The use of differential and non-local transformations for numerical integration of non-linear blow-up problems [1]

Andrei D. Polyanin[a,b,c,*], Inna K. Shingareva[d,**]

[a] *Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadsky Avenue, bldg 1, 119526 Moscow, Russia*
[b] *Bauman Moscow State Technical University, 5 Second Baumanskaya Street, 105005 Moscow, Russia*
[c] *National Research Nuclear University MEPhI, 31 Kashirskoe Shosse, 115409 Moscow, Russia*
[d] *University of Sonora, Blvd. Luis Encinas y Rosales S/N, Hermosillo C.P. 83000, Sonora, México*

Abstract

Two new methods of numerical integration of Cauchy problems for nonlinear ODEs of the first- and second-order, which have blow-up solutions are described. In such problems, the position of the singular point is not known in advance. The first method is based on obtaining an equivalent system of equations by applying a differential transformation, where the first derivative (given in the original equation) is chosen as a new independent variable, $t = y'_x$. The second method is based on introducing a new auxiliary non-local variable of the form $\xi = \int_{x_0}^{x} g(x, y, y'_x)\, dx$ with the subsequent transformation to the Cauchy problem for the corresponding system of coupled ODEs. Both methods lead to problems whose solutions are represented in parametric form and do not have blowing-up singular points; therefore the standard fixed-step numerical methods can be applied. The efficiency of the proposed methods is illustrated with a number of test problems that admit exact solutions. It is shown that the methods, based on spe-

*Principal corresponding author
**Corresponding author
Email addresses: polyanin@ipmnet.ru (Andrei D. Polyanin), inna@mat.uson.mx (Inna K. Shingareva)

[1] A. D. Polyanin and I. K. Shingareva, The use of differential and non-local transformations for numerical integration of non-linear blow-up problems, *Int. J. Non-Linear Mechanics*, 2017, Vol. 95, pp. 178–184.

cial exp-type transformations (which are particular cases of the general non-local transformation), are more efficient than the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation. The method, based on introducing a non-local variable, can be generalized to the n th-order ODEs and systems of coupled ODEs.

Keywords: non-linear differential equations, blow-up solutions, numerical integration, differential transformations, non-local transformations, arc-length transformation

1. Introduction

We will consider Cauchy problems for ODEs, whose solutions tend to infinity at some finite value of x, say $x = x_*$. Such x_* does not appear explicitly in the given differential equation and it is not known in advance. Similar solutions exist on a bounded interval (hereinafter in this article we assume that $x_0 \le x < x_*$) and are called blow up solutions. This raises the important question for practice: how to determine the position of a singular point x_* and the solution in its neighborhood with the aid of numerical methods.

In general, the blow-up solutions, that have a power singularity, can be represented in a neighborhood of the singular point x_* as

$$y \simeq A(x_* - x)^{-\beta}, \quad \beta > 0, \tag{1}$$

where A is a constant. For these solutions we have $\lim\limits_{x\to x_*} |y| = \infty$ and $\lim\limits_{x\to x_*} |y'_x| = \infty$.

For blow-up solutions with the power singularity (1) near the singular point x_* we have

$$y'_x/y \simeq \beta/(x_* - x), \tag{2}$$

i.e. the required function grows more slowly than its derivative. Therefore, we have $\lim\limits_{x\to x_*} y'_x/y = \infty$ (this is a common property of any blow-up solutions; it must be taken into account when carrying out numerical calculations).

The direct application of the standard fixed-step numerical methods in such problems leads to certain difficulties because of the singularity in the blow-up solutions and the unknown (in advance) range of variation of the independent variable x (see, for example, [1, 2]).

One of the basic ideas of numerical solution of such problems is to find a suitable transformation, leading to the equivalent problem for one differential equation or a system of coupled equations, the solutions of which have no singularities at unknown point.

Currently, two methods based on this idea are most commonly used. The first one is proposed by Acosta et al. [3]. They have suggested to apply a hodograph transformation $x = \bar{y}$, $y = \bar{x}$, where the independent and dependent variables are reversed. The second method, which is based on the arc-length transformation, has been proposed by Moriguti et al. [4] (for details, see below Items 2° in Sections 3.1 and 5.1 as well as reference [5]). This method is rather general and it can be applied for numerical integration of systems of ODEs.

The methods based on the hodograph and arc-length transformations for blow-up solutions with a power singularity of the form (1) lead to the Cauchy problems whose solutions tend to the asymptote with respect to the power law for large values of the new independent variable. This creates certain difficulties in some problems, since one has to consider large intervals of variation of the independent variable in numerical integration.

Based on other ideas, some special methods of numerical integration of blow-up problems are described, for example, in [1, 2, 5–9].

In this paper, we propose two new methods of numerical integration of nonlinear Cauchy problems for ODEs of the first- and second-orders, which have blow-up solutions. These methods are based on the differential and non-local transformations allowing us to obtain the equivalent systems of ODEs, whose solutions do not have singularities at some a priori unknown point. It is shown that special exp-type transformations (which are particular cases of the general non-local transformation) lead to the Cauchy problems whose solutions tend exponentially to the asymptote (which determines the position of the required singular point x_*) for large values of the new independent variable; therefore exp-type transformations are more preferable than the hodograph and arc-length transformations.

2. Problems for first-order equations. Differential transformations

2.1. Solution method based on a differential transformation

The Cauchy problem for the first-order differential equation has the form

$$y'_x = f(x, y) \quad (x > x_0), \tag{3}$$

$$y(x_0) = y_0. \tag{4}$$

In what follows we assume that $f = f(x, y) > 0$, $x_0 \geq 0$, $y_0 > 0$, and $f/y^{1+\varepsilon} \to \infty$ as $y \to \infty$, where $\varepsilon > 0$ (in such problems, blow-up solutions arise when the right-hand side of a non-linear ODE is quite rapidly growing as $y \to \infty$).

First, we present the ODE (3) as an equivalent system of differential-algebraic equations

$$t = f(x, y), \qquad y'_x = t, \tag{5}$$

where $y = y(x)$ and $t = t(x)$ are unknown functions to be determined.

By applying (5), we derive a system of equations of the standard form, assuming that $y = y(t)$ and $x = x(t)$. By taking the full differential of the first equation in (5) and multiplying the second one by dx, we get

$$dt = f_x\,dx + f_y\,dy, \qquad dy = t\,dx, \tag{6}$$

where f_x and f_y are the respective partial derivatives of $f = f(x, y)$. Eliminating first dy, and then dx from (6), we obtain a system of the first-order coupled equations

$$x'_t = \frac{1}{f_x + tf_y}, \qquad y'_t = \frac{t}{f_x + tf_y} \quad (t > t_0), \tag{7}$$

which must be supplemented by the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = f(x_0, y_0), \tag{8}$$

Conditions (8) are derived from (4) and the first equation of (5).

Assuming that the conditions $f_x + tf_y > 0$ at $t_0 < t < \infty$ are valid, the Cauchy problem (7)–(8) can be integrated numerically, for example, by applying the Runge–Kutta method or other standard numerical methods (see for example [10–15]). In this case, the difficulties (described in the introduction) will not occur since x'_t rapidly tends to zero as $t \to \infty$. The required critical value x_* is determined by the asymptotic behavior of the function $x = x(t)$ for large t.

2.2. Test problems. Exact and numerical solutions

Let us illustrate the method proposed in Section 2.1 with simple examples.

Example 1. Consider the model Cauchy problem for the first-order ODE with separated variables

$$y'_x = y^2 \quad (x > 0), \qquad y(0) = a, \tag{9}$$

where $a > 0$. The exact solution of this problem has the form

$$y = \frac{a}{1 - ax}. \tag{10}$$

It has a power-type singularity (a first-order pole) at a point $x_* = 1/a$ and does not exist for $x > x_*$.

By introducing a new variable $t = y'_x$ in (9), we obtain the following Cauchy problem for the system of equations:

$$\begin{gathered} x'_t = \frac{1}{2ty}, \quad y'_t = \frac{1}{2y} \quad (t > t_0); \\ x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = a^2, \end{gathered} \tag{11}$$

which is a particular case of the problem (7)–(8) with $f = y^2$, $x_0 = 0$, and $y_0 = a$. The exact solution of this problem has the form

$$x = \frac{1}{a} - \frac{1}{\sqrt{t}}, \quad y = \sqrt{t} \qquad (t \geq a^2). \tag{12}$$

It has no singularities; the function $x = x(t)$ increases monotonically for $t > a^2$, tending to to the desired limit value $x_* = \lim\limits_{t\to\infty} x(t) = 1/a$, and the function $y = y(t)$ monotonously increases with increasing t. The solution (12) for the system (11) is a solution of the original problem (9) in parametric form.

The maximum error of the numerical solution of the Cauchy problem for system of equations (11) with $a = 1$ obtained by the classical Runge–Kutta method of the fourth-order approximation for stepsize $h = 0.2$ does not exceed 0.017% for $y \leq 50$.

Remark 1. Here and in what follows, the numerical integration interval for the new variable t (or ξ) is usually determined, for demonstration calculations, from the condition $\Lambda_{\rm m} = 50$, where

$$\Lambda_{\rm m} = \min\{|y|,\, y'_x/y\} \quad \text{(for } |y_0| \sim 1 \text{ and } |y_1| \sim 1\text{)}, \tag{13}$$

and $y_1 = y'_x(x_0)$. In a few cases, the condition $\Lambda_{\rm m} = 100$ or $\Lambda_{\rm m} = 150$ is used, which is specially stipulated. In the definition of $\Lambda_{\rm m}$, a relation y'_x/y is included that takes into account the property (2). For first-order ODE problems of the form (3)–(4), the definition of $\Lambda_{\rm m}$ can be replaced by the equivalent definition $\Lambda_{\rm m} = \min\{|y|,\, f/y\}$.

Conditions $|y_0| \sim 1$ and $|y_1| \sim 1$ in (13) are not strongly essential, since the substitution $y = y_0 - 1 + (y_1 - 1)(x - x_0) + \bar{y}$ leads to an equivalent problem with the initial conditions $\bar{y}(x_0) = \bar{y}'_x(x_0) = 1$.

Example 2. For a more general two-parameter Cauchy problem,

$$y'_x = y^\gamma, \quad y(0) = a > 0,$$

having a blow-up solution for $\gamma > 1$, the introduction of a new variable $t = y'_x$ leads to the system of equations of the form (7), the solution of which is determined by the formulas

$$x = \frac{1}{\gamma - 1}\Big(a^{1-\gamma} - t^{\frac{1-\gamma}{\gamma}}\Big), \quad y = t^{\frac{1}{\gamma}} \qquad (t \ge a^\gamma). \tag{14}$$

This solution behaves qualitatively similar to the solution (12) as $t \to \infty$.

Remark 2. Solutions (12) and (14) slowly tend to the asymptotic values $x \to x_*$ as $t \to \infty$. To speed up the process of approaching the asymptotic behavior with respect to x in the system (7) is useful additionally to make the substitution of the exponential type

$$t = t_0 \exp(\kappa\tau), \quad \tau \ge 0, \tag{15}$$

where

$$\tau = (1/\kappa)\ln(t/t_0) = (1/\kappa)\ln(y'_x/t_0) \tag{16}$$

is a new independent variable and $\kappa > 0$ is a numerical parameter that can be varied. Transformations with a new independent variable of the form (16) will be called the *modified differential transformations*. See also Remarks 4 and 5.

3. Problems for first-order equations. Non-local transformations

3.1. Solution method based on non-local transformations

Introducing a new non-local variable [16, 17] according to the formula,

$$\xi = \int_{x_0}^{x} g(x, y)\, dx, \quad y = y(x), \tag{17}$$

leads the Cauchy problem for one equation (3)–(4) to the equivalent problem for the autonomous system of equations

$$\begin{aligned} &x'_\xi = \frac{1}{g(x,y)}, \quad y'_\xi = \frac{f(x,y)}{g(x,y)} \quad (\xi > 0); \\ &x(0) = x_0, \quad y(0) = y_0. \end{aligned} \tag{18}$$

Here, the function $g = g(x, y)$ has to satisfy the following conditions:

$$g > 0 \text{ if } x \geq x_0,\ y \geq y_0; \quad g \to \infty \text{ as } y \to \infty; \quad f/g = k \text{ as } y \to \infty, \tag{19}$$

where $k = \text{const} > 0$ (and the limiting case $k = \infty$ is also allowed); otherwise the function g can be chosen rather arbitrarily.

It follows from (17) and the second condition (19) that $x'_\xi \to 0$ as $\xi \to \infty$. The Cauchy problem (18) can be integrated numerically applying the Runge–Kutta method or other standard numerical methods.

Let us consider some possible selections of the function g in the system (18).

1°. The special case $g = f$ is equivalent to the hodograph transformation [3, 18] with an additional shift of the dependent variable, which gives $\xi = y - y_0$.

2°. Setting $g = \sqrt{1 + f^2}$, we arrive at the method of the arc-length transformation [4]. In this case, $k = 1$ in (19).

3°. Choosing $g = 1 + |f|$, we obtain the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{1}{1 + |f(x, y)|}, \quad y'_\xi = \frac{f(x, y)}{1 + |f(x, y)|}; \\ x(0) = x_0, \quad y(0) = y_0. \end{gathered} \tag{20}$$

Note that we use here the absolute value sign to generalize the results, since the system (20) can also be used in the case $f < 0$ for numerical integration of the problems having solutions with a root singularity [2]. In this case we have $k = 1$ in (19).

4°. We can take $g = f/y$ that corresponds to $k = \infty$ in (19). In this case, the second equation of the system is immediately integrated and, taking into account the initial condition, we get $y = y_0 e^\xi$. In addition, the variable x tends exponentially rapidly to a blow-up point x_* with increasing ξ. This transformation will be called the *special exp-type transformation*.

Remark 3. It follows from Items 1° and 2° that the method based on the hodograph transformation and the method of the arc-length transformation are particular cases of the method based on the non-local transformation of the general form, described in Section 3.1.

3.2. Test problems. Exact and numerical solutions

Example 3. For the test Cauchy problem (9) with $a = 1$, the equivalent the problem for the system of equations (20) admits an exact solution, which is expressed in terms of elementary functions in a parametric form as follows:

$$x = 1 + \tfrac{1}{2}\xi - \tfrac{1}{2}\sqrt{\xi^2 + 4}, \quad y = \tfrac{1}{2}\xi + \tfrac{1}{2}\sqrt{\xi^2 + 4} \qquad (\xi \geq 0). \tag{21}$$

This solution satisfies the initial conditions $x(0) = 0$ and $y(0) = 1$ and has no singularities. The function $x(\xi)$ is bounded, increases monotonically, and tends to its limiting value $x_* = \lim\limits_{\xi\to\infty} x(\xi) = 1$. The function $y(\xi)$ increases monotonically and tends to infinity as $\xi \to \infty$. At large ξ we have $x \approx 1 - \xi^{-1}$ and $y \approx \xi + \xi^{-1}$.

Example 4. For the test problem (9), in which $f = y^2$, we take $g = f/y = y$ (see Item 4° above). By substituting these functions in (18), we arrive at the Cauchy problem

$$\begin{aligned} &x'_\xi = \frac{1}{y}, \quad y'_\xi = y \quad (\xi > 0);\\ &x(0) = 0, \quad y(0) = a. \end{aligned} \tag{22}$$

The exact solution of this problem is written as follows:

$$x = \frac{1}{a}\bigl(1 - e^{-\xi}\bigr), \qquad y = ae^{\xi}. \tag{23}$$

We can see that the unknown quantity $x = x(\xi)$ exponentially tend to the asymptotic values $x = x_* = 1/a$ as $\xi \to \infty$.

The numerical solution of the problem (22) with $a = 1$, obtained by the fourth-order Runge–Kutta method for the stepsize $h = 0.2$, is presented in Fig. 1 (for the sake of clarity, a scale factor $\nu = 30$ is introduced for the function $x = x(\xi)$). In Fig. 1*b*, we also show the results of the numerical integration of the problem (11).

We can see (Fig. 1) that the numerical solutions are in a good agreement, but the speed of the process of approaching the asymptote (with respect to x) is different. For example, for system (11) it is required to take $t \in [1, 2400]$ to approach the asymptote, and for system (22), it is required to take a significantly smaller interval $\xi \in [0, 4]$. In this sense, the method based on a non-local transformation is more efficient than the method based on a differential transformation.

For comparison, similar calculations were performed according to the method based on the hodograph transformation (see Section 3.1, Item 1°) and the method of the arc-length transformation (see Section 3.1, Item 2° for $c = 1$ and $s = 2$). For both methods it is required to take $\xi \in [0, 49]$ to approach the asymptote. To

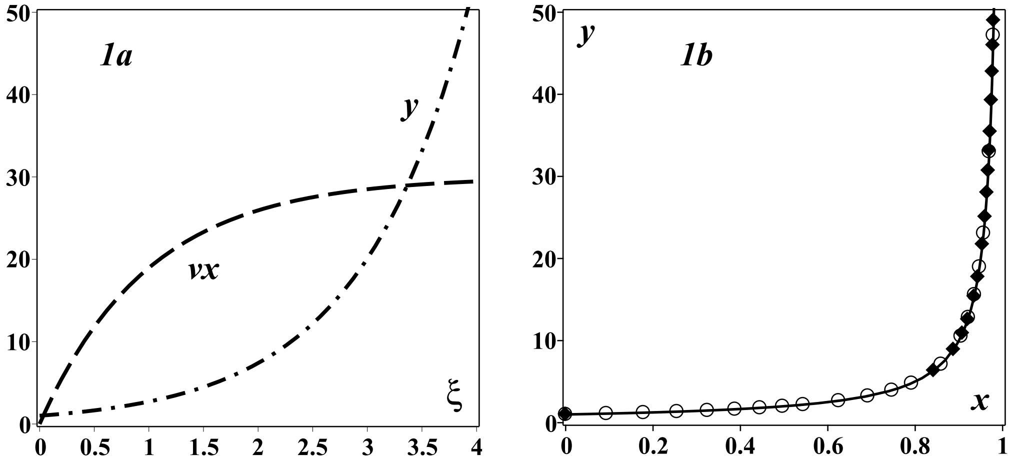

Figure 1: *1a*—the dependences $x = x(\xi)$ and $y = y(\xi)$, obtained by numerical solution of the problem (22) for $a = 1$ ($\nu = 30$); *1b*—numerical solution of the problem (22) (circle), numerical solution of the problem (11) (solid box), and exact solution (10) (solid line).

control the solution process, the calculations were performed with the aid of the three most powerful problem solving environments: Maple (2016), Mathematica (11), and MATLAB (2016a).

It can be observed that the method, based on the use of a special case of the system (18) with $g = f/y$ (see Item 4°), is a more efficient method compared to the method based on the hodograph transformation and the method of the arc-length transformation.

The absolute and percent errors of numerical integration of the problem (22) for stepsize $h = 0.2$ and $\Lambda_{\rm m} = 50$ are shown in Fig. 2. The maximum absolute and relative errors of numerical integration of the problem (22) for different values of stepsize h and $\Lambda_{\rm m}$ are given in Table 1. It can be seen that reducing the stepsize by one-half reduces the percent errors of numerical solutions by more than a factor of 14, and increasing $\Lambda_{\rm m}$ leads to an almost linear increasing of percent errors (increasing $\Lambda_{\rm m}$ by a factor of 3 increases the percent errors by a factor of 2.75).

Remark 4. The use of the transformation (15) with $\kappa = 2$ in the problem (11) leads to an exact solution in a parametric form, which coincides (up to renaming τ by ξ) with the exponential-type solution (23).

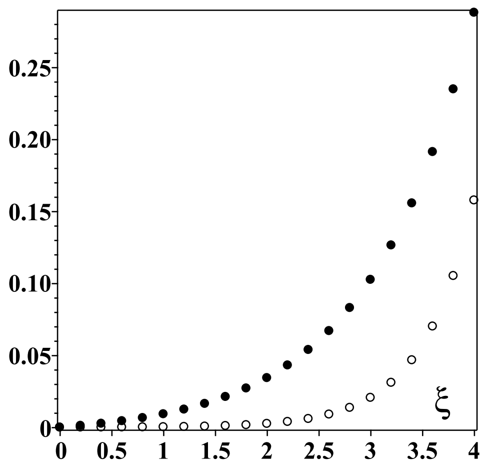

Figure 2: The absolute error (circle) and percent error (solid point) of numerical integration of the problem (22) for $h = 0.2$ and $\Lambda_{\rm m} = 50$.

Stepsize $h = 0.1$				Stepsize $h = 0.2$			
$\Lambda_{\rm m}$	$\xi_{\max}$	$\lvert\text{error}\rvert_{\max}$	$\text{error}_{\max}, \%$	$\Lambda_{\rm m}$	$\xi_{\max}$	$\lvert\text{error}\rvert_{\max}$	$\text{error}_{\max}, \%$
50	4.0	0.0109472	0.0200465	50	4.0	0.1577264	0.2880668
100	4.6	0.0366579	0.0368345	100	4.6	0.5293520	0.5293070
150	5.0	0.0818718	0.0551346	150	5.0	1.1851609	0.7922731

Table 1: The maximum absolute and percent errors of numerical solutions of the problem (22) for various values of $\Lambda_{\rm m}$ and stepsize h.

3.3. Comparison of efficiency of various transformations for numerical integration of first-order ODE blow-up problems

In Table 2, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (17) is presented by using the example of the test blow-up problem for the first-order ODE (9) with $f = y^2$ and $a = 1$. The comparison is based on the number of grid points needed to perform calculations with the same maximum error (approximately equal to 0.005). In all cases, for the integration of the transformed problems the standard fourth-order Runge–Kutta method was used.

It can be seen that for the first three transformations it is necessary to use a lot of grid points (the hodograph transformation is the least effective). This is due to

$\text{Error}_{\max}, \% = 0.005$				
Transformation	Function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Hodograph, Item 1°	$g{=}f$	49.035	0.1050	467
Arc-length, Item 2°	$g{=}\sqrt{1{+}f^2}$	49.266	0.1380	357
Nonlocal, Item 3°	$g{=}1{+}\lvert f\rvert$	50.135	0.1850	271
Special exp-type, Item 4°	$g{=}f/y$	3.9150	0.0725	54

Table 2: Various types of analytical transformations applied for numerical integration of the problem (9) for $f = y^2$ with a given accuracy (percent error $= 0.005$ and $\Lambda_{\rm m} \le 50$) and their basic parameters (maximum interval, stepsize, grid points number).

the fact that in these cases x tends to the point x_* rather slowly for large ξ ($x_*-x \sim 1/\xi$, $y \sim \xi$). The last transformation with $g = f/y$ require a significantly less number of grid points; in this case x tends exponentially rapidly to the point x_* for large ξ. We note that an analogous situation holds for $\text{Error}_{\max}, \% = 0.1$.

Remark 5. To get the percent error $= 0.005$, in the case of using the modified differential transformation (16) with $\kappa = 2$ to problem (9), it is necessary to take only 38 grid points.

4. Problems for second-order equations. Differential transformations

4.1. Solution method based on a differential transformation

The Cauchy problem for the second-order differential equation has the form

$$y''_{xx} = f(x, y, y'_x) \quad (x > x_0); \tag{24}$$

$$y(x_0) = y_0, \qquad y'_x(x_0) = y_1. \tag{25}$$

Note that the exact solutions of equations of the form (24), which can be used for the formulation of test problems with blow-up solutions, can be found, for example, in [18–21].

Let $f(x, y, u) > 0$ if $y > y_0 \ge 0$ and $u > y_1 \ge 0$, and the function f increases quite rapidly as $y \to \infty$ (for example, if f does not depend on y'_x, then $\lim\limits_{y\to\infty} f/y^{1+\varepsilon} = \infty$, where $\varepsilon > 0$).

First, as in Section 2.1, we represent ODE (24) as an equivalent system of differential-algebraic equations

$$y'_x = t, \qquad y''_{xx} = f(x, y, t), \tag{26}$$

where $y = y(x)$ and $t = t(x)$ are unknown functions.

Taking into account (26), we derive further a standard system of equations, assuming that $y = y(t)$ and $x = x(t)$. To do this, we differentiate the first equation in (26) with respect to t. We have $(y'_x)'_t = 1$. Taking into account the relations $y'_t = tx'_t$ (follows from the first equation of (26)) and $(y'_x)'_t = y''_{xx}/t'_x = x'_t y''_{xx}$, we get further

$$x'_t y''_{xx} = 1. \qquad (27)$$

If we eliminate the second derivative y''_{xx} by using a second equation of (26), we obtain the first-order equation

$$x'_t = \frac{1}{f(x, y, t)}. \qquad (28)$$

Considering further the relation $y'_t = tx'_t$, we transform (28) to the form

$$y'_t = \frac{t}{f(x, y, t)}. \qquad (29)$$

Equations (28) and (29) represent a system of coupled first-order equations with respect to functions $x = x(t)$ and $y = y(t)$. The system (28)–(29) should be defined with the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = y_1, \qquad (30)$$

which are derived from (25) and the first equation of (26).

The Cauchy problem (28)–(30) can be integrated numerically applying the standard numerical methods [10–15], without fear of blow-up solutions.

Remark 6. Systems of differential-algebraic equations (5) and (26) are particular cases of parametrically defined non-linear differential equations, which are considered in [22, 23]. In [23], the general solutions of several parametrically defined ODEs were obtained via the differential transformations, based on introducing a new independent variable $t = y'_x$.

4.2. Test problems. Exact and numerical solutions

Example 5. Consider a model Cauchy problem for the second-order ODE

$$y''_{xx} = 2yy'_x \quad (x > 0); \quad y(0) = a > 0, \quad y'_x(0) = a^2, \qquad (31)$$

which is obtained by differentiating equation (9). Exact solution of this problem is defined by the formula (10).

Introducing a new variable $t = y'_x$ in (31), we obtain the Cauchy problem, which exactly coincides with the problem (11). The exact solution of this problem is determined by the formulas (12).

Example 6. Let us now consider another Cauchy problem

$$y''_{xx} = 2y^3 \quad (x > 0); \quad y(0) = a, \quad y'_x(0) = a^2, \tag{32}$$

which is obtained by excluding the first derivative from the equations (9) and (31) (the latter one is a consequence of (9)). The exact solution of the problem (32) is defined by the formula (10).

Introducing a new variable $t = y'_x$ in (32), we transform (32) to the Cauchy problem for the system of the first-order ODEs

$$\begin{gathered} x'_t = \frac{1}{2y^3}, \quad y'_t = \frac{t}{2y^3} \quad (t > t_0); \\ x(t_0) = 0, \quad y(t_0) = a, \quad t_0 = a^2, \end{gathered} \tag{33}$$

which is a particular case of the problem (28)–(30) with $f = y^2$, $x_0 = 0$, and $y_0 = a$. The exact solution of the problem (33) is given by the formulas (12).

The maximum error of numerical solution of the Cauchy problem for the system of equations (33) with $a = 1$ obtained by the classical Runge–Kutta method of the fourth-order approximation for stepsizes $h = 0.1, 0.2, 0.4$, does not exceed, respectively, 0.001%, 0.022%, and 0.339% for $y \leq 50$ and $t \in [1, 2400]$.

The function $x(t)$ slowly tends to the asymptotic value x_*. Therefore to accelerate this process in the system (33) is useful additionally to make the exponential-type substitution (15).

5. Problems for second-order equations. Non-local transformations

5.1. Solution method based on non-local transformations

First, equation (24) can be represented as a system of two equations

$$y'_x = t, \quad t'_x = f(x, y, t),$$

and then we introduce the non-local variable of the general form [16, 17] by the formula

$$\xi = \int_{x_0}^{x} g(x, y, t)\, dx, \quad y = y(x), \quad t = t(x). \tag{34}$$

As a result, the Cauchy problem (24)–(25) can be transformed to following problem for the autonomous system of three equations:

$$x'_\xi = \frac{1}{g(x,y,t)}, \quad y'_\xi = \frac{t}{g(x,y,t)}, \quad t'_\xi = \frac{f(x,y,t)}{g(x,y,t)} \quad (\xi > 0);$$
$$x(0) = x_0, \quad y(0) = y_0, \quad t(0) = y_1. \tag{35}$$

For a suitable choice of the function $g = g(x, y, t)$ (not very restrictive conditions of the type (19) must be imposed on it), the Cauchy problem (35) can be numerically integrated applying the standard numerical methods [10–15], without fear of blow-up solutions.

Let us consider some possible choices of the function g in the system (35).

1°. The special case $g = t$ is equivalent to the hodograph transformation with an additional shift of the dependent variable, which gives $\xi = y - y_0$.

2°. We can take $g = \bigl(c + |t|^s + |f|^s\bigr)^{1/s}$ with $c \geq 0$ and $s > 0$. The case $c = 1$ and $s = 2$ corresponds to the method of the arc-length transformation [4].

3°. By taking $g = f$ in (35), after the integration of the third equation we arrive at the system (28)–(30). It follows that the method based on the non-local transformation (34) is a generalization of the method described in Section 4.1, which is based on the differential transformation.

4°. We can take $g = t/y$ or $g = kt/y$, where $k > 0$ is a numerical parameter that can be varied.

5°. Also, we can take $g = f/t$ or $g = kf/t$, where $k > 0$ is a free numerical parameter.

Remark 7. In the last two cases, 4° and 5°, the system (35) is much simplified, since one of its equations is directly integrated (for $g = t/y$ and $g = f/t$ we accordingly obtain the exponential components of the solutions, $y = ae^\xi$ and $t = a^2e^\xi$). The transformations corresponding to these two cases will be called *special exp-type transformations*, they lead to the solutions, in which the variable x tends exponentially rapidly to a blow-up point x_*.

Remark 8. It follows from Items 1°, 2°, and 3° that the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation are particular cases of the method based on the non-local transformation of the general form, described in this section.

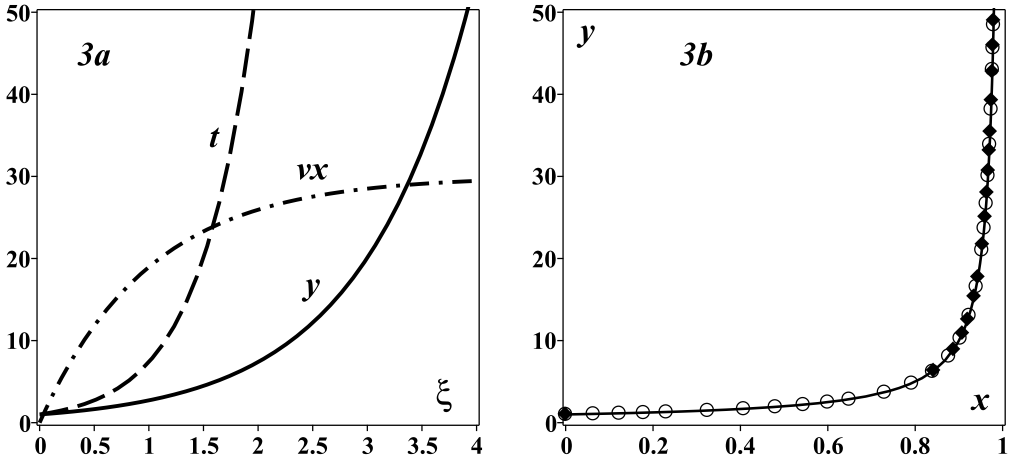

Figure 3: *3a*—the dependences $t = t(\xi)$, $x = x(\xi)$, $y = y(\xi)$, obtained by numerical solution of the problem (36) for $a = 1$ ($\nu = 30$); *3b*—numerical solution of the problem (36) (circle), numerical solution of the problem (33) (solid box), and exact solution (10) (solid line).

5.2. Test problems. Exact and numerical solutions

Example 7. For the test problem (32), in which $f = 2y^3$, we put $g = t/y$ (see Item 4° above). By substituting these functions in (35), we arrive at the Cauchy problem

$$x'_\xi = \frac{y}{t}, \quad y'_\xi = y, \quad t'_\xi = \frac{2y^4}{t} \quad (\xi > 0);$$
$$x(0) = 0, \quad y(0) = a, \quad t(0) = a^2. \tag{36}$$

The exact solution of this problem is written as follows:

$$x = \frac{1}{a}\bigl(1 - e^{-\xi}\bigr), \quad y = ae^{\xi}, \quad t = a^2 e^{2\xi}. \tag{37}$$

We can see that the unknown quantity $x = x(\xi)$ exponentially tend to the asymptotic values $x = x_* = 1/a$ as $\xi \to \infty$.

The numerical solution of the problem (36) with $a = 1$, obtained by the fourth-order Runge–Kutta method for the stepsize $h = 0.2$, is presented in Fig. 3. In Fig. 3*b*, we also show the results of the numerical integration of the problem (33). The numerical solutions are in a good agreement, but the speed of the process of approaching the asymptote (with respect to x) is different. For example, for system (33) it is required to take $t \in [1, 2400]$ to approach the asymptote, and for system (36), it is required to take a significantly smaller interval $\xi \in [0, 4]$. In this sense, the method based on the special exp-type transformation with $g = t/y$ is more efficient than the method based on a differential transformation.

For comparison, similar calculations were performed with the aid of Maple (2016) according to the method based on the hodograph transformation (see Section 5.1, Item 1°) and the method of the arc-length transformation (see Section 5.1, Item 2° for $c = 1$ and $s = 2$). To approach the asymptote, it is required to take $\xi \in [0, 49]$ for the method based on the hodograph transformation, and for the method of the arc-length transformation we have to take $\xi \in [0, 2500]$. To control the solution process, the calculations were performed also with the aid of the other two problem solving environments, Mathematica (11) and MATLAB (2016a). It can be observed that the method, based on the use of a special case of the system (35) with $g = t/y$ (see Example 7), is a more efficient method compared to the method based on the hodograph transformation and the method of the arc-length transformation.

The absolute and percent errors of numerical integration of the problem (36) for stepsize $h = 0.2$ and $\Lambda_m = 50$ are shown in Fig. 4. The maximum absolute and relative errors of numerical integration of the problem (36) for different values of stepsize h and Λ_m are given in Table 3. It can be seen that reducing the stepsize by one-half reduces the percent errors of numerical solutions by more than a factor of 14, and increasing Λ_m leads to an almost linear increasing of percent errors (increasing Λ_m by a factor of 3 increases the percent errors by a factor of 2.8).

Stepsize $h = 0.1$				Stepsize $h = 0.2$			
Λ_m	ξ_{max}	$\lvert\text{error}\rvert_{max}$	error_{max},%	Λ_m	ξ_{max}	$\lvert\text{error}\rvert_{max}$	error_{max},%
50	4.0	0.0221947	0.0406347	50	4.0	0.3233162	0.5887147
100	4.6	0.0741643	0.0744934	100	4.6	1.0851569	1.0790677
150	5.0	0.1655186	0.1114017	150	5.0	2.4339050	1.6135814

Table 3: The maximum absolute and percent errors of numerical solutions of the problem (36) for various values of Λ_m and stepsize h.

Example 8. For the test problem (32), in which $f = 2y^3$, we take $g = f/t$ (see Item 5° in Section 5.1). Substituting these functions into (35), we arrive at

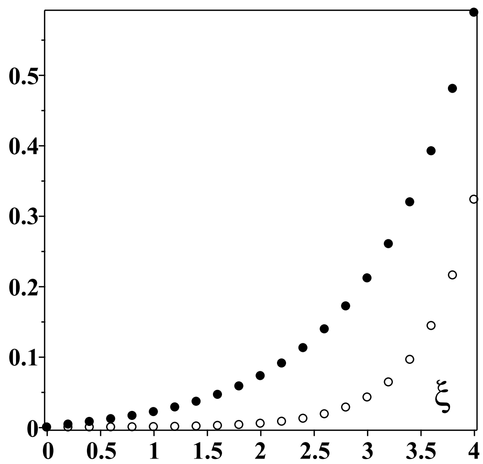

Figure 4: The absolute error (circle) and percent error (solid point) of numerical integration of the problem (36) for $h = 0.2$ and $\Lambda_{\rm m} = 50$.

the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{t}{2y^3}, \quad y'_\xi = \frac{t^2}{2y^3}, \quad t'_\xi = t \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \quad t(0) = a^2. \end{gathered} \tag{38}$$

The exact solution of this problem in parametric form has the form

$$x = \frac{1}{a}\bigl(1 - e^{-\xi/2}\bigr), \quad y = ae^{\xi/2}, \quad t = a^2 e^{\xi}. \tag{39}$$

The required value $x{=}x(\xi)$ tends exponentially to the asymptotic value $x = x_* = 1/a$ as $\xi \to \infty$. However, in comparison with the method applied in Example 7, in this case the rate of approximation of the parametric solution to the asymptote is less (which is not important for application of the standard numerical methods for solving similar problems). Note that the solution (39) coincides with (37) if we redenote ξ by 2ξ.

Remark 9. For applying non-local transformations is not necessary to compute the integrals (34) (or (17)).

5.3. Comparison of efficiency of various transformations for numerical integration of second-order blow-up ODE problems

In Table 4, a comparison of the efficiency of the numerical integration methods, based on various nonlocal transformations of the form (34) is presented by using the example of the test blow-up problem for the second-order ODE (32). The comparison is based on the number of grid points needed to make calculations with the same maximum error (approximately equal to 0.005). In all cases, for the integration of the transformed problems the standard fourth-order Runge–Kutta method was used.

$\text{Error}_{\max}, \% = 0.005$				
Transformation	Function g	Max. interval $\xi_{\max}$	Stepsize h	Grid points number N
Arc-length, Item 2°	$g{=}\sqrt{1{+}t^2{+}f^2}$	2500.0	0.200	12500
Nonlocal, Item 2°	$g{=}1{+}\lvert t\rvert{+}\lvert f\rvert$	2544.0	0.350	7268
Hodograph, Item 1°	$g{=}t$	49.0	0.125	392
Special exp-type, Item 5°	$g{=}f/t$	7.821	0.099	79
Special exp-type, Item 4°	$g{=}t/y$	3.9	0.060	65

Table 4: Various types of analytical transformations applied for numerical integration of the problem (32) for $f = 2y^3$ with a given accuracy (percent error $= 0.005$ and $\Lambda_{\mathrm{m}} \le 50$) and their basic parameters (maximum interval, stepsize, grid points number).

It can be seen that the arc-length transformation is the least effective, since the use of this transformation is associated with a large number of grid points (in particular, when using the last two transformations, you need about 160 and 190 times less of a number of grid points). The hodograph transformation has an intermediate (moderate) efficiency. The use of the exp-type transformations with $g = f/t$ and $g = t/y$ gives rather good results. This is due to the fact that for the last two transformations, the variable x tends exponentially rapidly to the singular point x_* for large ξ, while for the first three transformations, the variable x tends to the point x_* much slower (by the power law) for large ξ. We note that a similar situation holds for $\text{Error}_{\max}, \% = 0.1$.

6. Brief conclusions

We describe two new methods for numerical integration of non-linear Cauchy problems for ODEs of the first- and second-orders, which have blow-up solutions.

These methods are based on the differential and non-local transformations allowing us to obtain the equivalent systems of ODEs, whose solutions do not have singularities. It is shown that:

(i) the proposed method based on the general non-local transformation includes, as particular cases, the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation;

(ii) methods based on special exp-type and modified differential transformations are more efficient than the method based on the hodograph transformation, the method of the arc-length transformation, and the method based on the differential transformation.

It is important to note that the method described in Section 5.1 can be generalized to non-linear ODEs of arbitrary order and systems of coupled ODEs.

References

[1] M. Stuart and M. S. Floater. On the computation of blow-up, European J. Applied Math., 1990, Vol. 1, No. 1, pp. 47–71.

[2] E. A. Alshina, N. N. Kalitkin, and P.V. Koryakin. Diagnostics of singularities of exact solutions in computations with error control (in Russian), Zh. Vychisl. Mat. Mat. Fiz., 2005, Vol. 45, No. 10, pp. 1837–1847; http://eqworld.ipmnet.ru/ru/solutions/interesting/alshina2005.pdf

[3] G. Acosta, G. Durán, and J. D. Rossi. An adaptive time step procedure for a parabolic problem with blow-up. Computing, 2002, Vol. 68, pp. 343–373.

[4] S. Moriguti, C. Okuno, R. Suekane, M. Iri, and K. Takeuchi. Ikiteiru Suugaku — Suuri Kougaku no Hatten (in Japanese). Baifukan, Tokyo, 1979.

[5] C. Hirota and K. Ozawa. Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations — An application to the blow-up problems of partial differential equations, J. Comput. & Applied Math., 2006, Vol. 193, No. 2, pp. 614–637.

[6] R. Meyer-Spasche, Difference schemes of optimum degree of implicitness for a family of simple ODEs with blow-up solutions, *J. Comput. and Appl. Mathematics*, 1998, Vol. 97, pp. 137–152.

[7] N. R. Nassif, N. Makhoul-Karamb, and Y. Soukiassian, Computation of blowing-up solutions for second-order differential equations using re-scaling techniques, *J. Comput. Applied Math.,* 2009, Vol. 227, pp. 185–195.

[8] P. G. Dlamini and M. Khumalo. On the computation of blow-up solutions for semilinear ODEs and parabolic PDEs, Math. Problems in Eng., 2012, Vol. 2012, Article ID 162034, 15 p.

[9] A. Takayasu, K. Matsue, T. Sasaki, K. Tanaka, M. Mizuguchi, and S. Oishi. Numerical validation of blow-up solutions of ordinary differential equations. J. Comput. Applied Mathematics, 2017, Vol. 314, pp. 10–29.

[10] J. C. Butcher. The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods. Wiley-Interscience, New York, 1987.

[11] W. E. Schiesser. Computational Mathematics in Engineering and Applied Science: ODEs, DAEs, and PDEs. CRC Press, Boca Raton, 1994.

[12] U. M. Ascher and L. R. Petzold. Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations. SIAM, Philadelphia, 1998.

[13] G. A. Korn, T. M. Korn. Mathematical Handbook for Scientists and Engineers, 2nd Edition. New York: Dover Publ., 2000.

[14] I. K. Shingareva and C. Lizárraga-Celaya. Maple and Mathematica. A Problem Solving Approach for Mathematics, 2nd Edition. Springer, Wien–New York, 2009.

[15] D. Griffiths and D. J. Higham. Numerical Methods for Ordinary Differential Equations. Springer, Wien–New York, 2010.

[16] V. F. Zaitsev and A. D. Polyanin. Handbook of Nonlinear Differential Equations: Applications in Mechanics, Exact Solutions (in Russian), Nauka, Moscow, 1993.

[17] V. F. Zaitsev and A. D. Polyanin. Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics. CRC Press–Begell House, Boca Raton, 1994.

[18] A. D. Polyanin and V. F. Zaitsev. Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition. Chapman & Hall/CRC Press, Boca Raton–London, 2003.

[19] G. M. Murphy. Ordinary Differential Equations and Their Solutions. D. Van Nostrand, New York, 1960.

[20] E. Kamke. Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen. B.G. Teubner, Leipzig, 1977.

[21] N. A. Kudryashov. Analytical Theory of Nonlinear Differential Equations (in Russian). Institute of Computer Science, Moscow–Izhevsk, 2004.

[22] A. D. Polyanin and A. I. Zhurov. Parametrically defined nonlinear differential equations and their solutions: Applications in fluid dynamics. Applied Mathematics Letters, 2016, Vol. 55, pp. 72–80.

[23] A. D. Polyanin and A. I. Zhurov. Parametrically defined nonlinear differential equations, differential-algebraic equations, and implicit ODEs: Transformations, general solutions, and integration methods, Applied Mathematics Letters, 2017, Vol. 64, pp. 59–66.