Application of non-local transformations for numerical integration of singularly perturbed boundary-value problems with a small parameter*

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Abstract

Singularly perturbed boundary-value problems for second-order ODEs of the form $\varepsilon y''_{xx} = F(x, y, y'_x)$ with $\varepsilon \to 0$ are considered. We present a new method of numerical integration of such problems, based on introducing a new non-local independent variable ξ, which is related to the original variables x and y by the equation $\xi'_x = g(x, y, y'_x, \xi)$. With a suitable choice of the regularizing function g, this method leads to more appropriate problems that allow the application of standard numerical methods with fixed stepsize of ξ (in the whole range of variation of the independent variable x, including both the boundary-layer region and the outer region). It is shown that methods based on piecewise-uniform grids are a particular (degenerate) case of the method of non-local transformations with a piecewise-smooth regularizing function of special form. A number of linear and non-linear test problems with a small parameter (including convective heat and mass transfer type problems) that have exact or asymptotic solutions (both monotonic and non-monotonic), expressed in elementary functions, are presented. Comparison of numerical, exact, and asymptotic solutions showed the high efficiency of the

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method of non-local transformations for solving singularly perturbed problems with boundary layers. In addition to non-local transformations, examples of the use of point (local) transformations for numerical integration of singularly perturbed boundary-value problems are also given.

Keywords: singularly perturbed boundary-value problems, differential equations with a small parameter, boundary layers, non-local transformations, exact, asymptotic, and numerical solutions

1. Introduction

1.1. General remarks. Singularly perturbed boundary-value problems with a small parameter

Singularly perturbed boundary-value problems with a small parameter at the highest derivative are often encountered in hydro- and aerodynamics, theory of mass and heat transfer, theory of elasticity, non-linear mechanics and other applications (see, for example, [1–17]). The presence of small parameters in the equations can be due to both the characteristic values of physical-chemical parameters (for example, coefficients of viscosity, diffusion and thermal diffusivity), and the magnitude of dynamic characteristics of the phenomena or processes under consideration (for example, high fluid or gas velocities). An important qualitative feature of singularly perturbed boundary-value problems is that for the zero value of a small parameter the order of the differential equation under consideration decreases and some parts of the boundary conditions cannot be satisfied.

Solutions of singularly perturbed boundary-value problems with a small parameter have large gradients in the region of boundary layers, which leads to a loss of convergence of classical finite-difference schemes and makes them of little use or unsuitable for solving problems of this type. For the first time, the question about the inadmissibility of classical finite-difference schemes and the construction of special schemes (possessing the property of convergence regardless of the value of a small parameter), was raised in the works by Bakhvalov and Il'in [18, 19], who proposed two different approaches to solving linear and non-linear boundary-value problems with a boundary layer. In [19], an exponential fitting scheme was proposed, the coefficients of which are chosen so that the scheme is asymptotically exact on the boundary-layer component of the solution (this approach allows us to construct uniformly convergent finite-difference schemes on a uniform grid). The classical central-difference scheme with a grid, thickening near boundary layers, which has a uniform error in the approximation over the nodes, was applied in [18]. It is shown that such a grid possesses uniformly second-order accuracy with respect to a small parameter.

Various problems and methods of the numerical integration for linear and non-linear differential equations with a small parameter at the highest derivative are presented, for example, in [20–50]. Despite the large number of publications in this field, numerous applications dictate the need for development of new numerical methods with a wide range of applicability and finite-difference schemes of high order accuracy for singularly perturbed problems, especially non-linear ones.

We note that in [51] there is a very interesting collection of examples of exotic numerical solutions of boundary-value problems with a boundary layer, which are obtained on the basis of the use of inadequate numerical algorithms and schemes.

1.2. Numerical methods based on a piecewise-uniform grid (two-grid methods)

For the numerical solution of singularly perturbed boundary-value problems described by quasilinear equations with the Dirichlet boundary conditions,

$$\varepsilon y''_{xx} + p(x)y'_x + q(x,y) = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \tag{1}$$

where $\varepsilon > 0$ is a small parameter, many authors use methods with a piecewise-uniform grid that is characterized by a small stepsize in the boundary layer and a large stepsize outside it (see, for example, [22, 26, 32, 33, 35, 36, 41, 42, 45, 46]). The general idea of these methods is that the domain of the independent variable $x \in [0,1]$ is divided into two subdomains $\Omega_\mathrm{i} = [0,l]$ and $\Omega_\mathrm{e} = [l,1]$. If the boundary layer is located on the left (near the point $x = 0$), then the partition point l is chosen according to the condition $\varepsilon \ll l \ll 1$. In the outer domain Ω_e, a common fixed stepsize $h_\mathrm{e} = h$ is taken, and in the inner domain Ω_i, an essentially smaller, but fixed stepsize h_i is chosen.

Let N be the total number of intervals of the grid partition on the interval $[0,1]$, and $N_\mathrm{i} = \alpha N$ and $N_\mathrm{e} = (1-\alpha)N$ be the number of intervals of the grid partition on the segments $[0,l]$ and $[l,1]$. Then the stepsizes in these segments are

$$h_\mathrm{i} = \frac{l}{\alpha N}, \quad h_\mathrm{e} = \frac{1-l}{(1-\alpha)N} \qquad (N = N_\mathrm{i} + N_\mathrm{e},\ \ 0 < \alpha < 1). \tag{2}$$

The number of intervals N is initially set by the researcher, and then, based on additional non-strict considerations, it is necessary to choose two suitable values of the coefficients l and α. It is important to note that l and α depend on a specific class of equations under consideration and, in general, are semi-empirical constants (the optimal values of which are a priori unknown). The suitability of the selected values of l and α is usually established by comparing the numerical and exact solutions of several test problems.

For the numerical solution of singularly perturbed boundary-value problems, described by quasilinear equations of the form (1) with $p(x) > 0$, Shishkin [22] (see also [26, 32, 35, 42]) suggested to use a piecewise-uniform grid (2) with the parameters

$$\alpha = \frac{1}{2}, \quad l = \min\left(\frac{1}{2}, \frac{2\varepsilon}{p_{\rm m}} \ln N\right), \tag{3}$$

where $p_{\rm m} = \min\limits_{0\le x\le 1} p(x)$. The grid (2) with (3) usually leads to good results for this class of quasilinear equations. We note that for a sufficiently small ε we can also use a simpler grid, replacing $p_{\rm m}$ by $p_0 = p(0)$ in (3).

The assumption used in (3) on the equality of the number of partition intervals in the inner and outer regions $\alpha = 1/2$, in general, is not justified by anything. In [33], on the example of numerical integration of a specific linear boundary-value problem with a piecewise-uniform grid, the optimal (providing the greatest accuracy) values of l and α were determined, based on the use of an asymptotic solution (for some values of the determining parameters of the problem under consideration, the value $\alpha = 0.9$ was found to be optimal).

It is important to note that in the methods based on the use of a piecewise-uniform grid (as well as the methods developed in [18, 19]), a priori information on the structure and the rate of damping of the asymptotic solution is taken into account. For example, the grid (3) describes the characteristic sizes of the boundary layer qualitatively incorrectly when $p(x) \simeq p_0 x^n$ as $x \to 0$ and $n > 0$ (that is connected with another structure of the asymptotic solution, see further Section 6.2) or when $p(x)$ vanishes within the interval $0 < x < 1$. Certain difficulties with the choice of suitable values of l and α also occur for more complex nonlinear equations with a small parameter at the highest derivative

$$\varepsilon y''_{xx} = F(x, y, y'_x). \tag{4}$$

In this paper, for the numerical solution of singularly perturbed boundary-value problems, described by equations of the form (4) with the Dirichlet boundary conditions, it is proposed to use more general approach based on non-local transformations, which are applied at the first stage and allow further integration of the reduced problem by standard numerical methods with a uniform grid. It will be shown that methods based on the use of a piecewise-uniform grid are a particular (degenerate) case of the method based on non-local transformations. A comparison carried out between numerical, exact, and asymptotic solutions of a number of linear and non-linear test problems showed the high efficiency of the method of non-local transformations for solving singularly perturbed problems with boundary layers.

Remark 1. The method of non-local transformations was used in [52–56] for numerical integration of Cauchy problems having monotonic and non-monotonic blow-up solutions. In such problems there exists a singular point whose position is unknown a priori (for this reason, standard numerical methods for solving blow-up problems can lead to significant errors). The use of the method of non-local transformations leads to problems whose solutions are represented in parametric form and do not have blowing-up singular points; therefore, the transformed problems admit the use of standard fixed-step numerical methods. A comparison of the exact and numerical solutions of a number of test problems for differential equations of the first, second, third, and fourth orders showed the high efficiency of this method for numerical integration of blow-up problems.

Remark 2. Non-local transformations of a special kind were used in [57–60] to obtain exact solutions, first integrals, and linearizations of some second-order ordinary differential equations. The method for reducing the order of ODEs by means of non-local symmetries is described in [61, 62].

2. Qualitative features of boundary-layer type problems

2.1. Illustrating example of a linear boundary-value problem

Let us analyze qualitative features of singularly perturbed boundary-value problems with a small parameter at the highest derivative, that have solutions of the boundary-layer type, on an example of a specific problem.

Test problem 1. Consider the test boundary-value problem for a linear second-order equation with constant coefficients and Dirichlet boundary conditions

$$\varepsilon y''_{xx} + p y'_x + q y = 0 \quad (0 < x < 1); \tag{5}$$

$$y(0) = a, \quad y(1) = b, \tag{6}$$

where a, b, p, q, and ε are free parameters that satisfy the condition $p^2 - 4\varepsilon q > 0$.

The exact solution of the problem (5)–(6) has the form

$$\begin{gathered} y = \frac{a e^{\lambda_2} - b}{e^{\lambda_2} - e^{\lambda_1}} e^{\lambda_1 x} + \frac{b - a e^{\lambda_1}}{e^{\lambda_2} - e^{\lambda_1}} e^{\lambda_2 x}, \\ \lambda_1 = \frac{1}{2\varepsilon}\bigl(-p - \sqrt{p^2 - 4\varepsilon q}\bigr), \quad \lambda_2 = \frac{1}{2\varepsilon}\bigl(-p + \sqrt{p^2 - 4\varepsilon q}\bigr). \end{gathered} \tag{7}$$

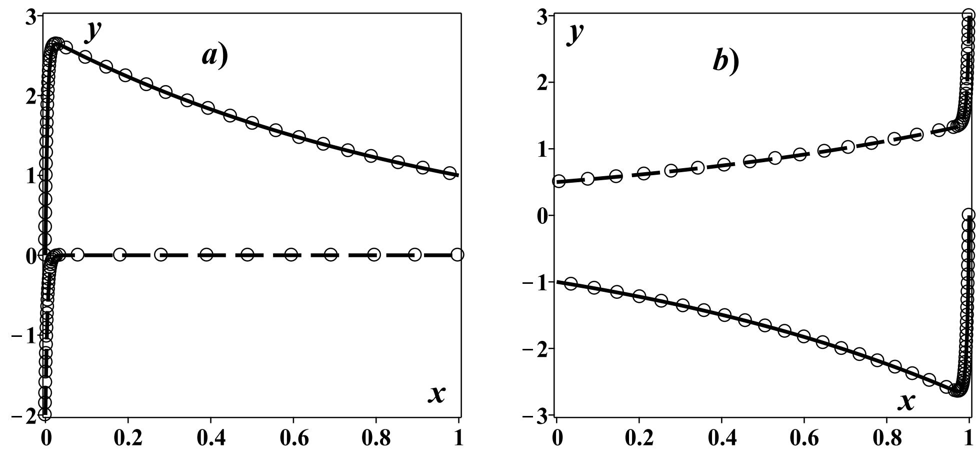

Figure 1: Exact solutions (7) of the problem (5)–(6) with $\varepsilon = 0.005$ for four sets of numerical values of the determining parameters: *a*) $p = q = 1$, $a = 0$, $b = 1$ (solid line) and $p = q = 1$, $a = -2$, $b = 0$ (dashed line) and *b*) $p = -1$, $q = 1$, $a = -1$, $b = 0$ (solid line) and $p = -1$, $q = 1$, $a = 0.5$, $b = 3$ (dashed line). The dots denote the corresponding numerical solutions of the transformed problem (20) for $g = (1 + |z| + |f|)^{1/2}$ and for the same sets of numerical values of the parameters.

Let $0 < \varepsilon \ll 1$ be a small parameter. For $p > 0$, the boundary layer is formed at the left boundary near the point $x = 0$, and for $p < 0$, at the right boundary near the point $x = 1$. For $p = 0$ and $q < 0$, we have two boundary layers that are located near the boundaries $x = 0$ and $x = 1$.

In Fig. 1, the exact solutions (7) of the problem (5)–(6) are shown for $\varepsilon = 0.005$ and four sets of numerical values of the determining parameters: *a*) $p = q = 1$, $a = 0$, $b = 1$ and $p = q = 1$, $a = -2$, $b = 0$ (the boundary layer is on the left) and *b*) $p = -1$, $q = 1$, $a = -1$, $b = 0$ and $p = -1$, $q = 1$, $a = 0.5$, $b = 3$ (the boundary layer is on the right). It can be seen that, depending on the values of the determining parameters, the solutions of the problem (5)–(6) can be either monotonic or non-monotonic.

For small ε (the constants $p > 0$ and q are of the order of unity), we have

$$\lambda_1 \simeq -\frac{p}{\varepsilon}, \quad \lambda_2 \simeq -\frac{q}{p}, \quad y'_x(0) \simeq \frac{p}{\varepsilon}(be^{q/p} - a), \tag{8}$$

and the corresponding asymptotic solution of the problem (5)–(6) is written as follows:

$$y_a = (a - be^{q/p})e^{-px/\varepsilon} + be^{(q/p)(1-x)}. \tag{9}$$

For concreteness, we further assume that $p > 0$, $q > 0$, $a \geq 0$, and $b \geq 0$. If $a > be^{q/p}$, the function (9) decreases monotonically. If $a < be^{q/p}$, the function (9) increases monotonically (and very quickly) in a narrow region $0 \leq x < x_*$, where

$$x_* \simeq \frac{\varepsilon}{p} \ln\left[\frac{p^2}{\varepsilon q}\left(1 - \frac{a}{b} e^{-q/p}\right)\right], \tag{10}$$

and in the remaining region $x_* \leq x \leq 1$ the solution decreases monotonically and changes slowly enough. Substituting (10) into (9), we find the maximum value of the required value $y_* = y(x_*) \simeq be^{q/p}$.

For $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, the asymptotic solution (9) of the problem (5)–(6) takes the form $y_a = e^{1-x} - e^{1-200x}$. In the narrow range $0 \leq x \leq 0.02649$ the function $y_a = y_a(x)$ increases rapidly, and for $0.02649 \leq x \leq 1$, it slow decreases. The maximum difference between the asymptotic solution and the exact solution (7) (in the interval $0 \leq x \leq 1$) in this case is 0.0127.

When applying direct numerical methods for solving problems of this type, to take into account the singularities of the solution in the boundary-layer region, it is necessary to take sufficiently many points in a small neighborhood of the left boundary. Therefore the use of uniform grids throughout the domain of variation of the independent variable is connected with the necessity of partitioning the segment $0 \leq x \leq 1$ into a large number of intervals of integration. Standard fixed-step programs require to involve unnecessarily many points $N = O(1/\varepsilon)$ for calculations [33].

We note that the derivatives of the solution (7) on the left boundary (in the boundary-layer region) are very large. In particular, for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, we have

$$(y_a)'_x|_{x=0} \simeq e\varepsilon^{-1} = 543.656, \qquad (y_a)''_{xx}|_{x=0} \simeq -e\varepsilon^{-2}.$$

Therefore, when using direct numerical methods for solving similar problems with boundary layers, the shooting procedure should begin with large values of the derivative (of order ε^{-1}), which is a complicating factor.

2.2. The characteristic sizes of the boundary layer

In applications, the local thickness of the boundary layer is sometimes introduced using the relation $\delta(x) = 1/|y'_x|$ (see, for example, [14–16]). For the linear boundary-value problem (5)–(6) with a small parameter, from formula (7) it follows that $\delta = O(\varepsilon)$ in the boundary-layer region.

It is convenient to determine the total length (thickness) of the entire boundary layer $\delta_{\rm t}$ according to the relation:

$$\left|1 - \frac{y_\varepsilon(\delta_{\rm t})}{y_0(\delta_{\rm t})}\right| = \sigma, \tag{11}$$

where $y_\varepsilon(x) = y(x,\varepsilon)$ is the solution of the problem under consideration, and $y_0(x) = y(x,0)$ is the solution of the degenerate problem for $\varepsilon = 0$ (which corresponds to an asymptotic solution in the outer region and satisfies one boundary condition), σ is a small quantity that is specified by the researcher. When $\sigma = 0.01$ the equation (11) means that on a certain boundary at the point $x = \delta_{\rm t}$ the solution in the boundary layer differs from the asymptotic solution in the outer region by 1%. In problems with a small parameter ε, it is also reasonable to take $\sigma = \varepsilon$ in the relation (11).

Example 1. In the problem (5)–(6) for $a = 0$ and $\varepsilon \ll 1$, from the formulas (7) and (11) we obtain

$$\delta_{\rm t} = \varepsilon\frac{|\ln\sigma|}{p}. \tag{12}$$

In the particular case when $p = q = 1$, $\varepsilon = 0.005$, the total length of the boundary layer, according to the formula (12) for $\sigma = 0.01$, is approximately equal to $\delta_{\rm t} \approx 0.023$.

Setting $\sigma = \varepsilon$ in (11), we obtain the formula $\delta_{\rm t} = \varepsilon|\ln\varepsilon|/p$ from (12), which is asymptotically equivalent to the right-hand side of the relation (10). For $p = q = 1$, $\varepsilon = 0.005$, this formula gives the length of the boundary layer $\delta_{\rm t} \approx 0.026$.

2.3. *The order relation between the derivatives*

We now derive one useful relation (which is a rather general) between the derivatives in the boundary layer. Without loss of generality, we consider the boundary layer on the left boundary.

We assume that inside the boundary layer, the principal term of the asymptotic expansion of the solution as $\varepsilon \to 0$ has the form

$$y_{\rm i} = \varphi\left(\frac{x}{\delta}\right), \tag{13}$$

where $\varphi = \varphi(z)$ is a smooth function having bounded and non-vanishing derivatives in some neighborhood of the point $z = 0$, and $\delta = \delta(\varepsilon)$ is a function having the property $\delta \to 0$ as $\varepsilon \to 0$. In the particular case of the linear problem (5)–(6), it is necessary to set $\varphi(z) = (a - be^{q/p})e^{-z} + be^{q/p}$ and $\delta = \varepsilon/p$ in the formula (13). By differentiating twice the formula (13) with respect to x, we find the derivatives

$$y'_x = \frac{1}{\delta}\varphi'_z(z), \qquad y''_{xx} = \frac{1}{\delta^2}\varphi''_{zz}(z). \tag{14}$$

Eliminating δ from this, we obtain the order relation

$$|y''_{xx}| = O(|y'_x|^2), \tag{15}$$

which we will need later.

3. Method of non-local transformations. Numerical solution of boundary-value problems

3.1. General description of the method of non-local transformations

We consider two-point problems for second-order differential equations with boundary conditions of the first kind, which in dimensionless variables have the form

$$y''_{xx} = f(x, y, y'_x) \quad (0 < x < 1); \tag{16}$$

$$y(0) = a, \qquad y(1) = b, \tag{17}$$

where the function f can also depend on the small parameter $\varepsilon > 0$.

We introduce a new non-local independent variable ξ by means of the first-order differential equation and the initial condition:

$$\xi'_x = g(x, y, y'_x, \xi), \qquad \xi(0) = 0. \tag{18}$$

Here $g = g(x, y, y'_x, \xi)$ is a regularizing function that can vary.

We represent the second-order equation (16) in the form of an equivalent system two equations of the first order

$$y'_x = z, \qquad z'_x = f(x, y, z). \tag{19}$$

Using (18), we pass from x to the new independent variable ξ in (19) and (17). As a result, the boundary-value problem (16)–(17) is transformed to the following problem for the system of three equations:

$$\begin{gathered} x'_\xi = \frac{1}{g(x, y, z, \xi)}, \quad y'_\xi = \frac{z}{g(x, y, z, \xi)}, \quad z'_\xi = \frac{f(x, y, z)}{g(x, y, z, \xi)} \quad (0 < \xi < \xi_1); \\ x(0) = 0, \qquad y(0) = a, \qquad y(\xi_1) = b, \end{gathered} \tag{20}$$

where the value ξ_1 is determined in the process of calculations according to the condition $x(\xi_1) = 1$.

If $g \equiv 1$, the first equation of the system (20), taking into account the initial condition $x(0) = 0$, gives $\xi = x$ and numerical integration of the remaining two equations is equivalent to the integration of the original problem (16)–(17). The successfully chosen regularizing function $g = g(x, y, z, \xi)$ itself will determine the location and density of integration points with respect to the original variables x and y and will allow to solve the problem (20) more accurately (for a given number of grid points) by applying standard fixed-step numerical methods with respect to ξ [63–71].

Remark 3. In the particular case, when $g = g(x, y, z)$, the new variable ξ, by integrating the equation (18), can be expressed in terms of the solution of the problem (16)–(17) in the form of an integral

$$\xi = \int_0^x g(x, y, z)\,dx, \qquad y = y(x), \qquad z = z(x). \tag{21}$$

Dependence (21) is non-local, since the variable ξ is expressed in terms of the integral of a function that depends on the initial variables x and y (we recall that local or point transformations are much simpler and have the form $\xi = g(x, y)$; some particular cases of such transformations are considered in Sections 3.3 and 4.4).

3.2. Conditions to be satisfied by regularizing functions. Examples of regularizing functions

For numerical solution of boundary-value problems, as well as for solving Cauchy problems, it is reasonable to use regularizing functions of the form [52–54]

$$g = G(|z|, |f|) \equiv G(|y'_x|, |y''_{xx}|), \tag{22}$$

where $f = f(x, y, z)$ is the right-hand side of the equation (16) and $z = y'_x$. The following conditions are imposed on the function $G = G(u, v)$:

$$G > 0; \quad G_u \geq 0, \quad G_v \geq 0; \quad G \to \infty \text{ as } u + v \to \infty; \quad G(0, 0) = 1, \tag{23}$$

where $u \geq 0$, $v \geq 0$. The last relation in (23) is the normalization condition.

For singularly perturbed boundary-value problems (16)–(17) with a small parameter at the highest derivative, for which the right-hand side of the equation (16) has the form $f = \varepsilon^{-1} F(x, y, z, \varepsilon)$, where $F(x, y, z, 0)$ is a smooth function that does not have singularities, when choosing regularizing functions, in addition to the conditions (22)–(23), some other considerations should be taken into account.

For $g = 1$ (in this particular case, non-local transformations are not applied) and $\varepsilon \to 0$ in the boundary-layer region, the right-hand sides of the last two equations of the system (20) will tend to infinity since $|z| \to \infty$ and $|f| \to \infty$; moreover, the order relation $|f| \sim z^2$ is valid, which follows from the formula (15). This circumstance considerably complicates numerical integration of the problem under consideration and leads to the need to proportionally refine the grid spacing as ε decreases.

It is possible to avoid refining the grid as $\varepsilon \to 0$ and to work with a fixed stepsize with respect to the non-local variable ξ by using regularizing functions satisfying the condition

$$|z|/g = O(1) \quad \text{as} \quad \varepsilon \to 0 \tag{24}$$

(in this case, the right-hand side of the second equation of the system (20) will not have singularities for small ε, and the third equation of this system in the boundary-layer region will have a substantially smaller singularity than for $g = 1$). In particular, we can choose regularizing functions having the asymptotics

$$g \to m_1|z| \qquad \text{as} \quad |z| \to \infty; \tag{25}$$

$$g \to m_2|f|^{1/2} \quad \text{as} \quad |f| \to \infty, \tag{26}$$

where m_1 and m_2 are positive constants of the order of unity.

Two-parameter regularizing functions of binomial form satisfy, for example, the asymptotic conditions (25) and (26)

$$g = (1 + k|z|^s)^{1/s}; \tag{27}$$

$$g = (1 + k|f|^{s/2})^{1/s}, \tag{28}$$

which in addition to the asymptotic conditions (25) and (26) also satisfy the normalization condition (see the last condition in (23)). In the formulas (27) and (28) there are positive constants k and s, which can vary.

We consider in more detail the regularizing function (27) for $k = 1$. In this case, the right-hand side of the second equation of the system (20) varies slightly and will be bounded for any values of the derivative (and, respectively, as $\varepsilon \to 0$), where $|y'_\xi| \leq 1$. In the boundary-layer region, where the derivative z is large, we have $|y'_\xi| \approx 1$, which leads to a linear dependence $y \approx \pm\xi + \text{const}$. The right-hand side of the third equation of the system (20) in the boundary-layer region, taking into account the relation $|f| \sim z^2$, becomes linear with respect to z; the right-hand side of the first equation of the system (20) in the boundary-layer region is small, which corresponds to small changes in x for finite changes in ξ, which corresponds to the essence of the matter. Outside the boundary layer, the system (20) does not have qualitative features. Thus, the use of regularizing functions (27) and (28) allows us to completely suppress the unbounded growth of the right-hand side of the second equation of the system (20) as $\varepsilon \to 0$ and to reduce (in comparison with $g = 1$) the right-hand side of the third equation.

For non-monotonic solutions having points at which the derivatives $y'_x = z$ or $y''_{xx} = f$ vanish, the regularizing functions (27) and (28) are not very effective. In such cases, more effective and sufficiently universal are, for example, the regularizing functions of the form

$$g = (k_1 + k_2|z| + k_3|f|)^{1/2}; \tag{29}$$

$$g = (k_1 + k_2 z^2 + k_3|f|)^{1/2}, \tag{30}$$

which also satisfy the condition (24). The formulas (29) and (30) include the three positive constants k_1, k_2, and k_3, which can vary.

Remark 4. A sufficiently tough condition (24) can be weakened by replacing it by

$$|z|/g = O(\varepsilon^{-\sigma}) \quad \text{as} \quad \varepsilon \to 0, \tag{31}$$

where $0 \le \sigma < 1$. Since in the standard boundary layer $|z| = O(\varepsilon^{-1})$, then under the condition (31), the right-hand side of the second equation of the system (20) will have a weaker singularity for small ε than the original equation. For example, for $g = (1+|f|)^{1/3}$ we have $\sigma = -1/3$, which substantially reduces the order of the singularity. Regularizing functions, satisfying the condition (31) for $0 < \sigma < 1$, can work well for moderately small values of ε and usually lead to significant errors for sufficiently small $\varepsilon < \varepsilon_{\text{min}}(\sigma)$.

Remark 5. The local conditions (24) and (31), which are superimposed on the regularizing functions in problems with a boundary layer, differ qualitatively from the more complex integral (non-local) conditions that are used in Cauchy problems with blow-up solutions [53].

Remark 6. If the right-hand side of the equation (16) does not depend on x (that is, the equation is autonomous) and the regularizing function is chosen in the form (22), then the second and the third equations of the system (20) form an independent subsystem that is integrated independently of the first equation.

3.3. Degenerate (local) transformations. Two-grid method

The regularizing functions of special form, $g = g(x, \xi)$, that do not depend on the unknown function and its derivative, define degenerate transformations (18), for which the first equation of the system (20) is an isolated equation and can be integrated independently of the other two equations. These transformations are equivalent to the point (local) transformations of special form $\xi = \xi(x)$.

Numerical integration of linear boundary-value problems based on a point transformation is considered in Section 4.4. Further in Section 7.1, we also describe the procedure of numerical integration of a boundary-value problem with two boundary layers based on a combination of point and non-local transformations.

Methods, based on the use of a piecewise-uniform grid of the form (2), are particular degenerate cases of the method of non-local transformations with a piecewise-smooth regularizing function

$$g = \begin{cases} \dfrac{\alpha h N}{l} & \text{if } 0 \le \xi \le \alpha h N, \\ \dfrac{(1-\alpha) h N}{1-l} & \text{if } \alpha h N \le \xi \le h N \end{cases}$$
$$= \begin{cases} \dfrac{\alpha(1-l)}{(1-\alpha) l} & \text{if } 0 \le \xi \le \dfrac{\alpha(1-l)}{1-\alpha}, \\ 1 & \text{if } \dfrac{\alpha(1-l)}{1-\alpha} < \xi \le \dfrac{1-l}{1-\alpha}, \end{cases}$$

where $h = h_{\mathrm{e}}$ is the stepsize with respect to the variable ξ.

3.4. Procedure of numerical integration of the transformed problem (20)

In this article, we apply a combination of the fourth-order Runge–Kutta method with a fixed stepsize and a specific shooting procedure with Maple implementation [56, 70, 72, 73]. To solve the transformed nonlinear boundary-value problem for the system of three equations (20) with different regularizing functions $g = g(x, y, z, \xi)$, we apply the shooting method, that is an iterative numerical method, the basic idea of which is to represent the original nonlinear boundary-value problem in a new form, i.e. as a related initial value problem with initial conditions at one endpoint, e.g.,

$$x(0) = 0, \quad y(0) = a, \quad z(0) = s, \tag{32}$$

which include a parameter s that has to be determined from the boundary condition at the other endpoint. At this stage, we apply the fourth-order Runge–Kutta method with a fixed stepsize. If this boundary condition is not satisfied to the desired accuracy, the process is repeated with other initial conditions until the desired accuracy is achieved.

We describe in more detail how the length of the transformed integration interval ξ_1 is determined. At first, for a given value of the parameter s, the solution of the ODE system (20) with the initial conditions (32) is determined numerically. To emphasize the dependence of this solution on s, for clarity, we denote it by $x = X(\xi, s)$, $y = Y(\xi, s)$, $z = Z(\xi, s)$. The computation process stops when ξ reaches the value $\xi_1 = \xi_1(s)$, which satisfies the condition $X(\xi_1, s) = 1$ with the given accuracy. Then we find the value $s = s_*$ for which the last condition in (20) is satisfied. For this, we solve the nonlinear equation $\mathcal{F}(s) \equiv Y(\xi_1, s) - b = 0$ by one of the root-finding numerical methods, for example, the Newton-Raphson method. At the final stage, using the found value of the parameter $s = s_*$, we apply the fourth-order Runge–Kutta method with a fixed step to obtain a numerical solution of the problem under consideration (20).

The number of iterations which we need to achieve an acceptable accuracy will depend only on the refinement algorithm for the parameter s (in our case, the Newton–Raphson method). Since the dimension of the space is low, the computational cost of the described procedure is also low.

Note that for the method of solving singularly perturbed boundary-value problems described above, the number of grid points $N = \xi_1/h$, where h is the stepsize in the variable ξ, is an auxiliary (not the most important) calculation parameter. The number N is useful for comparing the effectiveness of various regularizing functions.

3.5. How to specify the initial condition for the derivative in the shooting method

For numerical solution of the problem (20) by the shooting method (from the left), it is required to specify the initial condition for the derivative with respect to the non-local variable $y'_\xi(0) = y_0$. In what follows, we confine ourselves to an analysis of problems for which the boundary layer is located on the left (in the neighborhood of the point $x = 0$).

When choosing regularizing functions in the form (27), taking into account that the initial derivative $y'_x = z$ at the point $x = 0$ is large, from the second equation of the system (20) we have

$$y'_\xi(0) = (z/g)|_{x=0} \simeq k^{-1/s}\operatorname{sign} z.$$

Therefore, in particular, for $k = 1$, the modulus of the initial shooting value $y_0 = y'_\xi(0)$ it is necessary to choose approximately equal to unit, $|y_0| \simeq 1$ (but slightly less than 1).

When choosing the regularizing functions in the form (28), (29), or (30) with $k = O(1)$ and $k_n = O(1)$ ($n = 1,\, 2,\, 3$), for the shooting value of the derivative, in the general case, one can obtain only the order estimate $|y_0| = O(1)$ from the relation (15).

4. Numerical integration of linear boundary-value test problems. Comparison of the efficiency of various regularizing functions

4.1. Test problem for a linear homogeneous equation. Numerical integration based on non-local transformations

In Fig. 1, the results of numerical solutions of the transformed problem (20) (used for solving the original problem (5)–(6)) for $f = -\varepsilon^{-1}(pz + qy)$ and $\varepsilon = 0.005$ with the regularizing function $g = (1 + |z| + |f|)^{1/2}$, which are obtained by the shooting method with the fixed stepsize $h = 0.01$ by using Maple, are shown by circles for four sets of numerical values of the determining parameters: *a*) $p = q = 1$, $a = 0$, $b = 1$ and $p = q = 1$, $a = -2$, $b = 0$ and *b*) $p = -1$, $q = 1$, $a = -1$, $b = 0$ and $p = -1$, $q = 1$, $a = 0.5$, $b = 3$. For the first two sets of the numerical values, the shooting procedure is carried out from the left boundary (from the point $x = 0$), and for the last two sets, from the right boundary (from the point $x = 1$). It can be seen that there is a good coincidence between the numerical solutions and the corresponding exact solutions, which are determined by the formula (7) and are represented by solid and dashed lines.

Moreover, the numerical solution, $x = x(\xi)$ and $y = y(\xi)$, of this problem obtained by the method of non-local transformations for $g = (1 + |z| + |f|)^{1/2}$, $\varepsilon = 0.005$, and $p = q = 1$, $a = 0$, $b = 1$, is shown, respectively, by solid line and dashed line, in Fig. 2 *a*) on the ξ-interval. The exact solution (7) of this problem and numerical solution of the transformed problem (20) is shown, respectively, by solid line and circles, in Fig. 2 *b*) on the x-interval for the same values of the parameters. Also in Fig. 2 *c*) and *d*), we present the behavior of the absolute error E_a of the numerical solution of the transformed problem (20) for $g = (1 + |z| + |f|)^{1/2}$ and the same values of the parameters. The maximum absolute error in this case is equal to 5.5355809×10^{-7} and is located in the transition area between the boundary layer and the outer region (near the extremum of the function y).

Table 1 shows the maximum absolute errors of the non-monotonic numerical solutions[1] of the transformed problem (20), used for numerical integration of the

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[1] The absolute error of the numerical solution $y_{\rm n} = y_{\rm n}(x)$ of some problem on the interval $0 \le x \le 1$ is determined by the formula $E_{\rm a} = \max\limits_{0\le x\le 1} |y_{\rm n} - y_{\rm e}|$, where $y_{\rm e} = y_{\rm e}(x)$ is the exact solution of the same problem.

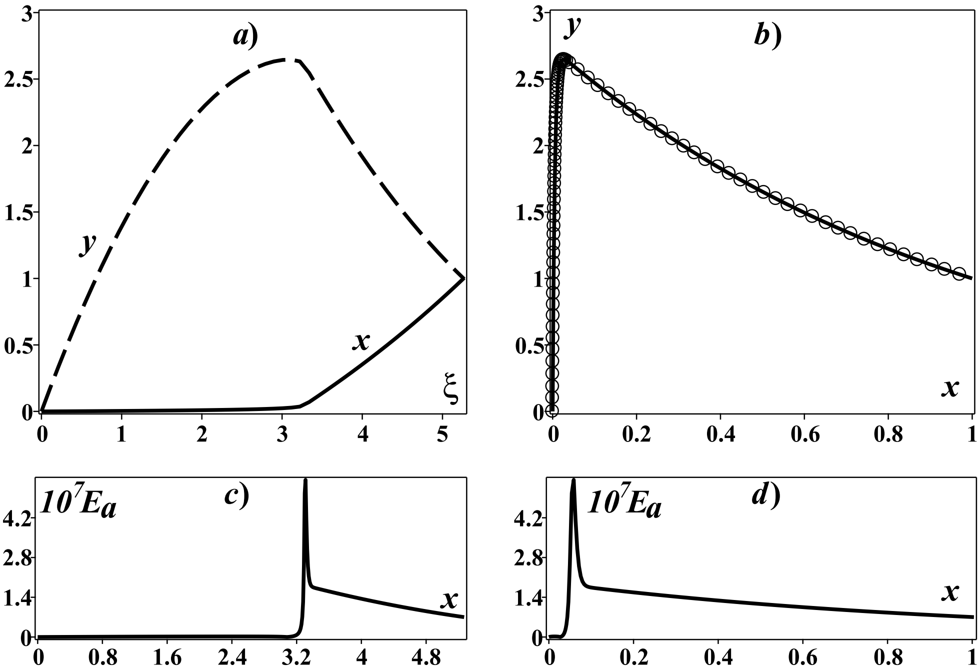

Figure 2: a) Numerical solution of the problem (5)–(6) with $\varepsilon = 0.005$ obtained by the method of non-local transformations (transformed problem (20)) with the regularizing function $g = (1 + |z| + |f|)^{1/2}$ for $p = q = 1$, $a = 0$, $b = 1$: $x = x(\xi)$ (solid line) and $y = y(\xi)$ (dashed line); *b*) Exact solution (7) (solid line) of the problem (5)–(6) with $\varepsilon = 0.005$ for $p = q = 1$, $a = 0$, $b = 1$ and numerical solution (circles) of the transformed problem (20) for $g = (1 + |z| + |f|)^{1/2}$, $\varepsilon = 0.005$, and $p = q = 1$, $a = 0$, $b = 1$; *c*) and *d*) Absolute errors E_a of numerical solutions of the transformed problem (20) for $g = (1 + |z| + |f|)^{1/2}$ and the same values of the parameters ($p = q = 1$, $a = 0$, $b = 1$), respectively, with respect to ξ and x.

The maximum absolute error of the numerical solutions of problem (5)–(6)				
No.	Regularizing function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g = 1 + \|z\|$	0.047029578	0.013710597	0.000713696
2	$g = (1 + z^2)^{1/2}$	0.072753172	0.021452406	0.002093477
3	$g = (1 + \|f\|)^{1/2}$	0.000824707	0.000249922	0.000001663
4	$g = (1 + f^2)^{1/4}$	0.001036473	0.000258455	0.000005012
5	$g = (1 + \|f\|)^{1/3}$	0.008648850	0.001222802	0.000003785
6	$g = (1 + \|f\|)^{1/2}/(1 + \ln(1 + \|f\|^{1/2}))$	0.021994979	0.002848291	0.000026123
7	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000570299	0.000115649	0.000000554
8	$g = (1 + \|z\| + \|f\|)^{1/3}$	0.008780481	0.001073051	0.000002329
9	$g = (1 + z^2 + f^2)^{1/2}$	0.003531054	0.000475305	0.000000102
10	$g = (1 + z^2 + f^2)^{1/4}$	0.000900140	0.000213801	0.000001389
11	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000559160	0.000109360	0.000000180
12	$g = 1$	process diverges	process diverges	0.528189578

Table 1: Comparison of the efficiency of various regularizing functions for the transformed problem (20) used for numerical solution of the original problem (5)–(6) by the method of non-local transformations for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, for three stepsizes.

original problem (5)–(6) by the method of non-local transformations for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$ for three stepsizes h and eleven different regularizing functions g. For comparison, similar data are also indicated for the case $g = 1$, which corresponds to the direct numerical solution (without using transformations) with the same stepsize with respect to x. It can be seen that four functions (Nos. 3, 7, 10, 11) make it possible to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to ξ) equal to $h = 0.1$.

Table 2 shows the length of the interval ξ_1 ($0 \le \xi \le \xi_1$) in the transformed problem (20), used for numerical integration of the original problem (5)–(6) by the method of non-local transformations for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$ for three stepsizes $h = 0.1, 0.05, 0.01$ and different regularizing functions g. It can be seen that the use of the function No. 9 leads to the maximum region of variation of the non-local variable ($0 \le \xi \le 547.9$ for $h = 0.1$ and $0 \le \xi \le 545.98$ for $h = 0.01$) and a large number of grid points. We note that the range of variation of the non-local variable ξ for other regularizing functions g in Table 2 for the same stepsizes ($h = 0.1$ and $h = 0.01$) varies considerably less (from $0 \le \xi \le 1.43$ to $0 \le \xi \le 10.6$).

The length of the interval ξ_1 in the transformed problem (20) obtained from the problem (5)–(6)				
No.	Regularizing function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g = 1 + \|z\|$	6.90	5.65	5.30
2	$g = (1 + z^2)^{1/2}$	6.40	5.25	4.58
3	$g = (1 + \|f\|)^{1/2}$	9.50	6.35	4.83
4	$g = (1 + f^2)^{1/4}$	9.50	6.35	4.61
5	$g = (1 + \|f\|)^{1/3}$	7.00	3.85	2.01
6	$g = (1 + \|f\|)^{1/2}/(1 + \ln(1 + \|f\|^{1/2}))$	6.90	3.75	1.43
7	$g = (1 + \|z\| + \|f\|)^{1/2}$	9.50	6.35	5.27
8	$g = (1 + \|z\| + \|f\|)^{1/3}$	7.00	3.85	2.24
9	$g = (1 + z^2 + f^2)^{1/2}$	547.90	546.30	545.98
10	$g = (1 + z^2 + f^2)^{1/4}$	9.50	6.35	4.80
11	$g = (1 + z^2 + \|f\|)^{1/2}$	10.60	7.50	6.67

Table 2: The length of the interval of variation of the non-local variable ξ_1 ($0 \le \xi \le \xi_1$) in the transformed problem (20) used for numerical solution of the original problem (5)–(6) for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, for various regularizing functions and three stepsizes $h = 0.1, 0.05, 0.01$.

The maximum absolute error of the numerical solutions of problem (5)–(6)				
No.	Regularizing function	$N = 100$	$N = 200$	$N = 500$
1	$g = 1 + \|z\|$	0.022065809	0.001390730	0.000470727
2	$g = (1 + z^2)^{1/2}$	0.022538096	0.002860760	0.001162406
3	$g = (1 + \|f\|)^{1/2}$	0.000685290	0.000129855	0.000006104
4	$g = (1 + f^2)^{1/4}$	0.000732160	0.000231898	0.000005539
5	$g = (1 + \|f\|)^{1/3}$	0.000093283	0.000004065	0.000000118
6	$g = (1 + \|f\|)^{1/2}/(1 + \ln(1 + \|f\|^{1/2}))$	0.000135996	0.000006656	0.000000148
7	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000481694	0.000019363	0.000000765
8	$g = (1 + \|z\| + \|f\|)^{1/3}$	0.000060213	0.000004002	0.000000070
9	$g = (1 + z^2 + f^2)^{1/2}$	0.029821881	0.027316539	0.022655875
10	$g = (1 + z^2 + f^2)^{1/4}$	0.000755412	0.000088372	0.000001114
11	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000762107	0.000039963	0.000000667
12	$g = 1$	0.528189578	0.018983935	0.000288408

Table 3: Comparison of the efficiency of various regularizing functions for the transformed problem (20) used for numerical solution of the original problem (5)–(6) by the method of non-local transformations for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, for a different number of grid points N.

Table 3 shows the results allowing to compare the efficiency of various regularizing functions that are used for numerical solution of the transformed problem (20) for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, for a different number of grid points N. Let us compare, for example, the maximum absolute errors of the numerical solutions obtained with regularizing functions Nos. 7 and 12 (the latter solution was obtained without using transformations) with the same number of grid points. In this case, for $N = 100$, the use of the method of non-local transformations makes it possible to increase the accuracy of the numerical solution approximately 1100 times, and for $N = 500$, approximately 380 times (as the number of grid points increases, the error of the numerical solution, obtained without using transformations, gradually decreases). The function No. 9, which corresponds to the arc length transformation [74, 75], is not very effective. The most effective of the considered regularizing functions for a given ε is the function No. 8 (the accuracy of which is approximately an order of magnitude higher than that of the function No. 7) which also provides high accuracy in the numerical solution of blow-up problems [52, 54–56].

Note that the functions No. 5 and 8, which give the best results in Table 3, but do not satisfy condition (24), can lead to significant errors with a further decrease of ε (this limited applicability of these functions remains valid for other test problems).

4.2. Test problem for a linear inhomogeneous equation. Numerical integration based on non-local transformations

Test problem 2. Let us investigate another test boundary-value problem with a small parameter for the second-order linear inhomogeneous equation:

$$\varepsilon y''_{xx} + p y'_x + r = 0 \quad (0 < x < 1); \qquad y(0) = a, \qquad y(1) = b, \tag{33}$$

where a, b, p, and r are free parameters. The exact solution of the problem (33) is written as follows:

$$y = \frac{a - b - (r/p)}{1 - e^{-p/\varepsilon}}\, e^{-px/\varepsilon} - \frac{r}{p}x + \frac{b + (r/p) - a e^{-p/\varepsilon}}{1 - e^{-p/\varepsilon}}. \tag{34}$$

We set $a = b = 0$, $p = 1$, $r = -1$. The solution (34) is non-monotonic in this case and is given by

$$y = \frac{1}{1 - e^{-1/\varepsilon}}\, e^{-x/\varepsilon} + x - \frac{1}{1 - e^{-1/\varepsilon}}, \tag{35}$$

and the transformed problem (20) takes the form

$$x'_\xi = \frac{1}{g}, \quad y'_\xi = \frac{z}{g}, \quad z'_\xi = \frac{f}{g} \quad (0 < \xi < \xi_1);$$
$$x(0) = 0, \quad y(0) = 0, \quad y(\xi_1) = 0. \tag{36}$$

Here $f = (1 - z)/\varepsilon$, and the value of ξ_1 is determined further in the solution process, according to the condition $x(\xi_1) = 1$.

The numerical solution, $x = x(\xi)$ and $y = y(\xi)$, of the transformed problem (36) obtained by the method of non-local transformations for $g = (1 + |z| + |f|)^{1/2}$, $\varepsilon = 0.005$, $a = b = 0$, $p = 1$, $r = -1$, with a fixed stepsize $h = 0.05$ and using Maple, is shown, respectively, by solid line and dashed line, in Fig. 3 *a*) on the ξ-interval. The exact solution (35) of the problem (33) and numerical solution of the transformed problem (36) is shown, respectively, by solid line and circles, in Fig. 3 *b*) on the x-interval for the same values of the parameters. Also in Fig. 3 *c*) and *d*), we present the behavior of the absolute error E_a of the numerical solution of the transformed problem (36) for $g = (1 + |z| + |f|)^{1/2}$ and the same values of the parameters. The maximum absolute error in this case is equal to 1.5983852×10^{-4} and is located in the transition area between the boundary layer and the outer region (near the extremum of the unknown function y).

Table 4 shows the maximum absolute errors of the numerical solutions of the transformed problem (36), used for the numerical solution of the original problem (33) for $a = b = 0$, $p = 1$, $r = -1$, $\varepsilon = 0.005$, for one hundred and two hundred grid points and for eleven different regularizing functions g. For comparison, similar data are also indicated for the case $g = 1$, which corresponds to the direct numerical solution of this problem (without using transformations) with the same stepsize with respect to x. Qualitatively, the situation is similar to that considered in (1).

Remark 7. The results of [33] obtained by the two-zone division of the region (using the asymptotic solution and additional optimization with respect to two grid parameters) give an absolute error in the numerical solution of the problem (33) for $a = b = 0$, $p = 1$, $r = -1$, $\varepsilon = 0.005$ and $N = 100$ equal to 0.00092 (this error is about 2–37 times higher than the error that can be obtained with the help of regularizing functions No. 3–10 of Table 4).

Table 5 shows the maximum absolute error of the numerical solutions of the transformed problem (36) for $f = (1 - z)/\varepsilon$ and the regularizing function $g = (1 + |z| + |f|)^{1/2}$, used for the numerical solution by the method of non-local

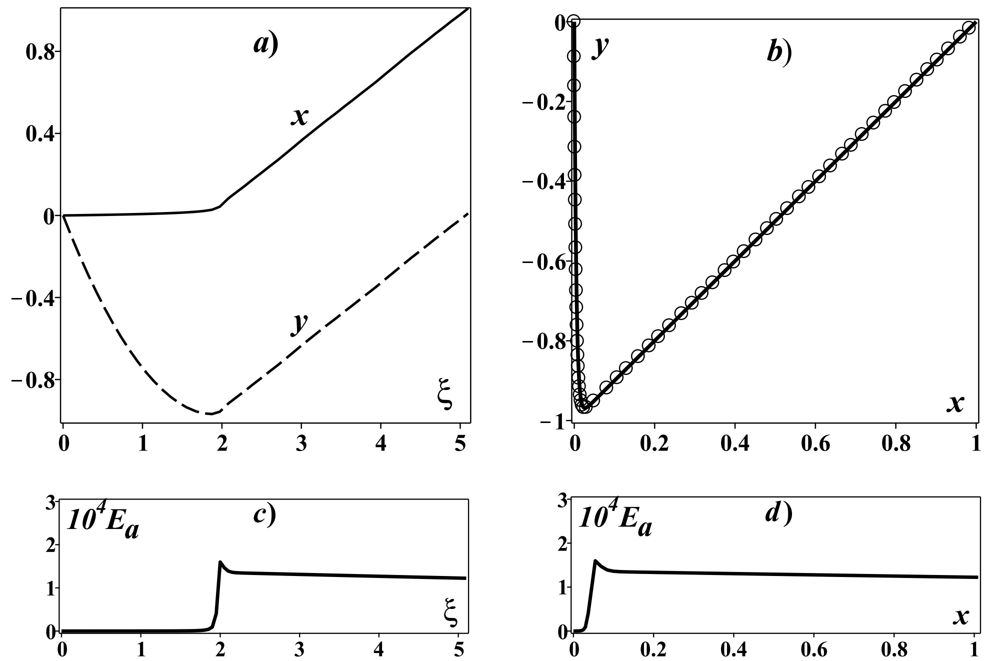

Figure 3: a) Numerical solution of the problem (33) obtained by the method of non-local transformations (transformed problem (36)) with the regularizing function $g = (1 + |z| + |f|)^{1/2}$ for $a = b = 0$, $p = 1$, $r = -1$, $\varepsilon = 0.005$: $x = x(\xi)$ (solid line) and $y = y(\xi)$ (dashed line); *b*) Exact solution (35) (solid line) of the problem (33) and the numerical solution of the transformed problem (36) (circles) for $g = (1 + |z| + |f|)^{1/2}$ and $a = b = 0$, $p = 1$, $r = -1$, $\varepsilon = 0.005$; *c*) and *d*) Absolute errors E_a of numerical solutions of the transformed problem (36) for $g = (1 + |z| + |f|)^{1/2}$ and the same values of the parameters ($a = b = 0$, $p = 1$, $r = -1$), respectively, with respect to ξ and x.

The maximum absolute error of the numerical solutions of problem (33)			
No.	Regularizing function	$N = 100$	$N = 200$
1	$g = 1 + \|z\|$	0.004600147	0.000069343
2	$g = (1 + z^2)^{1/2}$	0.005874977	0.000069009
3	$g = (1 + \|f\|)^{1/2}$	0.000162597	0.000008100
4	$g = (1 + f^2)^{1/4}$	0.000233108	0.000012153
5	$g = (1 + \|f\|)^{1/3}$	0.000024344	0.000000710
6	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000215369	0.000006823
7	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000205477	0.000009438
8	$g = (1 + \|z\| + \|f\|)^{1/3}$	0.000026290	0.000001062
9	$g = \theta/\ln\theta, \theta = (1 + \|z\| + \|f\|)^{1/2}$	0.000473114	0.000035329
10	$g = (1 + z^4 + f^2)^{1/4}$	0.000239924	0.000012717
11	$g = (1 + z^2 + f^2)^{1/2}$	0.002338633	0.002125528
12	$g = 1$	0.197998050	0.007120559

Table 4: Comparison of the efficiency of various regularizing functions for the transformed problem (36), for $p = 1$, $r = -1$, $a = b = 0$, $f = (1 - z)/\varepsilon$, used for the numerical solution of the original problem (33) with $\varepsilon = 0.005$ by the method of non-local transformations for a different number grid points N.

transformations of the original problem (33) for $p = 1$, $r = -1$, $a = b = 0$ and for different values of h and ε. Note that the length of the integration interval with respect to ξ ($0 \le \xi \le \xi_{\max}$) in this problem for $10^{-2} \le \varepsilon \le 5 \times 10^{-4}$ and the stepsize $h = 0.1$ varies in the range $5.0 \le \xi_{\max} \le 67.1$, and for the stepsize $h = 0.01$, in the range $3.3 \le \xi_{\max} \le 8.5$ (if $10^{-5} \le \varepsilon \le 10^{-4}$ and $0.1 \le h \le 0.01$, for a rough estimate we can use the formula $\xi_{\max} = 0.33h/\varepsilon$).

4.3. Variation of determining coefficients of the regularizing functions

Numerical solutions, obtained by the method of non-local transformations, can be refined using the variation of the appropriate determining coefficients of the regularizing functions. Let us demonstrate this with the example of three simple functions.

Table 6 shows the results of using regularizing functions of the form (27) (with $k = |\ln\varepsilon|^{-1}$) and (28) (with $k = |\ln\varepsilon|^{-1}$ and $k = |\ln\varepsilon|^{-2}$) that were applied for numerical solution of the transformed problem (20) in Test problem 1 for a different number of grid points N. Comparison with the data represented in Table 3 shows that in this case the maximum absolute errors are much less than in

The maximum absolute error of the numerical solutions of problem (36)				
ε	Stepsize 0.5	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
10^{-2}	0.01669480626141	0.00054035050047	0.00010017486513	0.00000011636415
5×10^{-3}	0.01725729158945	0.00062067791539	0.00013626986857	0.00000069674282
10^{-3}	0.01784086262805	0.00069322168092	0.00016824678413	0.00000549692469
5×10^{-4}	0.01792981095115	0.00070526113643	0.00017340917444	0.00000608694315
10^{-4}	0.01800349564889	0.00071699675486	0.00017847722961	0.00000692742385
5×10^{-5}	0.01801286411146	0.00071865293123	0.00017925906643	0.00000705440151
10^{-5}	0.01802038524283	0.00072001599693	0.00017991913800	0.00000717070013

Table 5: The maximum absolute error of the numerical solutions of the transformed problem (36) with the regularizing function $g = (1+|z|+|f|)^{1/2}$ (used for the numerical solution of the original problem (33) for $p = 1$, $r = -1$, $a = b = 0$) for three stepsizes with different ε.

the case of using the constant $k = 1$ in regularizing functions of the form (27) and (28).

The maximum absolute error of the numerical solutions of problem (5)–(6)			
No.	Regularizing function	$N = 100$	$N = 200$
1	$g = 1 + \|\ln\varepsilon\|^{-1}\|z\|$	0.001824574	0.000204348
2	$g = (1 + \|\ln\varepsilon\|^{-1}\|f\|)^{1/2}$	0.000415227	0.000024906
3	$g = (1 + \|\ln\varepsilon\|^{-2}\|f\|)^{1/2}$	0.000351332	0.000020161

Table 6: Comparison of the efficiency of various regularizing functions of the form (27) and (28) for the transformed problem (20) used for numerical solution of the original problem (5)–(6) by the method of non-local transformations for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$, for two different number of grid points N.

4.4. Numerical integration linear problems based on a point transformation

For numerical integration of singularly perturbed boundary-value problems one can also use point transformations that correspond to degenerate regularizing functions of the form $g = g(x, \xi)$. Without delving into this topic (which is not the main one here), we confine ourselves to an example of the use of concrete point transformation.

Example 2. Take the regularizing function

$$g = 1 + k(p/\varepsilon)e^{-px/\varepsilon}, \tag{37}$$

MAE of the numerical solutions of problem (5)–(6) with $a=1$, $b=0$, $p=q=1$			
	Regularizing function	$N=100$	$N=200$
1	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}$	0.002126032	0.000071843
2	$g{=}1+\varepsilon^{-1}e^{-x/\varepsilon}(1+\lvert\ln\varepsilon\rvert e^{-x/\varepsilon})^{-1}$	0.000896657	0.000026256
3	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}(1+e^{-x/\varepsilon})^{-1}$	0.001683419	0.000061345
MAE of the numerical solutions of problem (5)–(6) with $a=0$, $b=1$, $p=q=1$			
	Regularizing function	$N=100$	$N=200$
1	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}$	0.005807859	0.000196235
2	$g{=}1+\varepsilon^{-1}e^{-x/\varepsilon}(1+\lvert\ln\varepsilon\rvert e^{-x/\varepsilon})^{-1}$	0.002448877	0.000071669
3	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}(1+e^{-x/\varepsilon})^{-1}$	0.004598729	0.000167563
MAE of the numerical solutions of problem (33) with $a=b=0$, $p=1$, $r=-1$			
	Regularizing function	$N=100$	$N=200$
1	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}$	0.002134995	0.000071675
2	$g{=}1+\varepsilon^{-1}e^{-x/\varepsilon}(1+\lvert\ln\varepsilon\rvert e^{-x/\varepsilon})^{-1}$	0.000895610	0.000026040
3	$g{=}1+(\varepsilon\lvert\ln\varepsilon\rvert)^{-1}e^{-x/\varepsilon}(1+e^{-x/\varepsilon})^{-1}$	0.001690464	0.000061342

Table 7: Comparison of the efficiency of degenerate regularizing functions of the form (38) for the transformed problem (20) used for numerical solution of the original problems (5)–(6) and (33) by the method of non-local transformations for $\varepsilon = 0.005$ for a different number of grid points N. The MAE abbreviation, which stands for "maximum absolute error"', is used.

in which the expression $(p/\varepsilon)e^{-px/\varepsilon}$ is equal to the modulus of the derivative of the solution of the problem (5)–(6) for $q=0$, $a=1$, $b=0$, that is $(p/\varepsilon)e^{-px/\varepsilon} = |(e^{-px/\varepsilon})'_x|$, and k is a coefficient that can vary. The function (37) is a simplified analogue of the regularizing function (27) with $s=1$, in which the derivative of the required function $z=y'_x$ is formally replaced by the derivative of this simplest auxiliary problem (for quasilinear equations of the form (1) with boundary conditions (6) in the boundary-layer region, these derivatives differ only by the constant factor, if we set $p=p(0)$).

For the regularizing function (37), from the first equation of the system (20) we find the connection between the new and old independent variables

$$\xi = 1 + k - ke^{-px/\varepsilon}, \tag{38}$$

which is a point transformation.

In Table 7 we show the results for estimating the efficiency of degenerate regularizing functions of the form (34) that are used to integrate the system (20) obtained with the transformation (38) from the problems (5)–(6) and (33) for a different number of grid points N.

In (37), we choose the coefficient $k = 1/|\ln\varepsilon|$, which decrease sufficiently with decreasing ε (a similar factor is included in function No. 1 of Table 6). From the comparison of the data in Tables 3 and 4 it can be seen that degenerate regularizing functions of the form (37) also can be used, but they are somewhat inferior in efficiency to certain non-local regularizing functions.

In addition to the function (37), in Table 7, we also consider two other appropriate regularizing functions that have qualitatively similar asymptotic properties and correspond to point transformations.

5. On the choice of regularizing functions: a qualitative analysis

5.1. Problems with monotonic solutions. Properties of some regularizing functions

We first consider the boundary-value problem for the second-order linear equation with constant coefficients (5)–(6) for $a = 1, b = 0, p = q = 1, \varepsilon = 0.005$. The exact solution of this problem is monotonic and is determined by the relations (7), the asymptotic solution is described by the simple formula $y_a = e^{-200x}$. By introducing the non-local variable (18), the problem for one equation (5)–(6) is transformed to the problem for the system of equations (20), where $f = -\varepsilon^{-1}(z + y)$ and $z = y'_x$.

We carry out a qualitative analysis of the second equation of the system (20) for some regularizing functions g. To do this, we investigate the behavior of the function $\Phi = z/g$ (this is the right-hand side of the second equation) in the plane ξ, Φ for various functions g on the solution of the problem under consideration.

For $g = 1$, which corresponds to a direct numerical solution (without using transformations with $\xi = x$), the function Φ increases monotonically rapidly from -198.995 to 0 near the point $x = 0$, see Fig. 4*a* (solid line). Therefore, in order to obtain adequate numerical solutions on the basis of a uniform grid, in this case it is necessary to take a large number of points N. In Fig. 4*a*, similar curves are also shown for the regularizing functions $g = 1 + |z|$ (dashed line) and $g = (1 + z^2)^{1/2}$ (squares). Both curves change in a narrow band $-1 \le \Phi \le 0$ and have the form of a step, in the vicinity of which large gradients are observed (therefore, in calculations with a uniform grid with respect to ξ, we must also take a sufficiently large number of points N, but substantially less than for $g = 1$).

In Fig. 4*b*, we show the three curves $\Phi = \Phi(\xi)$ for functions $g = (1 + |f|)^{1/2}$, $g = (1 + |z| + |f|)^{1/2}$, and $g = (1 + z^2 + |f|)^{1/2}$ (the first two curves in this figure are indistinguishable). All these curves change rather slowly in a narrow band $-1 \le \Phi \le 0$ and have moderate derivatives. Therefore, for these regularizing

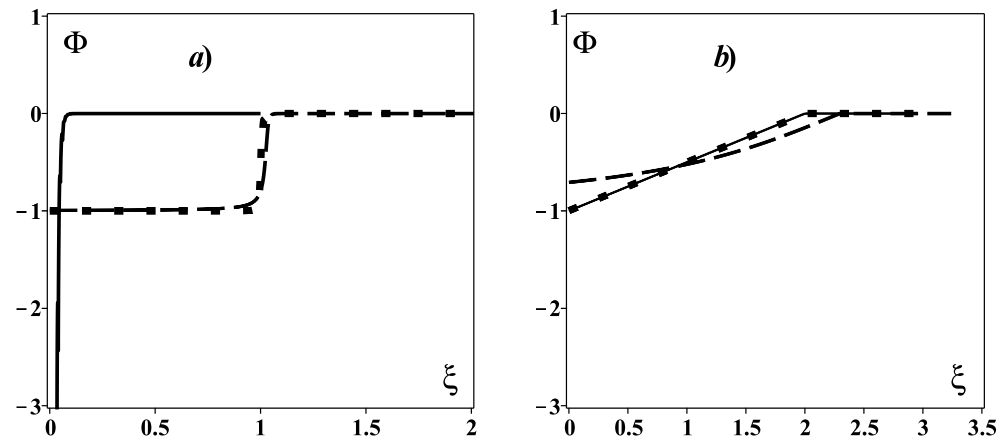

Figure 4: Dependency Φ on ξ in the problem (5)–(6) for $a = 1$, $b = 0$, $p = q = 1$, $\varepsilon = 0.005$ for the regularizing functions: a) $g = 1$ (solid line), $g = 1 + |z|$ (dashed line), and $g = (1 + z^2)^{1/2}$ (squares) and b) $g = (1 + |f|)^{1/2}$ (solid line), $g = (1 + |z| + |f|)^{1/2}$ (squares), and $g = (1 + z^2 + |f|)^{1/2}$ (dashed line).

functions, in the numerical integration of the system (20), one can use a uniform grid with respect to ξ with a sufficiently small number of points N.

5.2. Problems with non-monotonic solutions. Properties of some regularizing functions

We again consider the boundary-value problem for the second-order linear equation (5)–(6) for $a = 0$, $b = 1$, $p = q = 1$, $\varepsilon = 0.005$. The exact solution of this problem is non-monotonic and is determined by the relations (7), the asymptotic solution is given by the formula $y_a = e^{1-x} - e^{1-200x}$. Using the non-local variable (18), the problem for one equation (5)–(6) is transformed to the problem for the system of equations (20). As before, we will carry out a qualitative analysis of the second equation of the system (20) for $f = -\varepsilon^{-1}(z + y)$ and various functions g.

For $g = 1$, which corresponds to a direct numerical solution for $\xi = x$, the function $\Phi = z/g$ varies within a wide range of $-2.589 < \Phi < 540.917$, is non-monotonic and changes abruptly near the point $\xi = 0$, see Fig. 5a. In order to obtain adequate numerical solutions on the basis of a uniform grid, in this case it is necessary to take a large number of points N. In Fig. 5a, we show similar curves for the regularizing functions $g = 1 + |z|^{1/2}$ and $g = (1 + z^2)^{1/2}$. These

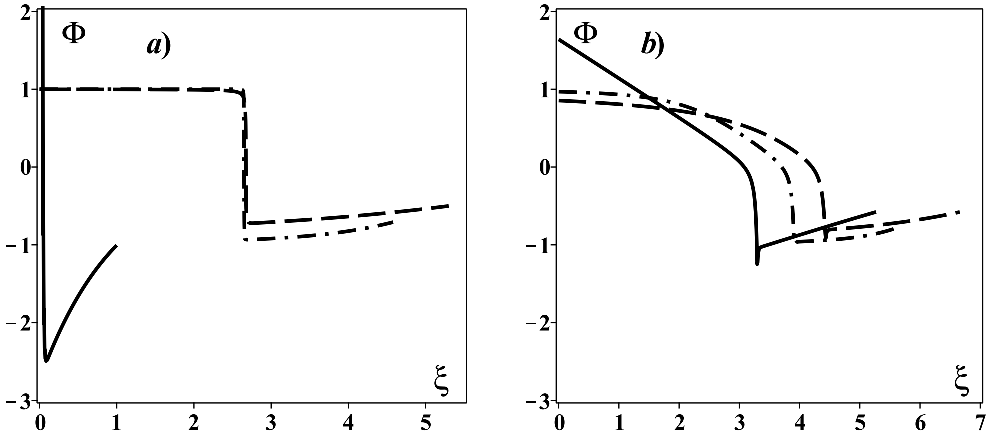

Figure 5: Dependency Φ on ξ in the problem (5)–(6) for $a = 0$, $b = 1$, $\varepsilon = 0.005$ and regularizing functions: *a)* $g = 1$ (solid line), $g = 1 + |z|$ (dashed line), and $g = (1 + z^2)^{1/2}$ (dash-dotted line), and *b)* $g = (1+|z|+|f|)^{1/2}$ (solid line), $g = (1+z^2+|f|)^{1/2}$ (dashed line) and $g = (1+z^4+f^2)^{1/4}$ (dash-dotted line).

curves change in the band $-1 \leq \Phi \leq 1$ and have the form of a non-smooth step, in the vicinity of which large gradients are observed; for a uniform grid with respect to ξ, it is also necessary to take a sufficiently large number of points N (qualitatively, the situation is similar to that described in Section 5.1 for these g).

The curve $\Phi = \Phi(\xi)$ for $g = (1 + |z| + |f|)^{1/2}$ (see Fig. 5*b*) is much more flat, than curves on Fig. 5a; test calculations show that we can use a uniform grid with respect to ξ with a sufficiently small number of grid points N. In Fig. 5*b* we also show the curves $\Phi = \Phi(\xi)$ for two other functions, $g = (1 + z^2 + |f|)^{1/2}$ and $g = (1+z^4+f^2)^{1/4}$. Both curves gradually change in a narrow band $-1 < \Phi < 1$, have the appearance of a smooth-step function, and have not too large values of the derivatives. Therefore, for these regularizing functions, in numerical integration of the transformed system (20) we can use a uniform grid with respect to ξ with a moderate or relatively small number of grid points N.

6. Asymptotic and numerical solutions of linear and non-linear singularly perturbed boundary-value problems with a small parameter

6.1. One class of non-linear boundary-layer type problems. Asymptotic solutions

Consider a two-point quasilinear boundary-value problem

$$\varepsilon y''_{xx} + p(x)y'_x + q(x,y) = 0 \quad (0 < x < 1); \tag{39}$$

$$y(0) = a, \quad y(1) = b, \tag{40}$$

which generalizes the linear problem (5)–(6). We assume that the functions $p(x) > 0$ and $q(x,y)$ are such that the problem (39)–(40) has a unique solution.

In the general case, it is impossible to represent the solution of the problem (39)–(40) in a closed analytical form. Therefore, to obtain an approximate solution as $\varepsilon \to 0$, we use the method of matched asymptotic expansions [4, 5, 8, 11, 56].

Let $p_0 = p(0)$. Then, as $\varepsilon \to 0$, near the left boundary $x = 0$, a boundary layer is formed, called the inner region. In this region, the last term of Eq. (39) can be neglected. The leading term of the asymptotic expansion of the solution in the boundary-layer region has the form

$$y_\mathrm{i} = c(1 - e^{-p_0\tau}) + a, \quad \tau = x/\varepsilon, \quad 0 \le \tau \le O(1), \tag{41}$$

where τ is the boundary-layer (stretched) variable, c is a constant that is determined further in the solution process.

In the outer region $O(\varepsilon) < x \le 1$, the first term of the equation (39) can be neglected also and the leading term of the asymptotic solution of the of the problem under consideration is determined from the truncated equation and the second boundary condition:

$$p(x)y'_x + q(x,y) = 0; \quad y(1) = b. \tag{42}$$

Let

$$y_\mathrm{e} = y_\mathrm{e}(x) \tag{43}$$

be the solution of the problem (42).

The internal and external solutions (41) and (43) must be consistent (matched), i.e. the condition

$$y_\mathrm{i}(\tau \to \infty) = y_\mathrm{e}(x \to 0), \tag{44}$$

must be satisfied, which allows us to determine the constant c occurring in (41):

$$c = y_{\rm e}(0) - a. \tag{45}$$

The composite asymptotic solution of the problem (39)–(40), which is uniformly applicable throughout the domain $0 \le x \le 1$, is defined by the formula

$$y = [a - y_{\rm e}(0)]e^{-p_0\tau} + y_{\rm e}(x) = [a - y_{\rm e}(0)]e^{-(p_0/\varepsilon)x} + y_{\rm e}(x). \tag{46}$$

Differentiating twice the formula (41) with respect to x, we find the derivatives in the boundary-layer region:

$$y'_x = \frac{cp_0}{\varepsilon}e^{-(p_0/\varepsilon)x}, \qquad y''_{xx} = -\frac{cp_0^2}{\varepsilon^2}e^{-(p_0/\varepsilon)x}. \tag{47}$$

It is seen as $\varepsilon \to 0$, both derivatives in the domain $0 \le x \le O(\varepsilon)$ are large, and there is a connection between the derivatives

$$|y''_{xx}| = c^{-1}e^{(p_0/\varepsilon)x}|y'_x|^2, \tag{48}$$

which is a particular case of the order relation (15).

In the problem (39)–(40) for $a = 0$ and $\varepsilon \ll 1$, the length of the boundary layer is determined by the formula (11) with $p = p_0$.

6.2. Asymptotic, exact, and numerical solutions of linear boundary-layer type problems

We consider the boundary-value problem for a second-order linear equation with variable coefficients

$$\varepsilon y''_{xx} + p(x)y'_x + q(x)y = r(x) \quad (0 < x < 1); \qquad y(0) = a, \quad y(1) = b, \tag{49}$$

which generalizes the problem (5)–(6). In the general case, it is impossible to obtain an exact solution of the problem (49) in a closed form. For some functions $p(x)$, $q(x)$, and $r(x)$, the exact solutions of the equation (49) can be found, for example, in [13, 56].

Let $p(x) > 0$ for $0 \le x \le 1$. Omitting the intermediate calculations (which are carried out according to the scheme described in [5, 11] and Section 6.1), we give the composite asymptotic solution of the problem (49) as $\varepsilon \to 0$:

$$\begin{gathered} y = [a - y_{\rm e}(0)]e^{-(p_0/\varepsilon)x} + y_{\rm e}(x), \qquad p_0 = p(0), \\ y_{\rm e}(x) = bE(x) - E(x)\int_x^1 \frac{r(\zeta)\,d\zeta}{p(\zeta)E(\zeta)}, \qquad E(x) = \exp\biggl[\int_x^1 \frac{q(\zeta)}{p(\zeta)}\,d\zeta\biggr]. \end{gathered} \tag{50}$$

We now consider, in more detail, a particular case of the problem (49) for $q(x) \equiv r(x) \equiv 0$:

$$\varepsilon y''_{xx} + p(x)y'_x = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \tag{51}$$

assuming that the conditions

$$p(x) > 0 \quad \text{at} \quad 0 < x \leq 1; \quad p(x) \simeq p_0 x^n \quad \text{as} \quad x \to 0; \quad n \geq 0 \tag{52}$$

are valid. We note that similar problems for $n = 1$ and $n = 2$ are often encountered in the theory of diffusion (thermal) boundary layer [14–16].

The exact solution of the problem (51) is determined by the formula

$$y = a + (b-a)\frac{I(x)}{I(1)}, \quad I(x) = \int_0^x P(t)\,dt, \quad P(x) = \exp\left[-\frac{1}{\varepsilon}\int_0^x p(t)\right] dt. \tag{53}$$

Test problem 3. For the power function $p(x) = p_0 x^n$, the solution (53) of the problem (51) takes the form

$$y = a + (b-a)\frac{I(x)}{I(1)}, \quad I(x) = \int_0^x \exp\left[-\frac{p_0}{\varepsilon(n+1)}t^{n+1}\right] dt \tag{54}$$

and can be expressed in terms of the incomplete gamma function as follows:

$$\begin{gathered} y = a + (b-a)\frac{\gamma(\nu, \kappa x^{n+1})}{\gamma(\nu, \kappa)}, \quad \gamma(\nu, z) = \int_0^x e^{-\tau}\tau^{\nu-1}\,d\tau, \\ \nu = \frac{1}{n+1}, \quad \kappa = \frac{p_0}{\varepsilon(n+1)}. \end{gathered} \tag{55}$$

Formula (55) defines an asymptotic solution of the problem (51) as $\varepsilon \to 0$ when the conditions (52) are satisfied.

We calculate the derivative on the left boundary of the solution:

$$\begin{aligned} y'_x|_{x=0} &= \frac{b-a}{I_1} = (b-a)p_0^{\frac{1}{n+1}}(n+1)^{\frac{n}{n+1}}\gamma^{-1}(\nu,\kappa)\varepsilon^{-\frac{1}{n+1}} \\ &\simeq (b-a)p_0^{\frac{1}{n+1}}(n+1)^{\frac{n}{n+1}}\Gamma^{-1}(\nu)\varepsilon^{-\frac{1}{n+1}}, \end{aligned} \tag{56}$$

where $\Gamma(\nu) = \int_0^\infty e^{-\tau}\tau^{\nu-1}\,d\tau$ is the gamma function. It can be seen that in this case the thickness of the boundary layer is proportional to $\varepsilon^{\frac{1}{n+1}}$ (and greater than ε for $n > 0$).

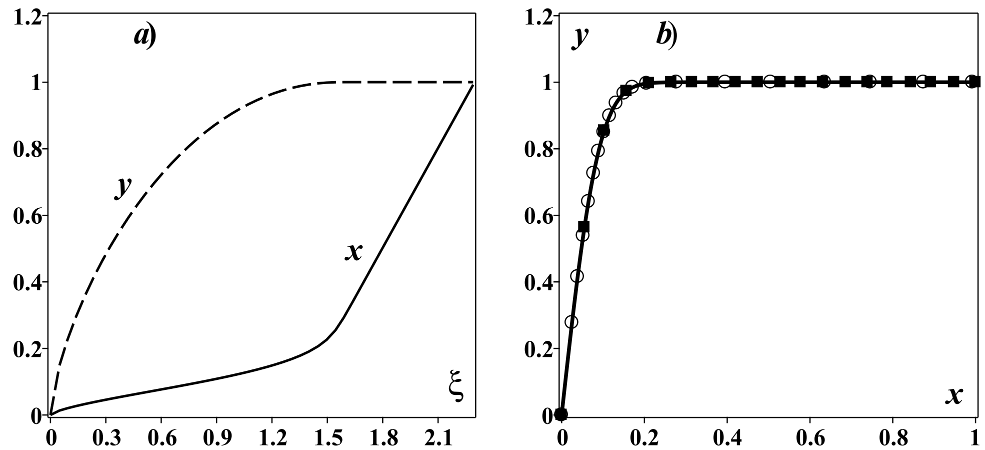

Figure 6: a)—numerical solution, $x(\xi)$ (solid line) and $y(\xi)$ (dashed line), of the problem (57) in the transformed variable domain (ξ); b)—The exact solution (solid line), the asymptotic solution (squares), and the numerical solution of the problem (57) (circles) in the original variable domain (x). Numerical solutions are obtained by the method of non-local transformations with the regularizing function $g = (1 + |f|)^{1/2}$ for $\varepsilon = 0.005$.

Test problem 4. We consider the boundary-value problem

$$\varepsilon y''_{xx} + \sin(x)y'_x = 0 \quad (0 < x < 1); \qquad y(0) = 0, \quad y(1) = 1, \tag{57}$$

which is a particular case of the problem (51) for $p(x) = \sin x$. The exact solution of the problem (57) is determined by the formula (53) with $I(x) = \int_0^x e^{(\cos t - 1)/\varepsilon} dt$ ($a = 0$, $b = 1$), and the asymptotic solution for small ε is determined by the formula (54) with $I(x) = \int_0^x e^{-t^2/(2\varepsilon)} dt$ ($a = 0$, $b = 1$).

In Fig. 6, the exact and asymptotic solutions of the problem (57) for $\varepsilon = 0.005$ are shown by the solid line and squares, the circles correspond to the numerical solution of this problem obtained by the method of non-local transformations using the system of equations (20) with the regularizing function $g = (1 + |f|)^{1/2}$ and the fixed stepsize $h = 0.01$. The maximum absolute error of the numerical solution in this case is 0.000334646.

6.3. Asymptotic, exact, and numerical solutions of non-linear boundary-layer type problems

Test problem 5. We consider the boundary-value problem for a second-order equation with exponential nonlinearity

$$\varepsilon y''_{xx} + p y'_x + k e^{\beta y} = 0 \quad (0 < x < 1); \tag{58}$$

$$y(0) = a, \quad y(1) = b. \tag{59}$$

The problem (58)–(59) is a particular case of the problem (39)–(40) for $p(x) = p = \text{const}$ and $q(x, y) = ke^{\beta y}$ (similar heat transfer problems with a kinetic function of exponential type occur in combustion theory [76, 77]). Note that the problem (58)–(59) for $p = 2$, $k = \beta = 1$, $a = b = 0$ was considered in [78, 79].

Let $p > 0$, $k \neq 0$, $0 < \varepsilon \ll 1$, and the condition $e^{-b\beta} - (k\beta)/p > 0$ be satisfied.

In the inner region (boundary-layer region), $0 \le x \le O(\varepsilon)$, the leading term of the asymptotic solution of the problem (58)–(59), satisfying the first boundary condition (59), is determined by the formula (41), in which it is necessary to set $p_0 = p$:

$$y_{\mathrm{i}} = c(1 - e^{-p\tau}) + a, \qquad \tau = x/\varepsilon, \tag{60}$$

where c is a constant, which is determined further in the solution process.

In the outer region, $O(\varepsilon) \le x \le 1$, the first term of the equation (58) can be neglected and the leading term of the asymptotic solution of the problem (58)–(59) is found from the truncated equation $py'_x + ke^{\beta y} = 0$. Its solution, satisfying the second boundary condition (59), has the form

$$y_{\mathrm{e}} = -\frac{1}{\beta}\ln\left[e^{-b\beta} + \frac{k\beta}{p}(x-1)\right]. \tag{61}$$

The inner and outer solutions (60) and (61) must satisfy the matching condition (44), which allows us to determine the constant c appearing in (60):

$$c = -\frac{1}{\beta}\ln\left(e^{-b\beta} - \frac{k\beta}{p}\right) - a. \tag{62}$$

The composite asymptotic solution of the problem (58)–(59), uniformly applicable throughout the domain $0 \le x \le 1$, is defined by the formula

$$y = \left[a + \frac{1}{\beta}\ln\left(e^{-b\beta} - \frac{k\beta}{p}\right)\right]e^{-px/\varepsilon} - \frac{1}{\beta}\ln\left[e^{-b\beta} + \frac{k\beta}{p}(x-1)\right]. \tag{63}$$

For the maximum value of the unknown value we obtain

$$y_* \simeq y_{\mathrm{e}}|_{x=0} = -\frac{1}{\beta}\ln\bigl[e^{-b\beta} - (k/p)\beta\bigr].$$

In Fig. 7, the asymptotic solution (63) of the problem (58)–(59) for $a = b = 0$, $p = 1$, $k = 0.9$, $\beta = 1$, $\varepsilon = 0.005$ is shown by the solid line. The circles

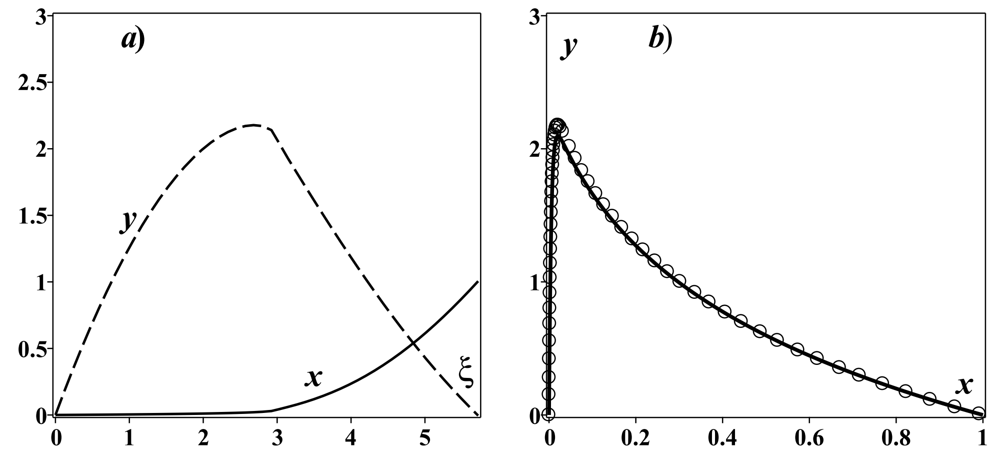

Figure 7: a)—numerical solution, $x(\xi)$ (solid line) and $y(\xi)$ (dashed line), of the transformed problem (20) in the transformed variable domain (ξ); b)—Asymptotic solution (63) (solid line) of the original problem (58)–(59) and numerical solution of the transformed problem (20) (circles) in the original variable domain (x). Numerical solutions are obtained by the method of non-local transformations with the regularizing function $g = (1 + |z| + |f|)^{1/2}$ for $a = 0$, $b = 0$, $p = 1$, $k = 0.9$, $\beta = 1$, $\varepsilon = 0.005$.

represent the results of the numerical solution of the corresponding transformed problem (20) with the regularizing function $g = (1 + |z| + |f|)^{1/2}$, obtained by the shooting method (from the point $x = 0$) with the stepsize $h = 0.01$ by using Maple. The maximum module of the difference between the asymptotic and numerical solutions is in the neighborhood of the extremum of the function $y = y(x)$ (at $x = 0.022350878$) and is equal to 0.082941959. The maximum module of the difference between the numerical solutions for $h = 0.01$ and $h = 0.005$ (the stepsize is reduced by half) is equal to $1.0 \cdot 10^{-9}$ (i.e., $0.082941959 - 0.082941960$).

Test problem 6. We now consider the boundary-value problem with power nonlinearity

$$\varepsilon y''_{xx} + p y'_x - k y^m = 0 \qquad (0 < x < 1); \tag{64}$$

$$y(0) = a, \qquad y(1) = b, \tag{65}$$

which is a particular case of the problem (39)–(40) for $p(x) = p = \text{const}$ and $q(x, y) = -ky^m$ (similar problems are encountered in the theory of convective mass transfer in the presence of a bulk chemical reaction of the mth order [15, 16]).

Let $a \geq 0$, $b \geq 0$, $p > 0$, $k > 0$, $m > 0$ ($m \neq 1$), and the condition $b^{1-m} + (k/p)(m-1) > 0$ be satisfied.

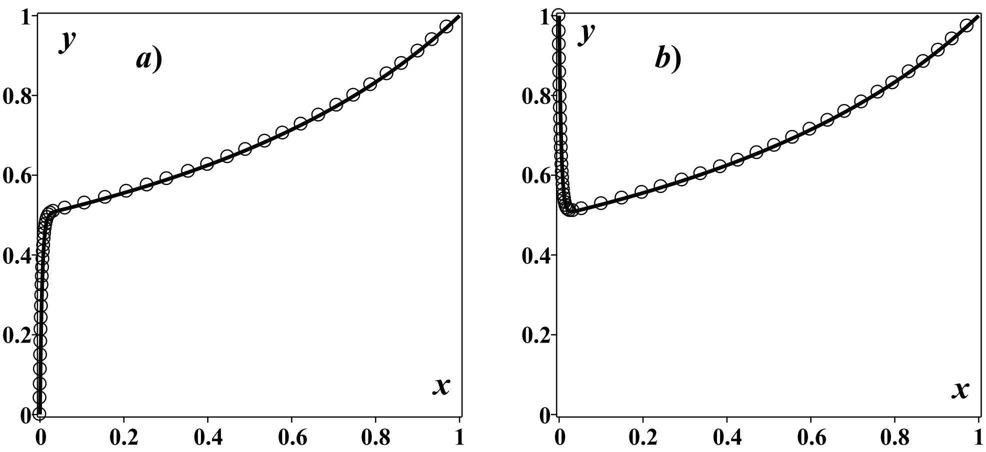

Figure 8: Asymptotic solutions (67) of the original problem (64)–(65) for $m = 2$, $p = k = 1$, $\varepsilon = 0.005$ (solid lines) and the corresponding numerical solutions of the transformed problem (20) (circles) obtained by the method of non-local transformations with the regularizing function $g = (1 + |f|)^{1/2}$ for two boundary conditions: *a*) $a = 0$, $b = 1$ and *b*) $a = b = 1$.

As $\varepsilon \to 0$, the solution of the problem (64)–(65) in the boundary-layer region, as in test problem 5, is given by the formula (60), and in the outer region the solution has the form

$$y = \left[b^{1-m} + \frac{k}{p}(1-m)(x-1)\right]^{\frac{1}{1-m}}. \tag{66}$$

The composite asymptotic solution of the problem (64)–(65), uniformly applicable throughout the domain $0 \le x \le 1$, is defined by the formula

$$y = \left[a - \left(b^{1-m} - \frac{k}{p}(1-m)\right)^{\frac{1}{1-m}}\right]e^{-px/\varepsilon} + \left(b^{1-m} + \frac{k}{p}(1-m)(x-1)\right)^{\frac{1}{1-m}}. \tag{67}$$

In Fig. 8, the asymptotic solutions (67) of the problem (64)–(65) are shown by solid lines for $p = k = 1$, $\varepsilon = 0.005$, for the second-order reaction $m = 2$, for the boundary conditions: *a*) $a = 0$, $b = 1$ (monotonic solution) and *b*) $a = b = 1$ (non-monotonic solution). The circles show the results of the numerical solutions of the corresponding transformed problem (20) with the regularizing function $g = (1 + |f|)^{1/2}$, obtained by the shooting method (from the point $x = 0$) with the stepsize $h = 0.01$ by using Maple. The maximum module of the difference between the asymptotic and numerical solutions for $a = 0$, $b = 1$ is equal to 0.00209386763545. The maximum module of the difference between the numerical solutions for $h = 0.01$ and $h = 0.005$ (the stepsize is reduced by half) is equal to $3.4286 \cdot 10^{-10}$ (i.e., $|0.00209386763545 - 0.00209386797831|$).

Test problem 7. We now consider a more complicated boundary-value problem

$$\varepsilon y''_{xx} + e^{y-x} y'_x - e^{y-x} = 0 \quad (0 < x < 1); \tag{68}$$

$$y(0) = a, \quad y(1) = b, \tag{69}$$

where the equation (68) does not belong to the class of equations (39).

The substitution $u = y - x$ transforms this equation to the autonomous equation $\varepsilon u''_{xx} + e^u u'_x = 0$ that is easily integrated: by introducing a new variable $v(u) = u'_x$ it is reduced to a first-order linear ODE $v'_u + e^u = 0$. The general solution of the equation (68) has the form

$$y = -\ln\left(ce^{-kx/\varepsilon} + \frac{1}{k}\right) + x. \tag{70}$$

The integration constants c and k are determined from the transcendental system of equations

$$c + \frac{1}{k} = e^{-a}, \quad ce^{-k/\varepsilon} + \frac{1}{k} = e^{1-b}, \tag{71}$$

obtained by substituting the expression (70) into the boundary conditions (69) and by performing elementary transformations.

As $\varepsilon \to 0$, the asymptotic solution of the system (71) is given by the formulas

$$c = e^{-a} - e^{1-b}, \quad k = e^{b-1}. \tag{72}$$

We note that the asymptotic solution (72) exactly satisfies the first the equation of the system (71), and the residual of the second equation of this system when $a \geq 0$ and $b \geq 0$ is less than $3e^{-1/(e\varepsilon)}$.

The numerical solution, $x = x(\xi)$ and $y = y(\xi)$, of the transformed problem (20) obtained by the method of non-local transformations for $g = (1 + z^2 + |f|)^{1/2}$, $\varepsilon = 0.005$, $a = 0$, $b = 0$, and applying the shooting method (from the point $x = 0$) with a fixed stepsize $h = 0.01$ by using Maple, is shown, respectively, by solid line and dashed line, in Fig. 9 *a*) on the ξ-interval. The asymptotic solution (70) with (72) of the problem (68)–(69) and numerical solution of the transformed

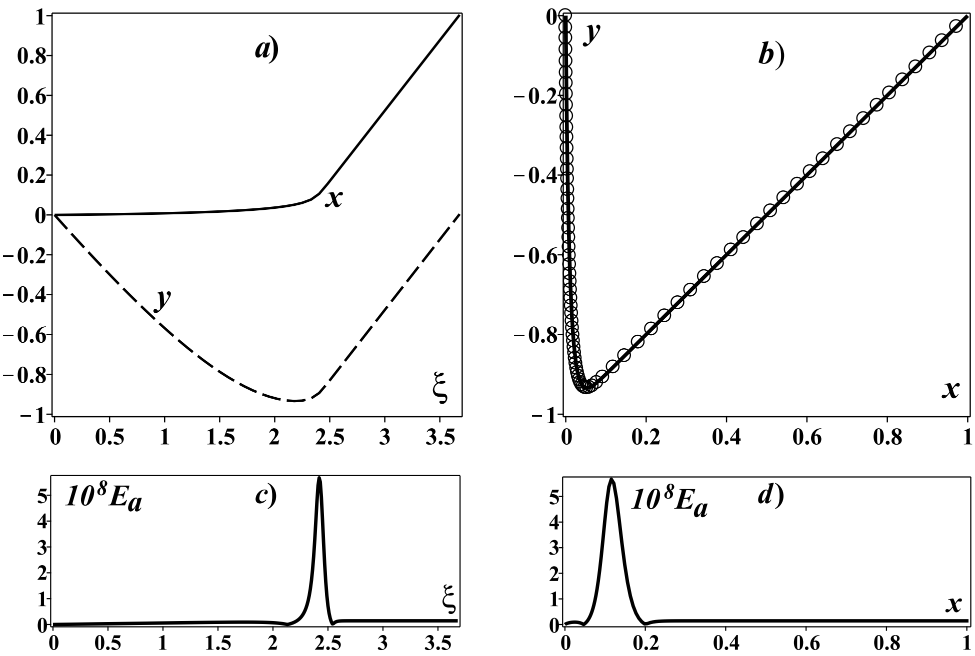

Figure 9: *a*)—numerical solution, $x = x(\xi)$ (solid line) and $y = y(\xi)$ (dashed line), of the transformed problem (20) in the transformed variable domain (ξ); *b*)—Asymptotic solution (70) with (72) (solid line) of the original problem (68)–(69) and numerical solution of the transformed problem (20) (circles) in the original variable domain (x). Numerical solutions are obtained by the method of non-local transformations with the regularizing function $g = (1 + z^2 + |f|)^{1/2}$ for $a = b = 0$, $\varepsilon = 0.005$ with the stepsize $h = 0.01$. *c*) and *d*) Absolute errors E_a of the numerical solutions of the transformed problem (20) for $g = (1 + z^2 + |f|)^{1/2}$ and the same values of the parameters, respectively, with respect to ξ and x.

problem (20) is shown, respectively, by solid line and circles, in Fig. 9 *b*) on the x-interval for the same values of the parameters. Also in Fig. 9 *c*) and *d*), we present the behavior of the absolute error E_a of the numerical solution of the transformed problem (20) for $g = (1 + z^2 + |f|)^{1/2}$ and the same values of the parameters. The maximum modulus of the difference between the asymptotic and numerical solutions is 5.6723604×10^{-8}. In this case, the difference between the asymptotic and the exact solutions is far beyond the limits of the accuracy of our calculations, and the corresponding curves for exact and asymptotic solutions coincide on the figure.

Table 8 shows the results allowing to compare the efficiency of various regularizing functions for numerical solutions of the transformed problem (20) used for the numerical integration of the original problem (68)–(69) by the method of non-local transformations for $a = b = 0$, $\varepsilon = 0.005$, for a different number of grid points N. It can be seen that five functions (Nos. 2–6) make it possible to obtain numerical solutions of the problem with high accuracy.

The maximum absolute error of the numerical solutions of problem (68)–(69)			
No.	Regularized function	$N = 100$	$N = 200$
1	$g = 1 + \|z\|$	0.001577514	0.000010978
2	$g = (1 + \|f\|)^{1/2}$	0.000022049	0.000000828
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000015480	0.000000615
4	$g = (1 + \|z\| + \|f\|)^{1/3}$	0.000004609	0.000000240
5	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000023103	0.000000841
6	$g = (1 + z^4 + f^2)^{1/4}$	0.000029709	0.000001437
7	$g = 1$	0.484249953	0.024531231

Table 8: Comparison of the efficiency of various regularizing functions for the transformed problem (20) used for the numerical solution of the original problem (68)–(69) by the method of nonlocal transformations for $a = b = 0$, $\varepsilon = 0.005$ for a different number grid points N.

Test problem 8. We now study the non-linear boundary-value problem

$$\varepsilon y''_{xx} + (py + q)y'_x = 0 \quad (0 < x < 1); \tag{73}$$

$$y(0) = a, \quad y(1) = b, \tag{74}$$

which also admits an exact solution. We note that the equation (73) is not included in the class of equations (39).

The substitution $u(y) = y'_x$ transforms the autonomous equation (73) to a first-order linear ODE $u'_y + py + q = 0$. Integrating it, we can obtain a general solution of the equation (73) in the form

$$y = \frac{c(1 - Ae^{-cx/\varepsilon})}{p(1 + Ae^{-cx/\varepsilon})} - \frac{q}{p}. \tag{75}$$

The constants of integration A and c are determined from the algebraic system of equations

$$c\,\frac{1 - A}{1 + A} - q = ap, \quad c\,\frac{1 - Ae^{-c/\varepsilon}}{1 + Ae^{-c/\varepsilon}} - q = bp, \tag{76}$$

which arises as a result of substituting the expression (75) into the boundary conditions (74). For $ap + q \geq 0$, $bp + q > 0$, and $\varepsilon \to 0$ the asymptotic solution of the system (76) is given by the formulas

$$A = \frac{(b-a)p}{(a+b)p+2q}, \qquad c = bp + q. \tag{77}$$

Note that for $a = b$, the asymptotic solution (77) is exact for any ε. In addition, the asymptotic solution (77) exactly satisfies the first equation of the system (76), and the residual of the second equation of this system is of the order of $e^{-(bp+q)/\varepsilon}$ as $\varepsilon \to 0$.

For $a = q = 0$, the asymptotic solution of the problem (73)–(74) has the form

$$y = \frac{b(1 - e^{-bpx/\varepsilon})}{1 + e^{-bpx/\varepsilon}}. \tag{78}$$

The numerical solution, $x = x(\xi)$ and $y = y(\xi)$, of the transformed problem (20) obtained by the method of non-local transformations for $g = (1 + z^2 + |f|)^{1/2}$, $\varepsilon = 0.005$, $a = 0$, $q = 0$, $b = 1$, $p = 1$, and applying the shooting method (from the point $x = 0$) with a fixed stepsize $h = 0.01$ by using Maple, is shown, respectively, by solid line and dashed line, in Fig. 10 *a*) on the ξ-interval. The asymptotic solution (78) of the problem (73) and numerical solution of the transformed problem (20) is shown, respectively, by solid line and circles, in Fig. 10 *b*) on the x-interval for the same values of the parameters. Also in Fig. 10 *c*) and *d*), we present the behavior of the absolute error E_a of the numerical solution of the transformed problem (20) for $g = (1 + z^2 + |f|)^{1/2}$ and the same values of the parameters. The maximum modulus of the difference between the asymptotic and numerical solutions is 1.7385554×10^{-6}. In this case, the difference between the asymptotic and the exact solutions is far beyond the limits of the accuracy of our calculations, and the corresponding curves for exact and asymptotic solutions coincide on the figure.

Table 9 shows the maximum absolute errors of the numerical solutions of the transformed problem (20), used for the numerical solution of the original problem (73)–(74) for $a = q = 0$, $b = p = 1$, $\varepsilon = 0.005$ for three stepsizes h and eight different regularizing functions g. For comparison, similar data are also indicated for the case $g = 1$, which corresponds to the direct numerical solution (without using transformations) with the same stepsize with respect to x. It can be seen that the functions Nos. 5–6 make it possible to obtain numerical solutions with high accuracy. Unsatisfactory results for function No. 2 can be explained by the fact that in this case a degeneracy occurs at the initial point, where the second derivative vanishes, $f|_{x=0} = 0$; therefore, the function $g = (1 + |f|)^{1/2}$ cannot

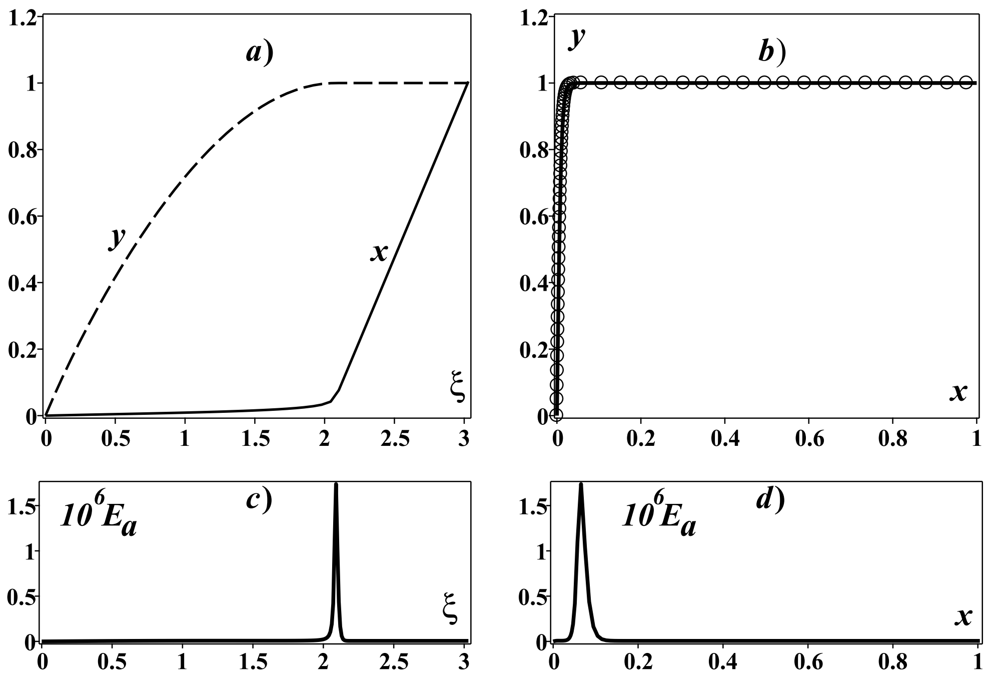

Figure 10: a)—numerical solution, $x(\xi)$ (solid line) and $y(\xi)$ (dashed line), of the transformed problem (20) in the transformed variable domain (ξ); b)—Asymptotic solution (78) of the original problem (73) (solid line) and numerical solution of the transformed problem (20) (circles) in the original variable domain (x). Numerical solutions are obtained by the method of non-local transformations with the regularizing function $g = (1 + z^2 + |f|)^{1/2}$ for $a = 0$, $q = 0$, $b = 1$, $p = 1$, $\varepsilon = 0.005$ with the stepsize $h = 0.01$. c) and d) Absolute errors E_a of the numerical solution of the transformed problem (20) for $g = (1 + z^2 + |f|)^{1/2}$ and the same values of the parameters, respectively, with respect to ξ and x.

suppress here the growth of the right-hand side of the second equation of the transformed problem (20).

The maximum absolute error of the numerical solutions of problem (73)–(74)				
No.	Regularized function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g = 1 + \|z\|$	0.018407343	0.004235485	0.000137385
2	$g = (1 + \|f\|)^{1/2}$	1.088190274	0.376921099	0.021473151
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.024998597	0.006396240	0.000193013
4	$g = (1 + \|z\| + \|f\|)^{1/3}$	0.055499931	0.017771942	0.000610856
5	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000854733	0.000181091	0.000001739
6	$g = (1 + z^4 + f^2)^{1/4}$	0.000614256	0.000159447	0.000003203
7	$g = 1$	process diverges	process diverges	0.019513818

Table 9: Comparison of the efficiency of various regularizing functions for the transformed problem (20) used for the numerical solution of the original problem (73)–(74) by the method of non-local transformations for $a = q = 0$, $b = p = 1$, $\varepsilon = 0.005$ for three steps.

Remark 8. The function (75) with constants (77) is the exact solution of the equation (73) (in the semi-bounded region $0 < x < \infty$) with the boundary conditions $y(0) = a$ and $y(\infty) = b$.

Remark 9. The equation (73) describes a family of exact travelling wave solutions of the Burgers equation $u_t + uu_z = \varepsilon u_{zz}$ [80], in which it is necessary to set $u = -py(x)$, where $x = z - qt$.

7. Boundary-value problems with two boundary layers

7.1. Problem for linear homogeneous equation. Numerical solution based on a combination of point and non-local transformations

Let us now demonstrate the possibility of applying a combination of point and non-local transformations for numerical integration of problems with two boundary layers.

We consider the boundary-value problem for the second-order linear homogeneous equation with variable coefficients

$$\varepsilon y''_{xx} - \varphi(x)y = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \tag{79}$$

which is a particular case of the problem (49) for $p(x) \equiv 0$, $q(x) = -\varphi(x)$, $r(x) = 0$. We assume that $\varphi(x) > 0$. In this case, two boundary layers of thickness of the order of $\varepsilon^{1/2}$ appear near the boundaries $x = 0$ and $x = 1$ as $\varepsilon \to 0$, and in the remaining region the solution of the problem tends to zero.

A composite asymptotic solution of the problem (79) as $\varepsilon \to 0$ can be represented in the form

$$y = ae^{-x\sqrt{\varphi_0/\varepsilon}} + be^{(x-1)\sqrt{\varphi_1/\varepsilon}}; \qquad \varphi_0 = \varphi(0), \qquad \varphi_1 = \varphi(1). \tag{80}$$

For numerical solution of the problem (79), at the preliminary stage it is useful to make the point transformation

$$y(x) = \lambda \exp\biggl[\varepsilon^{-1/2} \int_0^x \sqrt{\varphi(t)}\, dt\biggr] Y(x), \tag{81}$$

which contains a free parameter λ and leads to a qualitatively simpler problem with one boundary layer on the left end:

$$\begin{gathered} \varepsilon^{1/2} Y''_{xx} + 2\varphi^{1/2} Y'_x + \tfrac{1}{2}\varphi^{-1/2}\varphi'_x Y = 0; \\ Y(0) = a/\lambda, \qquad Y(1) = be^{-\beta}/\lambda, \end{gathered} \tag{82}$$

where $\beta = \varepsilon^{-1/2} \displaystyle\int_0^1 \sqrt{\varphi(x)}\, dx$. The problem (82) is a particular case of the problem (49) for $p(x) = 2\varphi^{1/2}$, $q(x) = \frac{1}{2}\varphi^{-1/2}\varphi'_x$, $r(x) = 0$, in which ε must be replaced by $\varepsilon^{1/2}$. The numerical solution of this problem can be obtained by using the method of non-local transformations at the first stage, and in the second stage the shooting method (from the point $x = 0$). It is convenient to choose the constant λ as follows:

$$\lambda = \begin{cases} be^{-\beta} & \text{if } b \neq 0, \\ a & \text{if } b = 0, \end{cases} \tag{83}$$

which leads to the normalized right boundary condition $Y(1) = 1$ (for $b \neq 0$).

Test problem 9. Consider the following test boundary-value problem with a small parameter for a linear equation with constant coefficients:

$$\varepsilon y''_{xx} - y = 0 \quad (0 < x < 1); \qquad y(0) = a, \qquad y(1) = b, \tag{84}$$

which is a particular case of the problem (79) and has the exact solution

$$y = \frac{ae^{\beta} - b}{e^{2\beta} - 1} e^{\beta(1-x)} + \frac{be^{\beta} - a}{e^{2\beta} - 1} e^{\beta x}, \qquad \beta = \varepsilon^{-1/2}. \tag{85}$$

As $\varepsilon \to 0$, we have two boundary layers of thickness $\sim O(\varepsilon^{1/2})$ near the end points $x = 0$ and $x = 1$, and in the remaining region the solution (85) tends to zero.

In this case (for $\varphi \equiv 1$), the point transformation (81) is simplified

$$y = \lambda e^{\beta x} Y(x), \quad \beta = \varepsilon^{-1/2}, \tag{86}$$

and converts the problem (84) to the following form:

$$\varepsilon^{1/2} Y''_{xx} + 2Y'_x = 0; \quad Y(0) = a/\lambda, \quad Y(1) = be^{-\beta}/\lambda. \tag{87}$$

The transformed problem (87) has one boundary layer on the left, its solution can be determined numerically by the shooting method (from the point $x = 0$).

The exact solution of the problem (87) is given by

$$Y = \frac{1}{\lambda} e^{-\beta x} \left[\frac{ae^{\beta} - b}{e^{2\beta} - 1} e^{\beta(1-x)} + \frac{be^{\beta} - a}{e^{2\beta} - 1} e^{\beta x} \right], \quad \beta = \varepsilon^{-1/2}. \tag{88}$$

The exact solution (85) of the problem (84) for $a = b = 1$, $\varepsilon = 0.005$ is shown by the solid line in Fig. 11; the points correspond to the numerical solution obtained by the method of non-local transformations with the regularizing function $g = (1 + |Y''_{xx}|)^{1/2}$ from the reduced problem (87) for $\lambda = 1$ with the stepsize $h = 0.01$ and the subsequent "reverse" recalculation by the formula (86); the squares show the composite asymptotic solution

$$y_a = e^{-\beta x} + e^{\beta(x-1)}, \quad \beta = \varepsilon^{-1/2}, \tag{89}$$

which is determined by substituting the values $a = b = \varphi_0 = \varphi_1 = 1$ into the formula (80). The maximum absolute errors of the numerical and asymptotic solutions in this case are respectively 1.04709×10^{-4} and 1.05423×10^{-4}.

7.2. *More general problem for linear inhomogeneous equation. Asymptotic solution*

We consider a boundary-value problem for the second-order linear inhomogeneous equation with variable coefficients

$$\varepsilon y''_{xx} - \varphi(x) y = g(x) \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \tag{90}$$

which is more general than the problem (79), it includes an additional function $g(x)$. We assume that $\varphi(x) > 0$. In this case, two boundary layers of thickness

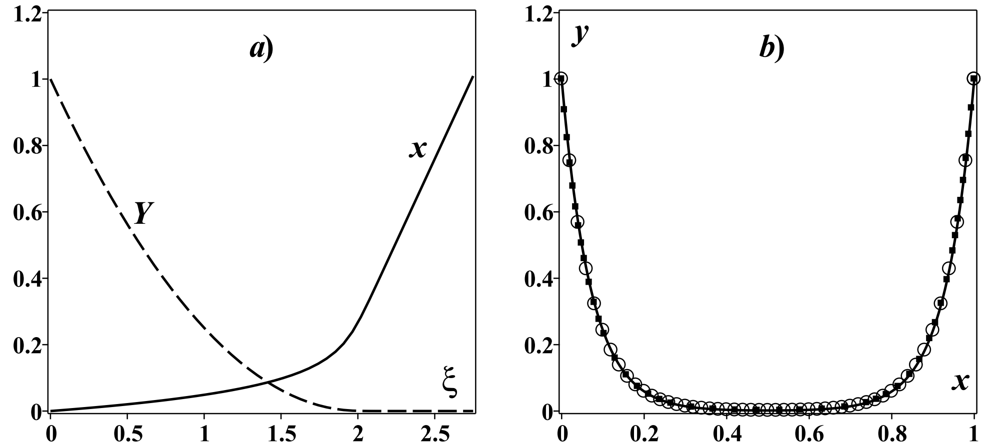

Figure 11: a)—numerical solution, $x = x(\xi)$ (solid line) and $Y = Y(\xi)$ (dashed line), of the problem (87) in the transformed variable domain (ξ); b)—Exact solution (85) of the original problem (84) (solid line), asymptotic solution (89) (squares), and numerical solution of the transformed problem (87) (circles) in the original variable domain (x).; Numerical solutions are obtained by the method of non-local transformations with the regularizing function $g = (1 + |Y''_{xx}|)^{1/2}$ for $a = b = 1$, $\varepsilon = 0.005$.

$\varepsilon^{1/2}$ appear near the boundaries $x = 0$ and $x = 1$ as $\varepsilon \to 0$, while in the remaining region the solution of the problem tends to the function $y_e = -g(x)/\varphi(x)$.

The composite asymptotic solution of the problem (90) as $\varepsilon \to 0$ can be represented in the form

$$y = -\frac{g(x)}{\varphi(x)} + \left(a + \frac{g_0}{\varphi_0}\right)e^{-\beta\sqrt{\varphi_0}\,x} + \left(b + \frac{g_1}{\varphi_1}\right)e^{\beta\sqrt{\varphi_1}\,(x-1)}, \quad \beta = \varepsilon^{-1/2}, \quad (91)$$

where $\varphi_0 = \varphi(0)$, $\varphi_1 = \varphi(1)$, $g_0 = g(0)$, $g_1 = g(1)$.

8. Brief conclusions

We consider singularly perturbed boundary-value problems for linear and nonlinear second-order ODEs of the form $\varepsilon y''_{xx} = F(x, y, y'_x)$. Such problems for $\varepsilon \to 0$ are characterized by boundary layers with large gradients in narrow regions and their solutions obtained by the standard fixed-step numerical methods can lead to significant errors. In this paper, we propose a new method of numerical integration of similar problems, based on the introduction of a non-local independent variable ξ that is related to the original variables x and y by the auxiliary differential equation $\xi'_x = g(x, y, y'_x, \xi)$. With a suitable choice of the regularizing function g, the proposed method leads to more appropriate problems that allow the application of standard methods with fixed stepsize of ξ (in the whole range of variation of the independent variable x, including both the boundary-layer region and the outer region). A number of test problems with a small parameter, which have simple exact or asymptotic solutions, expressed in elementary functions, are presented.

An extensive testing of the method of non-local transformations with more than ten different regularizing functions g is carried out on various problems for singularly perturbed ODEs with monotonic and non-monotonic solutions. Comparison of numerical, exact, and asymptotic solutions of a number of linear and non-linear test problems for ordinary differential equations of the second order has shown a high efficiency of the method of non-local transformations. For all the test problems considered, it is established that very good results are given, for example, by regularizing functions $g = (1+|y'_x|+|y''_{xx}|)^{1/2}$ and $g = (1+|y'_x|^2+|y''_{xx}|)^{1/2}$, where y''_{xx} can be replaced by $\varepsilon^{-1}F(x, y, y'_x)$. It is demonstrated that for numerical integration of singularly perturbed boundary-value problems one can also used point transformations corresponding to degenerate regularizing functions of the form $g = g(x, \xi)$ and that the method of non-local transformations is a generalization of the methods based on piecewise uniform grids. In addition to problems with a single boundary layer, we also consider some problems with two boundary layers.

9. Acknowledgments

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[1] M. Van Dyke, Perturbation Methods in Fluid Mechanics, Academic Press, New York, 1964.

[2] W. Eckhaus, Matched Asymptotic Expansions and Singular Perturbations, North-Holland, Amsterdam, 1973.

[3] J. Kevorkian, J. D. Cole, Perturbation Methods in Applied Mathematics, Springer, New York, 1981.

[4] A. H. Nayfeh, Introduction to Perturbation Techniques, Wiley–Interscience, New York, 1981.

[5] P. A. Lagerstrom, Matched Asymptotic Expansions. Ideas and Techniques, Springer, New York, 1988.

[6] R. E. O'Malley (Jr.), Singular Perturbation Methods for Ordinary Differential Equations, Springer, New York, 1991.

[7] J. A. Murdock, Perturbations. Theory and Methods, John Wiley & Sons, New York, 1991.

[8] A. M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, Amer. Math. Soc., Providence, 1992.

[9] M. H. Holmes, Introduction to Perturbation Methods, Springer, New York, 1995.

[10] J. Kevorkian, J. D. Cole, Multiple Scale and Singular Perturbation Methods, Springer, New York, 1996.

[11] A. H. Nayfeh, Perturbation Methods, Wiley–Interscience, New York, 2000.

[12] F. Verhulst, Methods and Applications of Singular Perturbations, Boundary Layers and Multiple Timescale Dynamics, Springer, New York, 2005.

[13] A. D. Polyanin, V. F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC Press, Boca Raton – London, 2003.

[14] V. G. Levich, Physicochemical hydrodynamics, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[15] A. D. Polyanin, V. V. Dilman, Methods of Modeling Equations and Analogies in Chemical Engineering, CRC Press/Begell House, Boca Raton – Ann Arbor, 1994.

[16] A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, D. A. Kazenin, Hydrodynamics, Mass and Heat Transfer in Chemical Engineering, Taylor & Francis, London, 2002.

[17] V. Ya. Neiland, V. V. Bogolepov, G. N. Dudin, I. I. Lipatov, Asymptotic Theory of Supersonic Viscous Gas Flows, Fizmatlit, Moscow, 2004 (in Russian).

[18] N. S. Bakhvalov, On the optimization methods for solving boundary value problems with boundary layers, Zh. Vychisl. Math. Fiz. 24 (1969) 841–859 (in Russian).

[19] A. M. Il'in, A difference scheme for a differential equation with a small parameter affecting the highest derivative, Mat. Zametki 6 (1969) 237–248 (in Russian).

[20] R. Vulanovic, A uniform numerical method for quasilinear singular perturbation problems without turning points, Computing 41 (1–2) (1989) 97–106.

[21] M. K. Jain, S. R. K. Iyengar, G. S. Subramanyam, Variable mesh methods for the numerical solution of two-point singular perturbation problems, Comput. Methods in Appl. Mech. Eng. 42 (3) (1984) 273–286.

[22] G. I. Shishkin, Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations, Ekaterinburg, Ural Branch of Russian Acad. Sci., 1992 (in Russian).

[23] L. Brugnano, D. Trigiante, High-order multistep methods for boundary value problems, Appl. Numer. Math. 18 (1–3) (1995) 79–94.

[24] J. R. Cash, Runge-Kutta methods for the solution of stiff two-point boundary value problems, Appl. Numer. Math. 22 (1–3) (1996) 165–177.

[25] G. Beckett, J. A. Mackenzie, Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem, Appl. Numer. Math. 35 (2) (2000) 87–109.

[26] P. Farrell, A. Hegarty, J. M. Miller, E. O'Riordan, G. I. Shishkin, Robust Computational Techniques for Boundary Layers, Chapman & Hall/CRC Press, Boca Raton – London, 2000.

[27] Y. Qiu, D. M. Sloan, T. Tang, Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution, analysis of convergence, J. Comput. Appl. Math. 116 (1) (2000) 121–143.

[28] W. L. Miranker, Numerical Methods for Stiff Equations and Singular Perturbation Problems, D. Reidel Publ., Dordrecht, 2001.

[29] T. Aziz, A. Khan, A spline method for second-order singularly perturbed boundary-value problems, J. Comput. Appl. Math. 147 (2) (2002) 445–452.

[30] M. K. Kadalbajoo, K. C. Patidar, A survey of numerical techniques for solving singularly perturbed ordinary differential equations, Appl. Math. Comput. 130 (2–3) (2002) 457–510.

[31] J. Vigo-Aguiar, S. Natesan, An efficient numerical method for singular perturbation problems, J. Comput. Appl. Math. 192 (1) (2006) 132–141.

[32] G. I. Shishkin, L. P. Shishkina, Difference Methods for Singular Perturbation Problems, Chapman & Hall/CRC Press, Boca Raton, 2009.

[33] T. Kolokolnikov, Boundary value problems with very sharp structures, numerical challenges, Dalhousie University, 2010, http://www.mathstat.dal.ca/ tkolokol/papers/birs2010.pdf

[34] M. K. Kadalbajoo, V. Gupta, A brief survey on numerical methods for solving singularly perturbed problems, Appl. Math. Comput. 217 (8) (2010) 3641–3716.

[35] N. Kopteva, E. O'Riordan, Shishkin meshes in the numerical solution of singularly perturbed differential equations, Int. J. Numer. Analysis and Modeling 7 (3) (2010) 393–415.

[36] L. G. Vulkov, A. I. Zadorin, Two-grid algorithms for an ordinary second order equation with an exponential boundary layer in the solution, Int. J. Numer. Analysis and Modeling 7 (3) (2010) 580–592.

[37] H. Ramos, J. Vigo-Aguiar, S. Natesan, R. Garcia-Rubio, M. A. Queiruga, Numerical solution of nonlinear singularly perturbed problems on nonuniform meshes by using a non-standard algorithm, J. Math. Chem. 48 (2010) 38–54.

[38] B. S. Attili, Numerical treatment of singularly perturbed two point boundary value problems exhibiting boundary layers, Commun. Nonlinear Sci. Numer. Simulat. 16 (9) (2011) 3504–3511.

[39] C.-S. Liu, The Lie-group shooting method for solving nonlinear singularly perturbed boundary value problems, Commun. Nonlinear Sci. Numer. Simulat. 17 (4) (2012) 1506–1521.

[40] S. A. Khuri, A. Sayfy, Self-adjoint singularly perturbed second-order two-point boundary value problems: A patching approach, Appl. Math. Modelling 38 (11) (2014) 2901–2914.

[41] H.-G. Roos, L. Teofanov, Z. Uzelac. A modified Bakhvalov mesh. *Appl. Math. Letters*, 2014, Vol. 31, pp. 7–11.

[42] P. Das, Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems, J. Comput. Appl. Math. 290 (2015) 16–25.

[43] V. Kumar, B. Srinivasan. An adaptive mesh strategy for singularly perturbed convection diffusion problems. Applied Math. Modelling 39 (7) (2015) 2081–2091.

[44] Q. Zheng, X. Li, Y. Gao. Uniformly convergent hybrid schemes for solutions and derivatives in quasilinear singularly perturbed BVPs, Appl. Numer. Math. 91 (2015) 46–59.

[45] P. Zhu, Y. Yang, Y. Yin, Higher order uniformly convergent NIPG methods for 1-d singularly perturbed problems of convection-diffusion type, Appl. Math. Modelling 39 (22) (2015) 6806–6816.

[46] M. Brdar, H. Zarin, A singularly perturbed problem with two parameters on a Bakhvalov-type mesh, J. Comput. Appl. Math. 292 (2016) 307–319.

[47] M. Kamranian, M. Dehghan, M. Tatari, An adaptive meshless local Petrov–Galerkin method based on a posteriori error estimation for the boundary layer problems, Appl. Numer. Math. 111 (2017) 181–196.

[48] R. K. Lodhi, H. K. Mishra, Quintic B-spline method for solving second order linear and nonlinear singularly perturbed two-point boundary value problems, J. Comput. Appl. Math. 319 (2017) 170–187.

[49] H. Zarin, Exponentially graded mesh for a singularly perturbed problem with two small parameters, Appl. Numer. Math. 120 (2017) 233–242.

[50] M. Ahmadinia, Z. Safari, Numerical solution of singularly perturbed boundary value problems by improved least squares method, J. Comput. Appl. Math. 331 (2018) 156–165.

[51] S. Franz, H.-G. Roos, The capriciousness of numerical methods for singular perturbations, SIAM Review 53 (1) (2011) 157–173.

[52] A. D. Polyanin, I. K. Shingareva, The use of differential and non-local transformations for numerical integration of non-linear blow-up problems, Int. J. Non-Linear Mech. 94 (2017) 178–184.

[53] A. D. Polyanin, I. K. Shingareva, The method of non-local transformations: Applications to blow-up problems, J. Physics: Conf. Series 937 (2017) 012042.

[54] A. D. Polyanin, I. K. Shingareva, Non-monotonic blow-up problems: Test problems with solutions in elementary functions, numerical integration based on non-local transformations, Appl. Math. Letters 76 (2018) 123–129.

[55] A. D. Polyanin, I. K. Shingareva, Non-linear problems with non-monotonic blow-up solutions: Non-local transformations, test problems, exact solutions, and numerical integration, Int. J. Non-Linear Mech. 99 (2018) 258–272.

[56] A. D. Polyanin, V. F. Zaitsev, Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems, CRC Press, Boca Raton – London, 2017.

[57] N. A. Kudryashov, D. I. Sinelshchikov, On the integrability conditions for a family of Liénard-type equations, Regular & Chaotic Dynamics 21 (5) (2016) 548–555.

[58] N. A. Kudryashov, D. I. Sinelshchikov, On the criteria for integrability of the Liénard equation, Appl. Math. Letters 57 (2016) 114–120.

[59] C. Muriel, J. L. Romero, Nonlocal transformations and linearization of second-order ordinary differential equations, J. Physics A, Math. Theor. 43 (43) (2010) 434025.

[60] S. V. Meleshko, S. Moyo, C. Muriel, J. L. Romero, P. Guha, A. G. Choudhury, On first integrals of second-order ordinary differential equations, J. Eng. Math. 82 (2013) 17–30.

[61] C. Muriel, J. L. Romero, New methods of reduction for ordinary differential equations, IMA J. Appl. Math. 66 (2001), 111–125.

[62] E. Pucci, G. Saccomandi, On the reduction methods for ordinary differential equations, J. Phys. A: Math. Gen. 35 (2002) 6145–6155.

[63] H. B. Keller, Numerical Solution of Two Point Boundary Value Problems, SIAM, Philadelphia, 1974.

[64] J. C. Butcher, The Numerical Analysis of Ordinary Differential Equations, Runge–Kutta and General Linear Methods, Wiley-Interscience, New York, 1987.

[65] L. Fox, D. F. Mayers, Numerical Solution of Ordinary Differential Equations for Scientists and Engineers, Chapman & Hall, London, 1987.

[66] W. E. Schiesser, Computational Mathematics in Engineering and Applied Science, ODEs, DAEs, and PDEs, CRC Press, Boca Raton, 1994.

[67] L. F. Shampine, Numerical Solution of Ordinary Differential Equations, Chapman & Hall/CRC Press, Boca Raton, 1994.

[68] U. M. Ascher, L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, Philadelphia, 1998.

[69] G. A. Korn, T. M. Korn, Mathematical Handbook for Scientists and Engineers, 2nd Edition, Dover Publ., New York, 2000.

[70] I. K. Shingareva, C. Lizárraga-Celaya, Maple and Mathematica. A Problem Solving Approach for Mathematics, 2nd Edition, Springer, Wien – New York, 2009.

[71] D. Griffiths, D. J. Higham, Numerical Methods for Ordinary Differential Equations, Springer, Wien – New York, 2010.

[72] J. H. Davis, Differential Equations with Maple: An Interactive Approach, Birkhäuser, Boston, 2001.

[73] B. R. Hunt, L. J. Lardy, et al., Differential Equations with Maple, 3rd Edition, John Wiley & Sons, New York, 2008.

[74] S. Moriguti, C. Okuno, R. Suekane, M. Iri, K. Takeuchi, Ikiteiru Suugaku – Suuri Kougaku no Hatten, Baifukan, Tokyo, 1979 (in Japanese).

[75] C. Hirota, K. Ozawa, Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations – An application to the blow-up problems of partial differential equations, J. Comput. Appl. Math. 193 (2) (2006) 614–637.

[76] D. A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics, Nauka, Moscow, 1987 (in Russian).

[77] Ya. B. Zel'dovich, G. I. Barenblatt, V. B. Librovich, G. M. Makhviladze, The Mathematical Theory of Combustion and Explosion, Division of Plenum Press, Consultants Bureau, New York, 1985.

[78] M. G. Gasparo, M. Macconi, Initial-value methods for second-order singularly perturbed boundary value problems, J. Optim. Theory Appl. 66 (1990) 197–210.

[79] Y. N. Reddy, P. P. Chakravarthy, An initial-value approach for solving singularly perturbed two-point boundary value problems, Appl. Math. Comput. 155 (2004) 95–110.

[80] A. D. Polyanin, V. F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, 2nd Edition, CRC Press, Boca Raton – London, 2012.