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Conservation laws and invariants of motion for nonlinear internal waves: part II

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Abstract In this paper, we derive three conservation laws and three invariants of motion for the generalized Gardner equation. These conserved quantities for internal waves are the momentum, energy, and Hamiltonian. The approach used for the derivation of these conservation laws and their associated invariants of motion is direct and does not involve the use of variational principles. It can be easily applied for finding similar invariants of motion for other general types of KdV, Gardner, and Boussinesq equations. The stability and conservation properties of discrete schemes for the simulations of internal waves propagation can be assessed and monitored using the analytical expressions of the constants of motion that are derived in this work.

Keywords Generalized Gardner equation · Conservation laws · Invariants of motion · Internal waves · Solitary waves

1 Introduction

The Gardner equation, which is given by

$$u_t + auu_x + bu^2u_x + \mu u_{xxx} = 0, \tag{1}$$

is an extension of the classical Korteweg-de Vries (KdV) equation. The small quadratic approximation of the nonlinear term in KdV equation is extended to a higher-order nonlinear expansion by adding an extra cubic nonlinear term. The Gardner equation exhibits basically the same properties as the classical KdV but extends its range of validity

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to a wider interval of the parameters of the internal wave motion for a given environment (Grimshaw 2001; Grimshaw et al. 1994, 1998, 2002, 2003, 2006; Holloway et al. 1997, 1999, 2001). In Eq. (1), $u = u(x, t)$ is a function of the two independent variables x and t that normally denote the space variable in the direction of wave propagation and time, respectively. As subscripts on u, x and t denote partial derivatives of the dependent variable u. In most applications, $u = u(x, t)$ represents the amplitude of the relevant wave mode (e.g., u may represent the vertical displacement of the pycnocline), the terms uu_x and u^2u_x represent nonlinear wave steepening, and the third-order derivative term u_{xxx} represents dispersive wave effects. The coefficients of the nonlinear terms a and b and the dispersive term μ are determined by the steady oceanic background density and flow stratification through the linear eigenmode (vertical structure function) of the internal waves.

In a previous paper (Hamdi et al. 2011), the authors derived solitary wave solutions for the more general Gardner equation that includes nonlinear terms of any order:

$$u_t + au^p u_x + bu^{2p} u_x + \mu u_{xxx} = 0, \quad p > 0. \tag{2}$$

In this paper, we derive analytical expressions for three conservation laws and three invariants of motion for the general Gardner equation (2).

The exact traveling wave solutions can be used to specify initial data for the incident waves in internal waves numerical models and for the verification and validation of the associated computed solution. The invariants of motion can be used as verification tools for the conservation properties of the numerical scheme (Hamdi et al. 2001, 2005; Schiesser 1994).

A conservation law in differential equation form can be written as $T_t + X_x = 0$, in which the "density" T and the "flux" X are polynomials in the solution u and its x-derivatives (Drazin and Johnson 1996). If both T and X_x are integrable over the domain $(-\infty, +\infty)$, then the assumption that $X \longrightarrow 0$ as $|x| \longrightarrow \infty$ implies that the conservation law can be integrated over all x to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{+\infty} T\mathrm{d}x\right) = 0 \quad \text{or} \quad \int_{-\infty}^{+\infty} T\mathrm{d}x = \text{constant}. \tag{3}$$

The integral of T over the entire spatial domain is therefore invariant with time and usually called an *invariant of motion* or a *constant of motion* (Verheest and Hereman 1994; Goktas and Hereman 1999). The KdV equation itself is already in conservation form (i.e. $T = u, X = u_{xx} + \frac{1}{2}u^2$). In the late 1960s, Miura et al. (1968) discovered that the KdV equation admits an infinite number of conservation laws and forms a completely integrable Hamiltonian system (see also a survey paper by Miura (1976). Olver (1979) has shown that the RLW equation, $(u_t + u_x + uu_x - \mu u_{xxt} = 0)$, has only three nontrivial conservation laws and therefore is not an integrable PDE. These conservation laws for water waves are the equivalents of the conservation of mass, momentum, and energy. Recently, Hamdi et al. (2004) identified three conservation laws for the generalized equal width (KdV) wave equation, $u_t + au^p u_x - \mu u_{xxt} = 0$, and derived analytical expressions for their corresponding invariants of motion.

Fontenelle and Gallas (1990) derived constants of motion for the KdV equation and the modified KdV equation (mKdV), which is a particular case of the Gardner equation. More recently, Yaşar (2010) derived conservation laws and invariant solutions of the mKdV equation using a nonlocal conservation theorem method and a partial Lagrangian approach.

In the following sections, we derive three conservation laws and three invariants of motion for the generalized Gardner equation (2). We will adopt the same approach that was devised by Hamdi et al. (2004a, b, 2005), which is based on integration by parts and does not involve the use of variational principles.

2 Conservation laws for generalized Gardner equation

The classical Gardner equation (1) is a particular case of the more general Gardner equation (2) that includes nonlinear terms of any order.

From (2), we can distinguish the following three important cases:

– When $p = 1$, $a \neq 0$ and $b \neq 0$, Eq. (2) becomes the classical Gardner equation, which is also known as the combined KdV and modified KdV equation (KdV-mKdV)

$$u_t + auu_x + bu^2 u_x + \mu u_{xxx} = 0. \tag{4}$$

– When $p = 1$, $a \neq 0$ and $b = 0$, Eq. (2) is further reduced to the KdV equation

$$u_t + auu_x + \mu u_{xxx} = 0. \tag{5}$$

– When $p = 1$, $a = 0$ and $b \neq 0$, Eq. (2) becomes the modified KdV equation (mKdV)

$$u_t + bu^2 u_x + \mu u_{xxx} = 0. \tag{6}$$

The main objective of this paper is to derive three conservation laws and invariants of motion to the general Gardner equation (2) including high-order nonlinear terms, from which we can easily deduce the conservation laws and invariants of motion to the classical Gardner equation (4), the KdV equation (5), and the mKdV equation (6), and also deduce explicit expressions of the invariants of motion for internal solitary waves and internal undular bores.

2.1 First conservation law for generalized Gardner equation

The first conservation law, which represents the conservation of momentum, can be obtained directly by rewriting the generalized Gardner equation ($u_t + auu^p u_x + bu^{2p} u_x + \mu u_{xxx} = 0$) in conservative form $T_t + X_x = 0$ as follows,

$$(u)_t + \left(\frac{a}{(p+1)} u^{p+1} + \frac{b}{(2p+1)} u^{2p+1} + \mu u_{xx} \right)_x = 0. \tag{7}$$

2.2 Second conservation law for generalized Gardner equation

The second conservation law, which represents the conservation of energy, is derived by multiplying the generalized Gardner equation ($u_t + au^p u_x + bu^{2p} u_x + \mu u_{xxx} = 0$) by $2u$. After multiplication and factorization, we obtain

$$2u_t u + 2au^{p+1} u_x + 2bu^{2p+1} u_x + 2\mu u u_{xxx} = 0. \tag{8}$$

The primitive with respect to time t of the first term of Eq. (8) is given by

$$2u_t u = \left(u^2\right)_t. \tag{9}$$

The primitives with respect to the independent spatial variable x of the second and third terms of Eq. (8) are respectively given by

$$2au^{p+1}u_x = \left(\frac{2a}{(p+2)}u^{p+2}\right)_x, \tag{10}$$

and

$$2bu^{2p+1}u_x = \left(\frac{2b}{(2p+2)}u^{2p+2}\right)_x. \tag{11}$$

The primitive with respect to the variable x of the last term of Eq. (8) is obtained using integration by parts:

$$2\mu u u_{xxx} = (2\mu u u_{xx})_x - \left(\mu (u_x)^2\right)_x. \tag{12}$$

After arranging the terms using these primitives, Eq. (8) is rewritten in conservative form $T_t + X_x = 0$ as follows:

$$\left(u^2\right)_t + \left(\frac{2a}{(p+2)}u^{p+2} + \frac{2b}{(2p+2)}u^{2p+2} + 2\mu u u_{xx} - \mu u_x u_x\right)_x = 0, \tag{13}$$

where the conserved density is the energy given by the functional T

$$T = u^2, \tag{14}$$

which is a polynomial conserved density. The associated conserved flux is given by

$$X = \frac{2a}{(p+2)}u^{p+2} + \frac{2b}{(2p+2)}u^{2p+2} + 2\mu u u_{xx} - \mu u_x u_x. \tag{15}$$

The flux X is also a polynomial in u and its x derivatives; therefore, the second conservation law (13) is a polynomial conservation law.

2.3 Third conservation law for generalized Gardner equation

The third conservation law is more complicated to derive as it requires less obvious substitutions. The first step is to multiply the generalized Gardner equation ($u_t + au^p u_x + bu^{2p}u_x + \mu u_{xxx} = 0$) by $\left(\frac{2a}{(p+1)}u^{p+1} + \frac{2b}{(2p+1)}u^{2p+1}\right)$. After multiplication and factorization, we obtain

$$\begin{aligned}
&\frac{2a}{(p+1)}u^{p+1}u_t + \frac{2b}{(2p+1)}u^{2p+1}u_t + \frac{2a^2}{(p+1)}u^{2p+1}u_x \\
&+ \frac{2b^2}{(2p+1)}u^{4p+1}u_x + \frac{2ab}{(p+1)}u^{3p+1}u_x \\
&+ \frac{2ab}{(2p+1)}u^{3p+1}u_x + \mu u_{xxx}\frac{2a}{(p+1)}u^{p+1} + \mu u_{xxx}\frac{2b}{(2p+1)}u^{2p+1} = 0.
\end{aligned} \tag{16}$$

The second step consists of differentiating the generalized Gardner equation ($u_t + au^p u_x + bu^{2p} u_x + \mu u_{xxx} = 0$) with respect to the variable x to obtain

$$u_{xt} + apu^{p-1} u_x u_x + au^p u_{xx} + 2bpu^{2p-1} u_x u_x + bu^{2p} u_{xx} + \mu u_{xxxx} = 0, \tag{17}$$

and multiplying the resulting Eq. (17) by $-2\mu u_x$. After multiplication and factorization, we obtain

$$\begin{aligned} &-2\mu u_t u_{xt} - 2\mu apu^{p-1} u_x u_x u_x - 2\mu au^p u_x u_{xx} - 2\mu b(2p)u^{2p-1} u_x u_x u_x - 2\mu bu^{2p} u_x u_{xx} \\ &\quad - 2\mu^2 u_x u_{xxxx} = 0. \end{aligned} \tag{18}$$

The third step consists of combining Eqs. (16) and (18). After several integrations by parts and tedious manipulations, the resulting quantity can be expressed in conservation form $T_t + X_x = 0$ as

$$\begin{aligned} &\left(\frac{2au^{(p+2)}}{(p+1)(p+2)} + \frac{2bu^{(2p+2)}}{(2p+1)(2p+2)} - \mu u_x u_x \right)_t \\ &\quad + \left(\frac{a^2 u^{2(p+1)}}{(p+1)^2} + \frac{b^2 u^{2(2p+1)}}{(2p+1)^2} + \frac{2abu^{(3p+2)}}{(p+1)(3p+2)} + \frac{2abu^{(3p+2)}}{(2p+1)(3p+2)} + \frac{2\mu au^{(p+1)} u_{xx}}{(p+1)} \right. \\ &\quad \left. + \frac{2\mu bu^{(2p+1)} u_{xx}}{(2p+1)} - 2\mu au^p u_x u_x - 2\mu bu^{2p} u_x u_x - 2\mu^2 u_x u_{xxx} + \mu^2 u_{xx} u_{xx} \right)_x = 0. \end{aligned} \tag{19}$$

where the conserved density is the Hamiltonian density given by the functional T

$$T = \frac{2au^{(p+2)}}{(p+1)(p+2)} + \frac{2bu^{(2p+2)}}{(2p+1)(2p+2)} - \mu u_x u_x, \tag{20}$$

which is also a polynomial conserved density. The associated conserved flux is given by

$$\begin{aligned} X = &\frac{a^2 u^{2(p+1)}}{(p+1)^2} + \frac{b^2 u^{2(2p+1)}}{(2p+1)^2} + \frac{2abu^{(3p+2)}}{(p+1)(3p+2)} + \frac{2abu^{(3p+2)}}{(2p+1)(3p+2)} \\ &+ \frac{2\mu au^{(p+1)} u_{xx}}{(p+1)} + \frac{2\mu bu^{(2p+1)} u_{xx}}{(2p+1)} - 2\mu au^p u_x u_x - 2\mu bu^{2p} u_x u_x - 2\mu^2 u_x u_{xxx} + \mu^2 u_{xx} u_{xx} \end{aligned} \tag{21}$$

The flux X is a polynomial in u and its x derivatives; therefore, the third conservation law (13) is also a polynomial conservation law.

3 Invariants of motion for generalized Gardner equation

The three conservation laws (7), (13), and (19) derived in the previous section lead to three invariants of motion. They can be integrated easily with respect to x over a large but finite spatial domain $[x_L, x_U]$ instead of $[-\infty, +\infty]$ to obtain the following intermediate results

$$\frac{\partial}{\partial t} \int_{x_L}^{x_U} u \mathrm{d}x + \frac{a}{(p+1)} \left(u_U^{(p+1)} - u_L^{(p+1)} \right) + \frac{b}{(2p+1)} \left(u_U^{(2p+1)} - u_L^{(2p+1)} \right) = 0, \tag{22}$$

$$\frac{\partial}{\partial t}\int_{x_L}^{x_U}(u^2)\mathrm{d}x+\frac{2a}{(p+2)}\left(u_U^{(p+2)}-u_L^{(p+2)}\right)+\frac{2b}{(2p+2)}\left(u_U^{(2p+2)}-u_L^{(p+2)}\right)=0, \tag{23}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{x_L}^{x_U}\left(\frac{2au^{(p+2)}}{(p+1)(p+2)}+\frac{2bu^{(2p+2)}}{(2p+1)(2p+2)}-\mu u_x u_x\right)\mathrm{d}x\\&+\frac{a^2}{(p+1)^2}\left(u_U^{2(p+1)}-u_L^{2(p+1)}\right)+\frac{b^2}{(2p+1)^2}\left(u_U^{2(2p+1)}-u_L^{2(2p+1)}\right)\\&+\frac{2ab}{(p+1)(3p+2)}\left(u_U^{(3p+2)}-u_L^{(3p+2)}\right)+\frac{2ab}{(2p+1)(3p+2)}\left(u_U^{(3p+2)}-u_L^{(3p+2)}\right)=0,\end{aligned} \tag{24}$$

after simplifications. In these equations, $u_L=u(x_L,t)$ and $u_U=u(x_U,t)$ are time-invariant constants at the domain boundaries. The terms $[u_{xx}]_{x_L}^{x_U}$, $[uu_{xx}]_{x_L}^{x_U}$, $[u_xu_x]_{x_L}^{x_U}$, $[u^{(p+1)}u_{xx}]_{x_L}^{x_U}$, $[u^{(2p+1)}u_{xx}]_{x_L}^{x_U}$, $[u^pu_xu_x]_{x_L}^{x_U}$, $[u^{2p}u_xu_x]_{x_L}^{x_U}$, $[u_xu_{xxx}]_{x_L}^{x_U}$, and $[u_{xx}u_{xx}]_{x_L}^{x_U}$ are zero at the boundaries if we assume localized solutions or constant solutions at the domain boundaries. Equations (22–24) can now be integrated with respect to t to yield

$$C_1=\int_{x_L}^{x_U}u\mathrm{d}x+\frac{a}{(p+1)}\left(u_U^{(p+1)}-u_L^{(p+1)}\right)t+\frac{b}{(2p+1)}\left(u_U^{(2p+1)}-u_L^{(2p+1)}\right)t, \tag{25}$$

$$C_2=\int_{x_L}^{x_U}(u^2)\mathrm{d}x+\frac{2a}{(p+2)}\left(u_U^{(p+2)}-u_L^{(p+2)}\right)t+\frac{2b}{(2p+2)}\left(u_U^{(2p+2)}-u_L^{(p+2)}\right)t, \tag{26}$$

$$\begin{aligned}C_3=&\int_{x_L}^{x_U}\left(\frac{2au^{(p+2)}}{(p+1)(p+2)}+\frac{2bu^{(2p+2)}}{(2p+1)(2p+2)}-\mu u_x u_x\right)\mathrm{d}x+\frac{a^2}{(p+1)^2}\left(u_U^{2(p+1)}-u_L^{2(p+1)}\right)t\\&+\frac{b^2}{(2p+1)^2}\left(u_U^{2(2p+1)}-u_L^{2(2p+1)}\right)t+\frac{2ab}{(p+1)(3p+2)}\left(u_U^{(3p+2)}-u_L^{(3p+2)}\right)t\\&+\frac{2ab}{(2p+1)(3p+2)}\left(u_U^{(3p+2)}-u_L^{(3p+2)}\right)t.\end{aligned} \tag{27}$$

These invariants of motion are determined over a large but finite length spatial domain when the solution $u(x, t)$ of (2) is constant but not necessarily zero at the domain boundaries. The extra terms follow directly from the convection of mass, momentum, and energy into and out of the lower and upper boundaries of the spatial domain.

4 Conclusion

This approach for the derivation of these invariants of motion (25–27) is general and can be easily applied for finding similar invariants of motion for other general types of KdV, Gardner and Boussinesq equations. During the whole period of time in the course of which internal waves propagate inside the domain $[x_L, x_U]$, these invariants of motion remain conserved and equal to their original values that are well determined initially at $t = 0$. Note that during numerical computations that provide solutions to (2), C_1, C_2 , and C_3 can be calculated after each successive time step k over the entire spatial domain $x_L \leq x_j \leq x_U$ that contains the wave motion by computing discretized functionals such as the momentum $\sum_j u_j^k$, the energy $\sum_j [u_j^k]^2$, and the discrete Hamiltonian so that the conservation properties of the numerical algorithm can be monitored and thereby assessed. Moreover, the conservation of energy provides valuable information regarding the boundness of the solutions and the detection of the occurrence of solution blow-ups. These verification tools for the accuracy and stability of numerical schemes for the solution of similar evolutionary partial differential equations are implemented in method of lines solvers (Hamdi et al. 2001, 2005; Schiesser 1994).

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