \tag{78}$$

where A and B are arbitrary constants, and $H=\int h(u)\,du$.

By setting $A=-2/k=\lambda$ and $B=1$ in the first three equations of (78), we get

$$f(u)=g(u)=e^{\lambda u},\quad h(u)=1. \tag{79}$$

Therefore, the equation

$$c(x)u_{tt}=[a(x)e^{\lambda u}u_x]_x+b(x)e^{\lambda u} \tag{80}$$

admits the exact solution in explicit form

$$u = -\frac{2}{\lambda}\ln t + \eta(x), \tag{81}$$

where the function $\eta = \eta(x)$ is determined by the ODE

$$(a\eta'_x)'_x + \lambda a(\eta'_x)^2 + b - \frac{2}{\lambda}ce^{-\lambda\eta} = 0. \tag{82}$$

Equations (80) and (82) include three arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

Solution 14. By setting $A = n + 1$, $B = 1 - k$, and $k = -2/n$ in the first three equations (78), we find

$$f(u) = u^n, \quad g(u) = u^{n+1}, \quad h(u) = 1/u. \tag{83}$$

As a result, we have the Klein–Gordon equation

$$c(x)u_{tt} = [a(x)u^n u_x]_x + b(x)u^{n+1} \tag{84}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, which admits an exact solution of the form $\ln u = -(2/n)\ln t + \eta(x)$. This solution can be represented explicitly as

$$u = t^{-2/n}\zeta(x), \quad \zeta(x) = e^{\eta(x)}, \tag{85}$$

where the function ζ is described by the ODE

$$[a(x)\zeta^n\zeta'_x]'_x + b(x)\zeta^{n+1} - \frac{2(n+2)}{n^2}c(x)\zeta = 0. \tag{86}$$

3.3. Functional separable solutions for $\omega(t) = kt^2$

Substituting $\xi = 1$ and $\omega(t) = kt^2$ in (10), we look for solutions in the form

$$\int h(u)\,du = kt^2 + \eta(x). \tag{87}$$

In this section we assume that $b(x) = c(x) = 1$. Eliminating t from (15), with $\omega = kt^2$, and (87), we obtain the functional-differential equation

$$(a\eta'_x)'_x\frac{f}{h} + a(\eta'_x)^2\frac{1}{h}\Big(\frac{f}{h}\Big)'_u + g - \frac{2k}{h} + \frac{4kh'_uH}{h^3} - 4k\eta\frac{h'_u}{h^3} = 0. \tag{88}$$

Solution 15. First, let us look at the degenerate case where the differential form $(a\eta'_x)'_x$ vanishes. In this case, equation (88) has solutions under the following conditions

$$(a\eta'_x)'_x = 0, \quad a(\eta'_x)^2 = C_1\eta,$$
$$C_1\Big(\frac{f}{h}\Big)'_u - 4k\frac{h'_u}{h^2} = 0, \quad g = \frac{2k}{h} - \frac{4kh'_u H}{h^3}, \tag{89}$$

where C_1 is an arbitrary constant. Integrating the first two equations (89), we obtain

$$a = \frac{C_2^2}{C_1C_3}\exp\Big(-\frac{C_1}{C_2}x\Big), \quad \eta = C_3\exp\Big(\frac{C_1}{C_2}x\Big), \tag{90}$$

where C_2 and C_3 are arbitrary constants. By integrating the third equation (89), we express the functions f and g through an arbitrary function h:

$$f = C_4h - \frac{4k}{C_1}, \quad g = \frac{2k}{h} - \frac{4kh'_u}{h^3}\int h(u)\,du. \tag{91}$$

Example 9. By setting $C_1 = C_3 = 1$, $C_2 = -1$, $C_4 = 0$, and $k = -\frac{1}{4}$ in (90) and (91), we have $a = e^x$, $\eta = e^{-x}$, $f = 1$, and $g = -\frac{1}{2}h^{-1} + h^{-3}h'_u\int h\,du$. As a result, we obtain the equation

$$u_{tt} = (e^x u_x)_x - \frac{1}{2h} + \frac{h'_u}{h^3}\int h\,du,$$

which for an arbitrary function $h = h(u)$ has the exact solution in implicit form

$$\int h\,du = e^{-x} - \frac{1}{4}t^2.$$

Solution 16. Equation (88) holds if we set

$$(a\eta'_x)'_x = C_1, \quad a(\eta'_x)^2 = C_2\eta,$$
$$C_2\Big(\frac{f}{h}\Big)'_u - 4k\frac{h'_u}{h^2} = 0, \quad g = \frac{2k}{h} - \frac{4kh'_u H}{h^3} - C_1\frac{f}{h}, \tag{92}$$

where C_1 and C_2 are arbitrary constants. The general solution of the system consisting of the first two equations in (92) has the form

$$a = \frac{1}{C_2C_4}(C_1x + C_3)^{2-(C_2/C_1)}, \quad \eta = C_4(C_1x + C_3)^{C_2/C_1}, \tag{93}$$

where C_3 and C_4 are arbitrary constants. Integrating the third equation (92), we find the functions f and g,

$$f = C_5 h - \frac{4k}{C_2}, \quad g = 2k\left(1 + \frac{2C_1}{C_2}\right)\frac{1}{h} - \frac{4kh'_u}{h^3}\int h\,du - C_1C_5, \tag{94}$$

where $h = h(u)$ is an arbitrary function.

Example 10. By setting $C_1 = 1$, $C_2 = 2 - m$, $C_3 = C_5 = 0$, $C_4 = 1/(2-m)$, and $k = \frac{1}{4}(m-2)$ in (93) and (94), we get $a = x^m$, $\eta = x^{2-m}/(2-m)$, $f = 1$, and $g = \frac{1}{2}mh^{-1} + (2-m)h^{-3}h'_u \int h\,du$. As a result, we obtain the equation

$$u_{tt} = (x^m u_x)_x + \frac{m-4}{2h} + (2-m)\frac{h'_u}{h^3}\int h\,du,$$

which for an arbitrary function $h = h(u)$ has the exact solution in implicit form

$$\int h\,du = \frac{1}{2-m}x^{2-m} + \frac{1}{4}(m-2)t^2, \qquad m \neq 2.$$

Solution 17. Equation (88) can be satisfied by setting

$$\begin{gathered}(a\eta'_x)'_x = C_1\eta, \quad a(\eta'_x)^2 = C_2^2,\\ f = \frac{4k}{C_1}\frac{h'_u}{h^2}, \quad g = \frac{2k}{h} - \frac{4kHh'_u}{h^3} - C_2^2\frac{1}{h}\left(\frac{f}{h}\right)'_u,\end{gathered} \tag{95}$$

where C_1 and C_2 are arbitrary constants. Eliminating the function a from the first two equations (95), we arrive at an autonomous ODE,

$$\eta''_{xx} + C_1C_2^{-2}\eta(\eta'_x)^2 = 0, \tag{96}$$

which admits the first integral $\eta'_x = C_3\exp(-\frac{1}{2}C_1C_2^{-2}\eta^2)$. It follows that the function η is expressed implicitly in terms of the integral of probabilities. The last two relations (95) determine the functions f and g in terms of an arbitrary function h.

4. Exact solutions in the case $h = f$

4.1. Generalized traveling-wave solutions for $\omega(t) = t$

For $(f/h)'_u = 0$, without loss of generality, we can set $h = f$. Substituting $h = f$ in (13)–(14), we get the equation

$$\omega''_{tt} - \xi\frac{f'_u}{f^2}(\omega'_t)^2 = \frac{(a\xi'_x)'_x}{c\xi}f\omega + \frac{1}{c\xi}\big[(a\eta'_x)'_x f + bfg\big]. \tag{97}$$

It is seen that when $\omega(t) = t$, the variables in (97) are separated. The situation under consideration corresponds to generalized traveling-wave solutions given in implicit form

$$\int f(u)\,du = \xi(x)t + \eta(x), \tag{98}$$

where $f(u)$ is the function included in equation (8), and $\xi(x)$ and $\eta(x)$ are functions to be found.

Having excluded $\omega = t$ from (97) by (98), after simple manipulations we obtain the equation

$$\xi(a\eta'_x)'_x - \eta(a\xi'_x)'_x + b\xi g + c\xi^3 f^{-3} f'_u + (a\xi'_x)'_x F = 0, \qquad F = \int f(u)\,du, \tag{99}$$

which is a special case of an equation of the form (11) with $N = 4$.

Solution 18. Let us first consider the degenerate case when two functional coefficients in (99) vanish at once, i.e.

$$(a\xi'_x)'_x = 0. \tag{100}$$

Integrating (100), we find the relationship between the functions $a = a(x)$ and $\xi = \xi(x)$:

$$\xi = C_1 \int \frac{dx}{a(x)} + C_2, \tag{101}$$

where C_1 and C_2 are arbitrary constants.

Equation (99) in the degenerate case (100) has solutions under the following conditions

$$g = k_1 + k_2 f^{-3} f'_u, \quad (a\eta'_x)'_x + k_1 b = 0, \quad k_2 b + c\xi^2 = 0, \tag{102}$$

where k_1 and k_2 are arbitrary constants. From the relations (101) and (102) it follows that the equation

$$c(x)u_{tt} = [a(x)f(u)u_x]_x - c(x)\xi^2(x)\left[k + \frac{f'_u(u)}{f^3(u)}\right], \quad k = \frac{k_1}{k_2}, \tag{103}$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, and the function $\xi = \xi(x)$ is defined by the formula (101), admits an exact solution of the form (98), in which the function $\eta(x)$ has the form

$$\eta(x) = k \int \frac{1}{a(x)} \left(\int c(x)\xi^2(x)\,dx \right) dx + C_3 \int \frac{dx}{a(x)} + C_4; \tag{104}$$

C_3 and C_4 are arbitrary constants.

Example 11. We set $a(x) = c(x) = 1$ in equation (103) and $C_1 = 1$, $C_2 = C_3 = C_4 = 0$ in formulas (101) and (104). As a result, we obtain the equation

$$u_{tt} = [f(u)u_x]_x - x^2\biggl[k + \frac{f'_u(u)}{f^3(u)}\biggr], \tag{105}$$

which contains an arbitrary function $f(u)$ and an arbitrary constant k and admits the generalized traveling-wave solution

$$\int f(u)\,du = xt + \frac{1}{12}kx^4.$$

In the particular case $f(u) = e^u$, equation (105) takes the form

$$u_{tt} = (e^u u_x)_x - x^2(k + e^{-2u}).$$

Its exact solution is $u = \ln(xt + \frac{1}{12}kx^4)$.

Solution 19. In the degenerate case (100), equation (99) also has other exact solutions under the following conditions:

$$g = k_1, \quad f^{-3}f'_u = -k_2, \quad (a\eta'_x)'_x + k_1 b - k_2 c\xi^2 = 0, \tag{106}$$

where k_1 and k_2 are arbitrary constants. From relations (101) and (106) with $k_1 = 1$ and $k_2 = \frac{1}{2}$, it follows that the equation

$$c(x)u_{tt} = [a(x)u^{-1/2}u_x]_x + b(x), \tag{107}$$

depending on arbitrary functions $a(x)$, $b(x)$, and $c(x)$, admits an exact solution of the form

$$u = \tfrac{1}{4}[\xi(x)t + \eta(x)]^2. \tag{108}$$

Here, the function $\xi = \xi(x)$ is defined by the formula (101), and the function $\eta(x)$ satisfies the ODE

$$[a(x)\eta'_x]'_x = \tfrac{1}{2}c(x)\xi^2 - b(x). \tag{109}$$

Since the right side of equation (109) is known, the function η is found by simple integration.

Remark 13. The equation

$$c(x)u_{tt} = [a(x)u^{-1/2}u_x]_x + b(x)u^{1/2} + p(x), \tag{110}$$

which is more general than (107), has an exact solution of the form (108). In the cases $p(x) = 0$ and $p(x)/b(x) = \text{const}$, equation (110) belongs to the class of equations (8) in question.

Remark 14. The non-linear delay PDE

$$c(x)u_{tt} = [a_1(x)u^{-1/2}u_x]_x + [a_2(x)w^{-1/2}w_x]_x + b_1(x)u^{1/2} + b_2(x)w^{1/2} + p(x),$$
$$w = u(x, t - \tau),$$

where τ is the delay time and $a_1(x)$, $a_2(x)$, $b_1(x)$, $b_2(x)$, $c(x)$, and $p(x)$ are arbitrary functions, also admits an exact solution of the form (108).

Solution 20. In the nondegenerate case, the functional differential equation (99) admits solutions under the following conditions:

$$g = k_1 + k_2 f^{-3} f'_u + k_3 F, \quad F = \int f(u)\,du, \tag{111}$$
$$\xi(a\eta'_x)'_x - \eta(a\xi'_x)'_x + k_1 b\xi = 0, \tag{112}$$
$$k_2 b + c\xi^2 = 0, \tag{113}$$
$$(a\xi'_x)'_x + k_3 b\xi = 0, \tag{114}$$

where $f(u)$ is an arbitrary function, and k_1, k_2, and k_3 are arbitrary constants.

Assuming that the functions $a = a(x)$ and $c = c(x)$ are given, and eliminating b from equations (113) and (114), we arrive at an Emden–Fowler type equation for ξ:

$$(a\xi'_x)'_x - \frac{k_3}{k_2} c\xi^3 = 0. \tag{115}$$

Equation (112) is a linear nonhomogeneous ordinary differential equation with respect to η, which has a particular solution $\eta_p = -k_1/k_3$ (with this value substituted, equation (112) reduces to (114)). The shortened linear homogeneous equation (112), corresponding to $k_1 = 0$, has a particular solution $\eta_0 = \xi$. Hence, the order of this equation can be reduced [61]. Considering the above, we find the general solution of equation (112):

$$\eta = C_1\xi + C_2\xi \int \frac{dx}{a\xi^2} - \frac{k_1}{k_3}, \tag{116}$$

where C_1 and C_2 are arbitrary constants. The functional coefficient b is determined from equation (113).

To sum up, we have obtained the non-linear Klein–Gordon type equation

$$c(x)u_{tt} = [a(x)f(u)u_x]_x - c(x)\xi^2(x)\left[k_1 + \frac{f'_u(u)}{f^3(u)} + k_3 \int f(u)\,du\right], \qquad (117)$$

where $a(x)$, $c(x)$, and $f(u)$ are arbitrary functions, the function $\xi = \xi(x)$ satisfies equation (115) with $k_2 = 1$. Equation (117) has the exact solution in implicit form

$$\int f(u)\,du = \xi(x)t + \eta(x),$$

where the function $\eta(x)$ is defined by formula (116).

Example 12. For $a(x) = c(x) = k_2 = 1$ equation (115) has exact solutions $\xi = \pm\sqrt{2/k_3}\,x^{-1}$. In this case, the function η in (116) takes the form $\eta = A_1x^{-1} + A_2x^2 - (k_1/k_3)$, where A_1 and A_2 are arbitrary constants that can be expressed in terms of C_1, C_2 and k_3.

4.2. Exact solutions for $\omega(t) = e^{\lambda t}$

Solution 21. We substitute $\omega(t) = e^{\lambda t}$ and $\eta = \eta_0 =$ const in (10) and (97) and then eliminate t. As a result, we obtain the functional-differential equation

$$-\lambda^2\frac{\bar{F}}{f} + \lambda^2\frac{f'_u}{f^3}\bar{F}^2 + \frac{(a\xi'_x)'_x}{c\xi}\bar{F} + \frac{b}{c}g = 0, \qquad \bar{F} = \int f\,du - \eta_0, \qquad (118)$$

which is a special case of an equation of the form (11) with $N = 3$.

Equation (118) holds if we set

$$(a\xi'_x)'_x = Ac\xi, \quad b = c, \quad g = \lambda^2\frac{\bar{F}}{f} - \lambda^2\frac{f'_u}{f^3}\bar{F}^2 - A\bar{F}. \qquad (119)$$

where A is an arbitrary constant.

By setting $b(x) = c(x) = 1$ and $\eta_0 = 0$ in (119), we obtain the equation

$$u_{tt} = [a(x)f(u)u_x]_x + \lambda^2\frac{F(u)}{f(u)} - \lambda^2\frac{f'_u(u)}{f^3(u)}F^2(u) - AF(u), \qquad F = \int f(u)\,du, \qquad (120)$$

which has the exact solution in implicit form $\int f(u)\,du = e^{\lambda t}\xi(x)$, where the function $\xi = \xi(x)$ satisfies the second-order linear ODE $(a\xi'_x)'_x = A\xi$.

Example 13. When $f(u) = u^k$, the equation (120) takes the form

$$u_{tt} = [a(x)u^k u_x]_x + \frac{\lambda^2}{(k+1)^2} u - \frac{A}{k+1} u^{k+1}, \tag{121}$$

where $a(x)$ is an arbitrary function and A, k, and λ are arbitrary constants ($k \neq -1$). It admits an exact solution that can be represented in the explicit form

$$u = [(k+1)e^{\lambda t}\xi(x)]^{1/(k+1)},$$

where the $\xi = \xi(x)$ is determined from the second-order linear ODE $(a\xi'_x)'_x = A\xi$.

Remark 15. Let us consider the equation

$$c(x)u_{tt} = [a(x)u^k u_x]_x + b(x)u^{k+1} + mc(x)u, \tag{122}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, and $m \neq 0$ is an arbitrary constant (equation (122) is a generalization of equation (121)).

For $m = \beta^2 > 0$, equation (122) admits the multiplicative separable solution

$$u = [C_1 \exp(-\beta t) + C_2 \exp(\beta t)]\theta(x),$$

where C_1 and C_2 are arbitrary constants, and the function $\theta = \theta(x)$ is described by the ODE

$$[a(x)\theta^k \theta'_x]'_x + b(x)\theta^{k+1} = 0, \tag{123}$$

which reduces to a linear equation by replacing $\zeta = \theta^{k+1}$,

$$[a(x)\zeta'_x]'_x + (k+1)b(x)\zeta = 0.$$

For $m = -\beta^2 < 0$, equation (122) admits exact solutions

$$u = [C_1 \cos(\beta t) + C_2 \sin(\beta t)]\theta(x),$$

where the function $\theta = \theta(x)$ is determined from ODE (123).

Remark 16. Note that more general than (122), the non-linear delay PDE

$$c(x)u_{tt} = [a(x)u^k u_x]_x + b(x)u^{k+1} + mc(x)w, \quad w = u(x,t) \tag{124}$$

has a solution of the form $u = \varphi(t)\theta(x)$, where the function $\theta = \theta(x)$ satisfies ODE (123) and the function $\varphi = \varphi(x)$ is described by the linear delay ODE $\varphi''_{tt}(t) = m\varphi(t - \tau)$.

Remark 17. The equation

$$c(x)u_{tt} = [a(x)e^{\lambda u}u_x]_x + b(x)e^{\lambda u} + mc(x), \tag{125}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, and $m \neq 0$ is an arbitrary constant, has the additive separable solution

$$u = \frac{1}{2}mt^2 + C_1 t + \theta(x).$$

Here, the function $\theta = \theta(x)$ is described by ODE

$$[a(x)e^{\lambda\theta}\theta'_x]'_x + b(x)e^{\lambda\theta} = 0.$$

The substitution $\zeta = e^{\lambda\theta}$ reduces this equation to a linear one,

$$[a(x)\zeta'_x]'_x + \lambda b(x)\zeta = 0.$$

4.3. Exact solutions for $\omega(t) = t^\beta$

Solution 22. We substitute $\omega(t) = t^{-2/k}$ and $\eta = \eta_0 = \text{const}$ in (10) and (97) and then eliminate t. As a result, we obtain the functional-differential equation

$$-\frac{2(k+2)}{k^2}\frac{\bar{F}^{k+1}}{f} + \frac{4}{k^2}\frac{f'_u}{f^3}\bar{F}^{k+2} + \frac{\xi^{k-1}(a\xi'_x)'_x}{c}\bar{F} + \frac{b\xi^k}{c}g = 0, \quad \bar{F} = \int f\,du - \eta_0, \tag{126}$$

which is a special case of an equation of the form (11) with $N = 3$.

Equation (126) holds if we set

$$(a\xi'_x)'_x = Ac\xi^{1-k}, \quad b = c\xi^{-k}, \quad g = \frac{2(k+2)}{k^2}\frac{\bar{F}^{k+1}}{f} - \frac{4}{k^2}\frac{f'_u}{f^3}\bar{F}^{k+2} - A\bar{F}, \tag{127}$$

where A is an arbitrary constant.

In particular, substituting $k = 1$ and $\eta_0 = 0$ in (127), we arrive at the Klein–Gordon type equation

$$c(x)u_{tt} = [a(x)f(u)u_x]_x - \frac{c(x)}{\xi(x)}\left[AF(u) - \frac{6}{f(u)}F^2(u) + 4\frac{f'_u(u)}{f^3(u)}F^3(u)\right], \tag{128}$$

where

$$\xi(x) = A\int\frac{1}{a(x)}\left(\int c(x)\,dx\right)dx + C_1\int\frac{dx}{a(x)} + C_2, \quad F(u) = \int f(u)\,du,$$

and C_1 and C_2 are arbitrary constants. This equation has the exact solution in implicit form $\int f(u)\,du = \xi(x)t^{-2}$.

5. Relationship of the direct method of functional separation of variables with other methods. Some generalizations and remarks

5.1. Relationship with the method of differential constraints

The direct method of constructing functional separable solutions in implicit form, based on formula (10), is closely related to the method of differential constraints [62] (which is based on the compatibility theory of PDEs). To show this, we differentiate formula (10), for example, with respect to t. As a result, we obtain the differential relation

$$u_t = \xi(x)\bar{\omega}(t)p(u), \tag{129}$$

where $\bar{\omega}(t) = \omega'_t(t)$ and $p(u) = 1/h(u)$.

Relation (129) can be treated as a first-order differential constraint, which can be used to find exact solutions of equation (8) through a compatibility analysis of the overdetermined pair of equations (8) and (129) with the single unknown u. The differential constraint (129) is equivalent to relation (10); at the initial stage, all functions included on the right-hand sides of (129) are considered arbitrary, and the specific form of these functions is determined in the subsequent analysis. In the general case, any PDE (or ODE, in a degenerate case) that depends on the same variables as the original equation can be treated as a differential constraint.

For a description of the method of differential constraints, its relationship with other methods, as well as a number of specific examples of its application, see, for example, [12, 18, 20, 37, 62–68].

It is important to note that the construction of exact solutions by the method of differential constraints is based on a compatibility analysis of PDEs and is carried out in several steps; on three of them, it is necessary to solve non-linear differential equations (for details, see, for example, [20, 50].

5.2. Relationship with the nonclassical method of symmetry reduction

The first-order differential constraint (129) is a special cases of the invariant surface condition [69], which characterizes the nonclassical method of symmetry reduction (in general, an invariant surface condition is a quasilinear first-order PDE of general form). This method, just like the method of different constraints, is also based on the compatibility analysis of two PDEs; specific examples of its use can be found, for example, in [18, 20, 60, 69–76]. For first-order differential constraints, the results of applying the method of differential constraint and the nonclassical method of symmetry reduction coincide (provided that the differential constraint coincides with the invariant surface condition [18, 20]).

In practice, it is technically much easier to use the direct method of functional separation of variables (since this method require less steps where it is necessary to solve intermediate differential equations and intermediate equations are simpler) than the method of differential constraints or the nonclassical method of symmetry reduction. In all cases, a complicating factor is the presence of arbitrary functions, if included in the equation in question (it is precisely such equations that are discussed in this article).

5.3. Notes on other methods

Let us now briefly discuss the direct method by Clarkson and Kruskal [77] (see also [20, 65, 75, 76, 78–80]), which is based on looking for exact solutions in the form $u = U(x,t,w(z))$ with $z = z(x,t)$. The functions $U(x,t,w)$ and $z(x,t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $w(z)$. The requirement that the function $w(z)$ must satisfy a single ODE greatly limits the capabilities of this method and does not allow it to be effectively used to construct exact solutions of equation (8).

Solutions of the form (10) usually cannot be obtained by applying the classical Lie group analysis of PDEs [9, 81].

Note also that in paper [82] it is discussed why invariance with respect to certain symmetry transformations in physical applications should be used with care.

5.4. Some generalizations and remarks

Let's see what happens if instead of (10) we look for solutions in a more general form,

$$\int h(u)\,du = Q(x,t). \tag{130}$$

Using (130) we calculate partial derivatives, u_x, u_t, u_{xx}, and u_{tt}. Then, substituting these partial derivatives into equation (8), we obtain

$$-cQ_{tt} + cQ_t^2\frac{h'_u}{h^2} + (aQ_x)_x f + aQ_x^2\Big(\frac{f}{h}\Big)'_u + bgh = 0. \tag{131}$$

For $h = 1$, equation (131) coincides with the original equation (8). Therefore, at this stage, no solutions have been lost.

Introducing the functions

$$\begin{aligned} &\varphi_1 = -cQ_{tt}, \quad \varphi_2 = cQ_t^2, \quad \varphi_3 = (aQ_x)_x, \quad \varphi_4 = aQ_x^2, \quad \varphi_5 = b;\\ &\psi_1 = 1, \quad \psi_2 = h'_u/h^2, \quad \psi_3 = f, \quad \psi_4 = (f/h)'_u, \quad \psi_5 = gh, \end{aligned} \tag{132}$$

we rewrite equation (131) as

$$\sum_{j=1}^{5} \varphi_j \psi_j = 0. \tag{133}$$

Equation (133) is similar in form to the functional-differential equation (11), but in equation (133) the functions φ_j and ψ_j depend on the same independent variables x and t (whereas in equation (11) they depend on different variables). Therefore, in this case it is not possible to make full use of the splitting principle formulated in Section 2.3, as there may also be other exact solutions. However, one can try to construct exact solutions of equation (133) by equating several linear combinations of the functions φ_j (and ψ_j) to zero (in addition, one can also consider degenerate cases in which, in addition to the linear combinations, the individual φ_j or ψ_j vanish). We will call this approach of constructing exact solutions the *generalized splitting principle*.

Example 14. Equation (133) holds if we set

$$\begin{aligned} &\varphi_1 = -k_1\varphi_5, \quad \varphi_2 = -k_2\varphi_5, \quad \varphi_3 = -k_3\varphi_5; \\ &\psi_4 = 0, \quad \psi_5 = k_1\psi_1 + k_2\psi_2 + k_3\psi_3, \end{aligned} \tag{134}$$

where k_1, k_2, and k_3 are arbitrary constants. Substituting (132) in (134), we obtain

$$\begin{aligned} &cQ_{tt} = k_1 b, \quad cQ_t^2 = -k_2 b, \quad (aQ_x)_x = -k_3 b; \\ &(f/h)'_u = 0, \quad gh = k_1 + k_2 h^{-2} h'_u + k_3 f. \end{aligned} \tag{135}$$

For $b(x) = c(x) = 1$, analysis of the system of equations (135) leads to equation (19) and its solution (20) (if $k_1 = 0$, $k_2 = -1$, and $k_3 = -A$).

To construct more complex solutions containing integrals of type $\int f(u)\,du$ or $\int h(u)\,du$ (which are included in the solutions obtained in Sections 3.2, 3.3, and 4.1), we need to use a more specific (not too general) function $Q(x,t)$. For a suitable function $Q(x,t)$, the variable t can be excluded from (130) and (131) and a bilinear functional-differential equation of the form (11) can be obtained. A good example of a suitable (more specific) function is $Q(x,t) = \xi(x)\omega(t) + \eta(x)$, which is used in this article.

Remark 18. A wider class of exact solutions can be obtained if, instead of equation (131), we consider the equivalent equation

$$\sum_{j=1}^{5} \varphi_j \psi_j + \lambda(Q - H) = 0, \quad H = \int h\,du,$$

in which the function $\lambda(Q - H)$ is added, vanishing on solutions of the form (130). Here, λ is a functional coefficient that can depend on the functions a, b, c, Q, f, g, h (and their derivatives) included in equation (131) (the functional coefficient λ can vary at the discretion of the researcher). A similar approach can also be used to find exact solutions to other nonlinear PDEs.

6. Conclusions

The direct method for constructing functional separable solutions to non-linear equations of mathematical physics has been described. The solutions are sought in the form of an implicit relation containing several free functions (these functions are determined in the subsequent analysis). Different classes of non-linear Klein–Gordon type equations with variable coefficients, which admit exact solutions (in closed form), have been considered. Special attention has been paid to non-linear Klein–Gordon type equations of general form, which depend on one or several arbitrary functions. Many new generalized traveling-wave solutions and functional separable solutions have been obtained.

Acknowledgments

The study was supported within the framework of the Russian State Assignment (State Registration Number AAAA-A17-117021310385-6) and was partially supported by the Russian Foundation for Basic Research (project No. 18-29-10025).

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