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Analytical solutions of long nonlinear internal waves: Part I

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Abstract The Gardner equation is an extension of the Korteweg–de Vries (KdV) equation. It exhibits basically the same properties as the classical KdV, but extends its range of validity to a wider interval of the parameters of the internal wave motion for a given environment. In this paper, we derive exact solitary wave solutions for the generalized Gardner equation that includes nonlinear terms of any order. Unlike previous studies, the exact solutions are derived without assuming their mathematical form. Illustrative examples for internal solitary waves are also provided. The traveling wave solutions can be used to specify initial data for the incident waves in internal waves numerical models and for the verification and validation of the associated computed solutions.

Keywords Extended KdV equation · Generalized Gardner equation · Cubic nonlinearity · Internal waves · Solitary waves

1 Introduction

The well-known Korteweg–de Vries (KdV) equation, $u_t + auu_x + \mu u_{xxx} = 0$, is a generic evolutionary Partial Differential Equation (PDE) used frequently to model weakly nonlinear long waves, incorporating a certain balance of leading-order nonlinearity and dispersion (Drazin and Johnson 1996). In particular, weakly nonlinear KdV-type theories have played a pivotal role in elucidating the salient features of unsteady internal waves evolution in coastal oceans. In situ and remote observations have shown that long solitary-like waves are common features of internal waves in density stratified shallow water

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(Helfrich and Melville 2006; Lamb and Yan 1996; Lee and Beardsley 1974). Recent studies have demonstrated that although KdV models are generally valid for a wide range of parameters of the given region of study, there are instances where the KdV equation is not applicable such as the critical case where the coefficients of the common quadratic nonlinear term in the KdV equation modeling long internal solitary waves vanishes when a symmetrical stratification occurs. It is often sufficient to extend the small quadratic approximation of the nonlinear term in KdV equation to a higher-order nonlinear expansion by adding an extra cubic nonlinear term.

This extension leads to the combined KdV and modified KdV equation (KdV–mKdV), also commonly known as the extended KdV (eKdV) or Gardner equation given by

$$u_t + auu_x + bu^2u_x + \mu u_{xxx} = 0, \tag{1}$$

which is written in dimensionless form. It exhibits basically the same properties as the classical KdV, but extends its range of validity to a wider interval of the parameters of the internal wave motion for a given environment (Grimshaw 2001; Grimshaw et al. 1994, 1998, 2002b; 2003, 2006; Holloway et al. 1997, 1999, 2001). In Eq. 1, $u = u(x, t)$ is a function of the two independent variables x and t that normally denote the space variable in the direction of wave propagation and time, respectively. As subscripts on u, x, and t denote partial derivatives of the dependent variable u. In most applications, $u = u(x, t)$ represents the amplitude of the relevant wave mode (e.g., u may represent the vertical displacement of the pycnocline), the terms uu_x and u^2u_x represent nonlinear wave steepening and the third-order derivative term u_{xxx} represents dispersive wave effects. The coefficients of the nonlinear terms a and b and the dispersive term μ are determined by the steady oceanic background density and flow stratification through the linear eigenmode (vertical structure function) of the internal waves.

The Gardner equation (1), like the Korteweg–de Vries, is a completely integrable PDE. As an integrable system, the initial value problem of the Gardner equation with a localized initial condition is exactly solvable. The most important implication of this integrability for Eq. 1 is that its solitary wave solutions are solitons which have the particular property of elastic wave interaction without shedding any oscillatory decaying tails or disturbance around the transmitted waves. This last property and the remarkable stability properties of solitons propagation explain why long nonlinear internal solitary-like waves are so commonly observed and represent ubiquitous features of oceanic shelves.

Exact soliton solutions of the Gardner equation were given by several authors (Apel 2003; Apel et al. 2007; Grimshaw et al. 1999, 2002a, 2003, 2004; Kakutani and Yamasaki 1978; Miles 1979, 1981; Nakoulima et al. 2004; Ostrovsky and Stepanyants 1989). For internal waves, solitary wave solutions depend essentially on the sign of the coefficient of the leading cubic nonlinear term b. It is worth noting for this model that the dispersive coefficient μ, is always positive, whereas the nonlinear coefficients, a and b, can be of either sign. In particular, if the coefficient $b > 0$, Eq. 1 admits two families of solitons, and oscillating wave packets (called breathers), whereas if $b < 0$ only one family of solitons exists.

In previous studies of the Gardner equation, the methods used for the derivation of internal waves solitary wave solutions were based on a priori knowledge of the general mathematical expression of the expected solutions (Wazwaz 2007). The assumed ansatze of the form of the solution is dictated in most cases from physical observations of internal wave evolution or obtained simply by trial and error. A criterion to specify which ansatze would lead to real solutions cannot be determined without a trial by direct substitution to show the existence of such analytical solutions.

In this paper, we derive exact solitary wave solutions for the generalized Gardner equation that includes nonlinear terms of any order. Unlike previous studies, the exact solutions are derived without assuming their mathematical form. These exact traveling wave solutions can be used to specify initial data for the incident waves in numerical models for internal waves and for the verification and validation of the associated computed solution.

2 Generalized Gardner equation

The classical Gardner equation (1) is a particular case of a more general Gardner equation that includes nonlinear terms of any order:

$$u_t + au^p u_x + bu^{2p} u_x + \mu u_{xxx} = 0, \quad p > 0. \tag{2}$$

From this generalized Gardner equation (2), we can distinguish the following three important cases:

- When $p = 1$, $a \neq 0$ and $b \neq 0$, Eq. 2 becomes the classical Gardner equation, which is also known as the combined KdV and modified KdV equation (KdV–mKdV)

$$u_t + auu_x + bu^2 u_x + \mu u_{xxx} = 0. \tag{3}$$

- When $p = 1$, $a \neq 0$ and $b = 0$, Eq. 2 is further reduced to the classical KdV equation

$$u_t + auu_x + \mu u_{xxx} = 0. \tag{4}$$

- When $p = 1$, $a = 0$ and $b \neq 0$, Eq. 2 becomes the modified KdV equation (mKdV)

$$u_t + bu^2 u_x + \mu u_{xxx} = 0. \tag{5}$$

The main objective of this paper is to derive exact traveling wave solutions to the general Gardner equation (2) including high-order nonlinear terms, from which we can easily deduce the solutions to the classical Gardner equation (3), the KdV equation (4), and the mKdV equation (5).

3 Derivation of exact solutions of generalized Gardner equation

Hamdi et al. (2004b) recently presented exact solitary wave solutions for general types of equal width (EW) wave equations with nonlinear terms of any order p,

$$u_t + au^p u_x - \mu u_{xxt} = 0. \tag{6}$$

and the generalized EW-Burgers equation with nonlinear terms of any order and dissipative effects modeled by the term δu_{xx},

$$u_t + au^p u_x + \delta u_{xx} - \mu u_{xxt} = 0. \tag{7}$$

More recently, Hamdi et al. (2005b) introduced a new evolution equation by including a fifth-order dispersion term u_{txxxx} in the generalized KdV equation (gKdV). This equation can model the effects of a high-order singular perturbation (in the limit $\varepsilon \longrightarrow 0$) to the gKdV equation,

$$u_t + au^p u_x + \mu u_{xxx} + \varepsilon u_{txxxx} = 0. \tag{8}$$

Exact solitary wave solutions, conservation laws and invariants of motion for Eq. 8 were also derived by Hamdi et al. (2005b).

In this section, we will adopt the same approach that was devised by Hamdi et al. (2004a, b, 2005b) to derive analytical traveling wave solutions to the general Gardner equation (2). The key idea of this approach consists of reducing the PDE (2) to a first-order nonlinear ordinary differential equation (ODE) that can be integrated exactly using symbolic computation and without assuming a priori any mathematical form of the analytical solution.

We concentrate on finding an exact solitary wave solution of PDE (2) of the form

$$u(x,t) = u(x - x_0 - Ct) \equiv u(\xi). \tag{9}$$

This corresponds to a traveling wave propagating with steady celerity C. We are interested in solutions depending only on the moving coordinate $\xi = x - x_0 - Ct$. Substituting into Eq. 2, the function $u(\xi)$ satisfies a third-order nonlinear ODE,

$$-Cu' + au^p u' + bu^{2p} u' + \mu u''' = 0, \tag{10}$$

where the derivatives are defined with respect to the coordinate ξ.

Integrating once, we obtain

$$-Cu + \frac{a}{p+1}u^{p+1} + \frac{b}{2p+1}u^{2p+1} + \mu u'' = k_1, \tag{11}$$

where k_1 is a constant of integration. If we assume that the solitary wave solution and its derivatives have the following asymptotic values,

$$u(\xi) \longrightarrow u_{\pm} \quad \text{as } \xi \longrightarrow \pm\infty, \tag{12}$$

and for $n \geq 1$,

$$u^{(n)}(\xi) \longrightarrow 0 \quad \text{as } \xi \longrightarrow \pm\infty, \tag{13}$$

(where the superscript denotes differentiation to the order n, with respect to ξ), and if we also assume that $u_{\pm}$ satisfies the following algebraic equation

$$-Cu_{\pm} + \frac{a}{p+1}u_{\pm}^{p+1} + \frac{b}{2p+1}u_{\pm}^{2p+1} = 0, \tag{14}$$

then the constant of integration k_1 is equal to zero and Eq. 11 reduces to

$$-Cu + \frac{a}{p+1}u^{p+1} + \frac{b}{2p+1}u^{2p+1} + \mu u'' = 0. \tag{15}$$

From Eq. 14 we also have the relation

$$\frac{a}{p+1}\left(u_+^{p+2} - u_-^{p+2}\right) + \frac{b}{2p+1}\left(u_+^{2p+2} - u_-^{2p+2}\right) = C\left(u_+^2 - u_-^2\right). \tag{16}$$

To allow another integration, we first multiply Eq. 15 by 2 u'. Then, each term can be integrated separately to obtain,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)}u^{p+2} + \frac{2b}{(2p+1)(2p+2)}u^{2p+2} + \mu(u')^2 = k_2, \tag{17}$$

where k_2 is a second constant of integration.

Using the asymptotic boundary conditions (12) and (13) at infinity we have

$$-Cu_+^2 + \frac{2a}{(p+1)(p+2)}u_+^{p+2} + \frac{2b}{(2p+1)(2p+2)}u_+^{2p+2} = k_2, \quad \text{as } \xi \longrightarrow +\infty, \tag{18}$$

and

$$-Cu_-^2 + \frac{2a}{(p+1)(p+2)}u_-^{p+2} + \frac{2b}{(2p+1)(2p+2)}u_-^{2p+2} = k_2, \quad \text{as } \xi \longrightarrow -\infty. \tag{19}$$

By combining (18) and (19) we obtain,

$$\frac{2a}{(p+1)(p+2)}\left(u_+^{p+2} - u_-^{p+2}\right) + \frac{2b}{(2p+1)(2p+2)}\left(u_+^{2p+2} - u_-^{2p+2}\right) = C\left(u_+^2 - u_-^2\right). \tag{20}$$

From relations (16) and (20) we conclude that $\left(u_+^2 - u_-^2\right) = 0$.

Therefore, the traveling wave solutions for (2) that satisfy the assumptions (12)–(14) should also satisfy the condition $|u_-| = |u_+|$. Under these conditions, the solutions are bell-shaped solitary wave solutions and could not have kink-profiles since such profiles require $|u_-| \neq |u_+|$.

If we also assume that $|u_-| = |u_+| = 0$, which is a valid assumption for the case of localized solutions, then the second constant of integration k_2 is equal to zero. In this case, Eq. 17 reduces to the following nonlinear ODE,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)}u^{p+2} + \frac{2b}{(2p+1)(2p+2)}u^{2p+2} + \mu(u')^2 = 0. \tag{21}$$

There are several approaches for analytically integrating this type of nonlinear ODE (Hamdi et al. 2004a, b, 2005b). This ODE can also be solved using symbolic computation. First, we make the following change of the dependent variable

$$u(\xi) = \phi^{1/p}(\xi), \tag{22}$$

in order to reduce the power of the nonlinear terms u^{p+2} and u^{2p+2} in (21). Using the above transformation we obtain,

$$-C\phi^2 + \frac{2a}{(p+1)(p+2)}\phi^3 + \frac{2b}{(2p+1)(2p+2)}\phi^4 + \frac{\mu}{p^2}(\phi')^2 = 0. \tag{23}$$

For simplification purposes, we consider the following new coefficients for the previous equation (23)

$$\mathbb{A} = \frac{p^2 C}{\mu}, \tag{24}$$

$$\mathbb{B} = -\frac{2a}{(p+1)(p+2)}\frac{p^2}{\mu}, \tag{25}$$

$$\mathbb{C} = -\frac{2b}{(2p+1)(2p+2)}\frac{p^2}{\mu}, \tag{26}$$

then we have

$$(\phi')^2 = \mathbb{A}\phi^2 + \mathbb{B}\phi^3 + \mathbb{C}\phi^4. \tag{27}$$

This auxiliary nonlinear ODE is an elliptic Riccati equation, which can be solved using symbolic computation. If $\mathbb{A} > 0$ and $\sqrt{\mathbb{B}^2 - 4\mathbb{A}\mathbb{C}} > 0$, the general exact solution of the ODE (27) for internal waves that satisfy asymptotic boundary conditions (12) and (13) at infinity and the condition $|u_-| = |u_+|$ is given by:

$$\phi(\xi) = \frac{2\mathbb{A}\mathrm{sech}\left(\sqrt{\mathbb{A}}(\xi - \mathrm{x}_0)\right)}{\sqrt{\mathbb{B}^2 - 4\mathbb{A}\mathbb{C}} - \mathbb{B}\mathrm{sech}\sqrt{\mathbb{A}}(\xi - x_0)}, \tag{28}$$

in which x_0 is a constant of integration.

As $\mathrm{sech} = \frac{1}{\cosh}$, it follows that

$$\phi(\xi) = \frac{2\mathbb{A}}{-\mathbb{B} + \sqrt{\mathbb{B}^2 - 4\mathbb{A}\mathbb{C}}\cosh\left(\sqrt{\mathbb{A}}(\xi - x_0)\right)}, \tag{29}$$

After substitutions using the relations (24)–(26) and the change of variables ($\xi = x - C\,t$), and ($u(\xi) = \phi^{1/p}(\xi)$), we obtain an explicit expression of the exact solution for the generalized Gardner equation:

$$u(x,t) = \left[\frac{(p+1)(p+2)C}{a}\,\frac{1}{1 + \sqrt{1 + \left(\frac{p^3+5p^2+8p+4}{2p+1}\right)\frac{bC}{a^2}}\cosh\left(p\sqrt{\frac{C}{\mu}}(x - Ct - x_0)\right)}\right]^{\frac{1}{p}}, \tag{30}$$

where the amplitude A (peak value) is given by

$$A = \left[\frac{(p+1)(p+2)C}{a}\,\frac{1}{1 + \left[\sqrt{1 + \left(\frac{p^3+5p^2+8p+4}{2p+1}\right)\frac{bC}{a^2}}\right]}\right]^{\frac{1}{p}}. \tag{31}$$

The analytical solution (30) is a bell-profile solitary wave. It is a single pulse of amplitude given by (31), initially centered at x_0, and traveling without change of shape at a steady celerity C and a wave number $\kappa = p\sqrt{\frac{C}{\mu}}$, which is inversely proportional to the width of the solitary wave.

4 Illustrative examples for internal solitary waves

Internal waves in stratified oceanic shelves or in two-layer fluids can be modeled using the classical Gardner equation, which is also known as the combined KdV and modified KdV equation (KdV–mKdV). It is a particular case of the generalized Gardner equation (2), when $p = 1$, $a \neq 0$, $b \neq 0$. It can be rewritten with the coefficient of the dispersive term $\mu = 1$ without loss of generality using a re-scaling of variables (Hamdi et al. 2004a):

$$u_t + auu_x + bu^2u_x + u_{xxx} = 0. \tag{32}$$

If $b = 0$ and $a \neq 0$, the cubic nonlinear term is dropped and in this case the Gardner equation reduces to the well-known Korteweg–de Vries equation. From the analytical solution (30), we obtain

$$u(x,t) = \frac{6C}{a}\frac{1}{1+\cosh\left(\sqrt{C}(x-Ct-x_0)\right)}. \tag{33}$$

Using the half-angle formula of the hyperbolic cosine $(1+\cosh x) = 2\cosh^2\frac{1}{2}x$, we have

$$u(x,t) = \frac{3C}{a}\frac{1}{\cosh^2\left(\frac{1}{2}\sqrt{C}(x-Ct-x_0)\right)}. \tag{34}$$

From the definition of the hyperbolic secant sech = 1/cosh, it follows that the solitary wave solution is:

$$u(x,t) = \frac{3C}{a}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{C}(x-Ct-x_0)\right). \tag{35}$$

The relation between the wave amplitude A and the wave celerity C of the solitary wave is obtained from Eq. 31:

$$A = \frac{3C}{a}. \tag{36}$$

The solitary wave solution (35) can also be expressed using the wave amplitude instead of the wave celerity

$$u(x,t) = A\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{aA}{3}}\left(x-\frac{aA}{3}t-x_0\right)\right), \tag{37}$$

which is the famous KdV solitary wave solution.

If $a = 0$ and $b \neq 0$, the quadratic nonlinear term is dropped and in this case the Gardner equation reduces to the well-known modified Korteweg–de Vries equation. In this case since $\mathbb{B} = 0$, Eq. 28 reduces to:

$$\phi(\xi) = \frac{2\mathbb{A}\mathrm{sech}\left(\sqrt{\mathbb{A}}(\xi - \mathrm{x}_0)\right)}{\sqrt{-4\mathbb{A}\mathbb{C}}}, \tag{38}$$

from which we obtain the solitary wave solution after substituting the explicit expressions (24) and (26), for the coefficients $\mathbb{A}$ and $\mathbb{C}$, respectively,

$$u(x,t) = \sqrt{\frac{6C}{b}}\mathrm{sech}\left(\sqrt{C}(x-Ct-x_0)\right). \tag{39}$$

The wave amplitude is given by

$$A = \sqrt{\frac{6C}{b}}. \tag{40}$$

Using the wave amplitude instead of the wave celerity, we obtain

$$u(x,t) = A\mathrm{sech}\left(\sqrt{\frac{bA^2}{6}}\left(x-\frac{bA^2}{6}t-x_0\right)\right), \tag{41}$$

and we retrieve the well-known solitary wave solution of the modified KdV equation, which has only cubic nonlinearity.

If $a \neq$ and $b \neq 0$, and for all sign combinations of these coefficients of the quadratic and cubic nonlinear terms, respectively, the exact internal solitary wave solution of the Gardner equation (32) can be obtained by a straightforward application of the analytical solution (30)

$$u(x,t) = \frac{6C}{a}\frac{1}{1+\left[\sqrt{1+\frac{6bC}{a^2}}\right]\cosh\left(\sqrt{C}(x-Ct-x_0)\right)}, \tag{42}$$

which can also be written in the following form

$$u(x,t) = \frac{6C}{a\left[1+B\cosh\left(\sqrt{C}(x-Ct-x_0)\right)\right]}, \tag{43}$$

where we introduced a new parameter B defined by

$$B^2 = 1 + \frac{6bC}{a^2}. \tag{44}$$

The internal solitary wave solution for the Gardner equation is a localized bell pulse with a peak amplitude given by:

$$A = \frac{6C}{a(1+B)}. \tag{45}$$

It is initially centered at x_0, and traveling without change of shape at a steady celerity C and a wave number $\sqrt{C}$, which also characterizes the inverse width of the solitary wave. The wave celerity is positive and left as an arbitrary free parameter and the two other dependent parameters B and A are expressed through it using the relations (44) and (45), which can be combined to obtain an alternate expression of the wave amplitude without an explicit dependence on C.

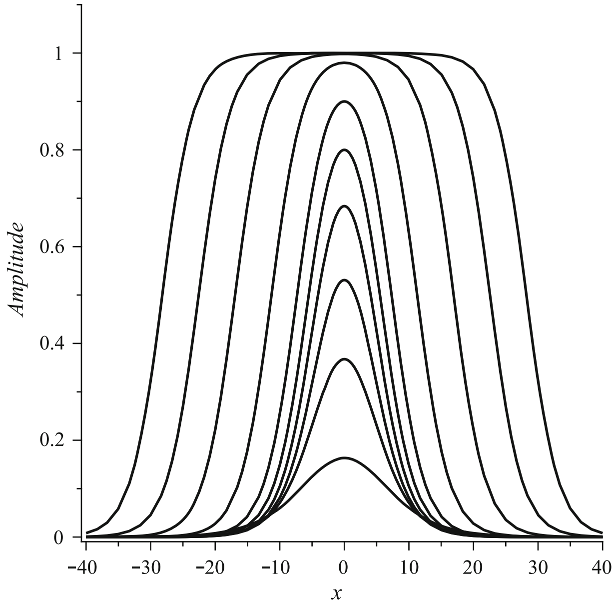
$$A = \frac{a}{b}(B-1). \tag{46}$$

Note that since C is positive, according to (45), the polarity of this solitary wave is defined by the sign of $a(1+B)$. In physical models, it is easier to prescribe the wave amplitude A rather than imposing the wave celerity C. We can distinguish two families of possible solitary wave solutions of the Gardner equation (43) depending on the sign of the coefficient b of the cubic nonlinear term in Eq. 32.

4.1 Negative cubic nonlinearity ($b < 0$)

For this case, there is only one possible branch of solitary waves ($0 < B < 1$). The sign of the coefficient a of the quadratic nonlinear term dictates the polarity of such solitons. The solitary wave shapes for $a = 1$ and $b = -1$ are depicted in Fig. 1. As shown in this figure, the solitary waves have positive polarity since $a > 0$. For small wave amplitude ($6\,|b|C \ll a^2$), or $B \longrightarrow 1$, at this limit the solitary wave solution of the Gardner equation (32) tends asymptotically to the KdV solitary wave given by Eq. 37

Fig. 1 The shape of solitary wave solutions (43) of the dimensionless Gardner equation for negative cubic nonlinearity ($b < 0$)



$$u(x,t) = A\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{aA}{3}}\left(x - \frac{aA}{3}t - x_0\right)\right), \quad \text{where } A = \frac{3C}{a}. \tag{47}$$

As the wave amplitude increases, it remains bounded by an upper limit given by

$$A_{\text{critical}} = \frac{a}{|b|}. \tag{48}$$

This critical value is obtained from the limit of Eq. 46 when $B \longrightarrow 0$. At this limit, the length of the solitary wave tends to infinity and such solitary waves are called "thick" or "table-top" solitons as shown in Fig. 1.

4.2 Positive cubic nonlinearity ($b > 0$)

For this case, there are two possible families of solitary waves of opposite polarity corresponding both to $B^2 > 1$. The first family corresponds to $1 < B < \infty$. The solitary waves have the same polarity as the coefficient of the quadratic nonlinear term a and are not bounded by a limiting critical amplitude. At small amplitudes, these solitary waves approach solutions of the KdV equation (37), whereas for large amplitudes (when $B \longrightarrow \infty$) these solitary waves tend to solutions of the modified KdV equation (41), which includes only cubic nonlinearity.

Possible shapes for this family of solitary wave solutions are shown in Fig. 2 for the coefficients $a = 1$ and $b = +1$ of the quadratic and cubic nonlinear terms, respectively. The second family of solitary waves corresponds to $-\infty < B < -1$ with opposite polarity to a and has negative B. They are negative solitary waves as shown in Fig. 2. For large negative amplitudes (when $B \longrightarrow -\infty$), these solitary waves also tend to solutions of the modified KdV equation (41). The amplitude of these negative solitons are bounded by a lower limit given by Eq. 46 when $B \longrightarrow -1$

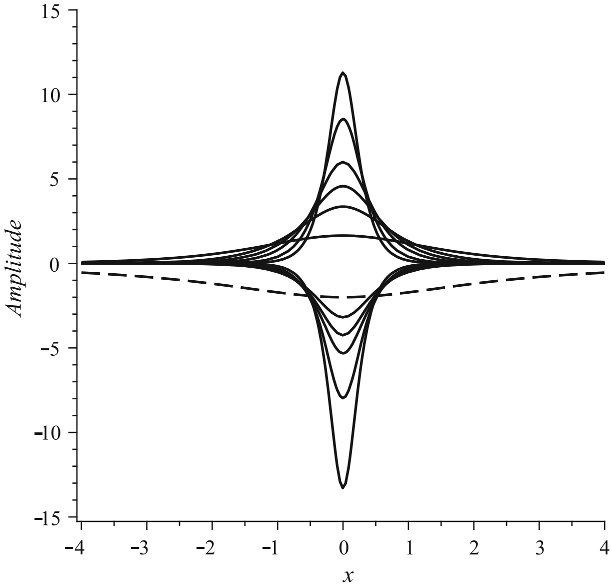


Fig. 2 The shape of solitary wave solutions of Gardner equation (43) of the dimensionless Gardner equation for positive cubic nonlinearity ($b > 0$)

$$A_{\text{algebraic}} = -2\frac{a}{b}. \tag{50}$$

In this limit, the wave amplitude tends to a critical value $A_{\text{algebraic}}$, and the wave solution tends to an algebraic soliton given by:

$$u(x) = \frac{A_{\text{algebraic}}}{1 + x^2/6b} \tag{51}$$

This algebraic soliton is shown by the dash line in Fig. 2. It is structurally unstable, and since $C = 0$ it does not propagate (steady state soliton).

5 Conclusion

A simple and direct method is devised for finding exact and explicit solitary wave solutions for the generalized Gardner equation for modeling internal waves in oceanic shelves. The exact solutions are derived without assuming their mathematical form. The accuracy and stability of numerical schemes for the solution of these general model equations for internal waves evolution can be assessed using, as test problems, the new exact solutions. These verification tools for similar evolutionary partial differential equations were implemented in a method of lines solver (Hamdi et al. 2001, 2005a; Schiesser 1994). The approach that we introduced for finding exact solitary wave solutions, is general and can be used for a wide class of nonlinear dispersive wave equations, such as general types of KdV-like equations.

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