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A robust false transient method of lines for elliptic partial differential equations

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HIGHLIGHTS

- A more robust method of solving elliptic PDEs is developed and discussed.
- A comparison of the false transient and the proposed method is explained.
- Several engineering/transport examples are considered.
- Linear solutions are described using matrix algebra and matrix exponentials.
- Nonlinear problems are solved, including unstable steady state solutions.

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ABSTRACT

Elliptic partial differential equations (PDEs) are frequently used to model a variety of engineering phenomena, such as steady-state heat conduction in a solid, or reaction-diffusion type problems. However, computing a solution can sometimes be difficult or inefficient using standard solvers. Techniques have been developed, including the method of lines (Schiesser, 1991), which can solve parabolic PDEs using well developed numerical solvers, but are not directly applicable to elliptic PDEs. The method of false transients overcomes this limitation by arbitrarily introducing a pseudo time derivative to modify the elliptic PDE to a parabolic PDE. However, this technique diverges for certain problems, such as when the solution is an unstable equilibrium point. A Jacobian-based perturbation approach is presented as an alternative for situations when the standard false-transient method fails. Two examples are shown to demonstrate the robustness of the proposed method over the false transient method.

1. Introduction

A wide variety of partial differential equations arise when describing engineering systems. For examples, variations on Laplace's equation arise frequently in problems of transport phenomena (Bird et al., 2006). In order to solve such a wide range of problems, several numerical methods have been developed to solve partial differential equations. The choice of method is dependent on the desired accuracy, as well as concerns about the stability and robustness of the system, while maintaining computational efficiency. Furthermore, these characteristics are dependent on the form of the partial differential equation to be solved, i.e. elliptic, parabolic, or hyperbolic. For parabolic equations such as the heat equation, several numerical methods exist that can be used to find a solution (Dehghan, 2006). For example, the method of lines is one such efficient routine in which the spatial dimensions are discretized using any of a number of techniques, such as finite difference, finite element, finite volume, or collocation methods (Berzins et al., 1989; Constantinides and Mostoufi, 1999; Cutlip and Shacham, 1998; Dehghan, 2006; Sadiku and Obiozor, 2000; Schiesser, 1991, 1994a, 1994b; Schiesser and Griffiths, 2009; Schiesser and Silebi, 1997; Taylor, 1999). This converts the partial differential equation (PDE) to an initial value problem (IVP) system of ordinary differential equations (ODE) or differential algebraic equations (DAEs). Software packages have been developed to specifically solve problems using the method of lines (Berzins et al., 1989). Alternatively, the resulting DAEs can be solved using standard efficient time integrators (Cash, 2005), including FORTRAN solvers such as DASKR or DASSL or in a computer algebra system such as Matlab (MathWorks, 2012)

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(dsolve), Maple (Maplesoft, 2012) (dsolve), Mathematica (Wolfram, 2012) (ndsolve), etc. The versatility and simplicity of the method of lines has led to its use in a wide range of engineering applications, including fracture problems (Bao et al., 2001), heat transfer (Labuzov and Potapov, 1985), solving Navier-Stokes equations (Erşahin et al., 2004) and electromagnetics (Pregla and Vietzorreck, 1995; Sadiku and Obiozor, 2000). Furthermore, Pregla and Cietzorreck used the method of lines in conjunction with the source method to handle inhomogeneous boundary conditions and discontinuities in microstrip lines and antennas (Pregla and Vietzorreck, 1995).

The solution of elliptic partial differential equations, such as Laplace's equation, is more difficult because there is not a simple way to convert the equations to an initial value problem to allow the use of the method of lines. A Newton–Raphson method, or another approach to solving a system of nonlinear equations, can be used if the system of algebraic equations resulting from the discretization is sufficiently well behaved and a reasonable initial guess is available. A semianalytical method of lines, valid for linear elliptic PDEs and certain quasilinear elliptic PDEs has been presented previously (Subramanian and White, 2004). However, a more popular choice has been the method of false transients, partially due to its ability to handle some nonlinear problems, and ease of implementation. In the false transient method the variables are discretized in the spatial or boundary value independent variables (x and y), and a pseudo time derivative is arbitrarily added to the problem statement (Mallinson and de Vahl Davis, 1973; Schiesser, 1991, 1994a; Schiesser and Griffiths, 2009; Schiesser and Silebi, 1997; White and Subramanian, 2010). The addition of this fictitious time derivative converts the elliptic PDE to a parabolic PDE and allows the solution to be determined by marching in pseudo time to a steady state condition. By doing this, the efficient IVP/DAE solvers can be applied in a matter analogous to the method of lines (Schiesser and Griffiths, 2009).

Like the method of lines, the method of false transients is used to solve a variety of engineering problems. For example, Xu, et al., used the false transient method to describe the concentration and temperature profiles of catalyst particles (Xu, 1993). This approach has also been used to numerically solve for three dimensional velocity profiles by solving the Navier-Stokes equation (Lo et al., 2005), as well as solving the convective diffusion equation for axial-diffusion problems in laminar-flow reactors (Nauman and Nigam, 2004). Other researchers have used the false transient method for analyzing mass transfer in porous media (Singh et al., 1999) or laminar film boiling (Srinivasan and Rao, 1984).

However, as shown in this paper, the system of ODE/DAEs resulting from the use of the false transient method can be unstable and may not converge to the desired (or any) solution. This problem can sometimes be rectified by modifying the form of the equations or boundary conditions using intuition and trial and error. In other cases, the system cannot be made to converge, regardless of how the problem is presented. An alternative, Jacobian-based perturbation approach is proposed in this paper, which is robust and does not suffer from the same stability issues which befall the false transient method. A similar approach has been used as a superior method for the initialization of the algebraic variables in systems of DAEs (Methekar et al., 2011).

2. Generic formulation of the false transient method and the perturbation method

Consider a general PDE of the form

$$D(\phi(\mathbf{x}))=0 \tag{1}$$

where $\phi(\mathbf{x})$ is the (continuous) dependent variable of interest, $\mathbf{x}$ is the vector of independent variables, and D is a generic linear differential operator with the form:

$$D=\sum_i\sum_j a_{ij}\frac{\partial^i}{\partial x_j^i} \tag{2}$$

Eq. (1) can be discretized using any of a number of techniques, such as finite difference, finite element, finite volume, or collocation, among others. This results in a system of algebraic equations of the form

$$\mathbf{g}(\mathbf{\Phi})=0 \tag{3}$$

where $\mathbf{\Phi}$ is the vector of the discretized dependent variables. In linear systems, Eq. (3) can be solved directly, though this is not the case in highly nonlinear problems. Both the method of false transients and the perturbation method introduce a pseudo time variable, τ, such that Eq. (3) is represented as:

$$\mathbf{g}(\mathbf{\Phi}(\tau))=0 \tag{4}$$

when using the method of false transients, this is done by introducing a first order pseudo-time derivative into Eq. (4) such that it becomes:

$$\mathbf{g}(\mathbf{\Phi}(\tau))=\frac{d\mathbf{\Phi}}{d\tau} \tag{5}$$

This allows the use of efficient time adaptive ODE solvers to be used. In order for convergence to occur, the right hand side must go to zero as τ goes to infinity:

$$\lim_{\tau\to\infty}\frac{d\mathbf{\Phi}}{d\tau}=0 \tag{6}$$

This reduces Eq. (5) to Eq. (3) and ensures that the original problem is satisfied. However, the method of false transients can fail if Eq. (6) does not hold, as can occur in an unstable system. Therefore, an alternative perturbation approach is shown here. A small perturbation parameter, ϵ, can be applied in time to Eq. (4) such that

$$\lim_{\epsilon\to 0}\mathbf{g}(\mathbf{\Phi}(\tau+\epsilon))=0 \tag{7}$$

Eq. (7) can be expanded using a Taylor series to give

$$\mathbf{g}(\mathbf{\Phi}(\tau))+\epsilon\frac{d\mathbf{g}(\mathbf{\Phi}(\tau))}{d\tau}+O(\epsilon^2)=0 \tag{8}$$

Assuming that ϵ is sufficiently small that the higher order terms can be neglected, Eq. (8) reduces to

$$\mathbf{g}(\mathbf{\Phi}(\tau))+\epsilon\frac{d\mathbf{g}(\mathbf{\Phi}(\tau))}{d\tau}=0 \tag{9}$$

The total derivative in Eq. (9) can be rewritten using the chain rule with partial derivatives

$$\mathbf{g}(\mathbf{\Phi}(\tau))+\epsilon\left[\frac{\partial\mathbf{g}}{\partial\mathbf{\Phi}}\frac{\partial\mathbf{\Phi}}{\partial\tau}+\frac{\partial\mathbf{g}}{\partial\tau}\right]=0 \tag{10}$$

Noting that $\partial\mathbf{g}/\partial\mathbf{\Phi}=\mathbf{J}$, where $\mathbf{J}$ is the Jacobian representing the algebraic system. Also, note that from Eq. (3), $\mathbf{g}$ is not a function of pseudo time directly; only indirectly through the dependent variables, $\mathbf{\Phi}$, are functions of pseudo time. Therefore, $\partial\mathbf{g}/\partial\tau=0$ above and Eq. (10), can be rearranged to give

$$\mathbf{g}(\mathbf{\Phi}(\tau))=-\epsilon\mathbf{J}\frac{\partial\mathbf{\Phi}}{\partial\tau} \tag{11}$$

Eq. (11) can be considered as an application of Davidenko's Method (Schiesser, 1994a). Note that the choice of ϵ is somewhat arbitrary, and must be chosen with consideration to the system. Ideally ϵ must be sufficiently small that the assumption that the higher order terms in Eq. (8) can be neglected is valid. Here, $\epsilon=10^{-3}$ is used throughout the remainder of this work. This choice is somewhat arbitrary as changing $\epsilon=10^{-3}$ by an order of magnitude in either direction does not affect the steady state results. Eq. (11) is similar to Eq. (5) given above for the method of false transients, and similarly allows for the use of efficient DAE solvers, although the right hand side consists of a linear combination of time derivatives of several of the dependent variables, $\mathbf{\Phi}$. The use of the Jacobian ensures that Eq. (11) is stable and more robust than Eq. (5). This will be shown for linear models using matrix algebra and considering the exponential matrix solution that Eq. (6) is always valid and Eq. (11) converges to Eq. (3) at infinite times irrespective of the initial conditions. The concepts can then be extended to nonlinear models by considering the eigenvalues of the resulting system of equations. In contrast, the false transient method may or may not converge to Eq. (3) depending on the eigenvalues of the Jacobian. This will be explained in more detail in a later section.

3. Implementation and comparison of the false transient method and the perturbation method

Several examples will be shown to compare the performance of the false transient method with the proposed Jacobian approach, as well as to note the conditions which cause failure of the method of false transients. The examples will be explored in 2-dimensional space in Cartesian coordinates, although extensions to other coordinate systems and to 3-dimensional space are appropriate and can be applied analogously. In this paper, the system of ODEs given in Eqs. (5) and (11) were written to a FORTRAN file and simulated using DASKR for computational efficiency. Furthermore, all symbolic calculations for the calculation of the Jacobian when using the perturbation approach were performed in Maple (Maplesoft, 2012).

3.1. Solving Laplace's equation

The simplest example to be considered is Laplace's equation, which is given in 2 dimensional rectangular coordinates as:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{12}$$

Laplace's equation is used in numerous engineering disciplines such as steady state heat/mass transfer or when calculating potential fields. The following boundary conditions are considered, as shown in Fig. 1.

$$\frac{\partial\phi(0,y)}{\partial x}=0 \tag{13}$$

$$\frac{\partial\phi(x,0)}{\partial y}=0 \tag{14}$$

$$\phi(1,y)=0 \tag{15}$$

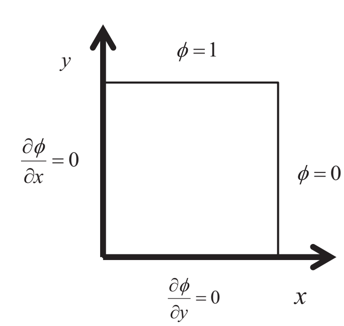

Fig. 1. Boundary conditions used for solving Example 1.

$$\phi(x,1)=1 \tag{16}$$

Notice that Eq. (16) is made to be non-homogeneous in order to avoid the trivial solution. This problem can be solved analytically using the standard separation of variables technique to yield:

$$\phi=\sum_{n=0}^{\infty}\frac{(-1)^n 4\cos(((2n+1)/2)\pi x)\cosh(((2n+1)/2)\pi y)}{\pi(2n+1)\cosh\left(((2n+1)/2)\pi\right)} \tag{17}$$

Since an analytical solution can be found only for limited cases (e.g. linear problems), Eq. (17) is used to benchmark the accuracy of the proposed approach.

A numerical solution can be found by discretizing Eq. (12) into M interior node points in x and N interior node points in y. This discretizes the domain into $(N+2)\times(M+2)$ node points when the surface points are considered. The following finite difference schemes of order h^2 are used:

$$\frac{\partial^2\phi(x,y)}{\partial x^2}\approx\frac{\phi_{m-1}^n-2\phi_m^n+\phi_{m+1}^n}{(\Delta x)^2} \tag{18}$$

$$\frac{\partial^2\phi(x,y)}{\partial y^2}\approx\frac{\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}}{(\Delta y)^2} \tag{19}$$

with m as the node index in the x-direction and n as the node index in the y-direction. When these approximations are applied to Eq. (12), the following equation is obtained for each interior node point, (m,n):

$$\frac{\phi_{m-1}^n-2\phi_m^n+\phi_{m+1}^n}{(\Delta x)^2}+\frac{\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}}{(\Delta y)^2}=0,\quad \text{for } m=1\ldots M,\quad n=1\ldots N \tag{20}$$

A second order forward finite difference is applied for the Neumann boundary conditions given in Eqs. (13) and (14):

$$\frac{\partial\phi(0,y)}{\partial x}\approx\frac{-\phi_2^n+4\phi_1^n-3\phi_0^n}{2\Delta x}=0,\quad \text{for } m=0\ldots M+1 \tag{21}$$

$$\frac{\partial\phi(x,0)}{\partial y}\approx\frac{-\phi_m^2+4\phi_m^1-3\phi_m^0}{2\Delta y}=0,\quad \text{for } m=0\ldots M+1 \tag{22}$$

The Dirichlet boundary conditions from Eqs. (15) and (16) can be expressed simply as

$$\phi_{M+1}^n=0,\quad \text{for } n=0\ldots N+1 \tag{23}$$

$$\phi_m^{N+1}=1,\quad \text{for } m=0\ldots M+1 \tag{24}$$

Eqs. (20)–(24) are a system of linear algebraic equations which can be solved trivially using a variety of solvers. However, for nonlinear systems which cannot be solved so simply, other methods must be utilized to arrive at a solution, and thus this is used as a verifiable test problem. When the method of false transients is applied to Eqs. (20)–(24) the following ordinary differential equations (ODEs) are obtained.

$$\frac{d\phi_n^m}{d\tau}=\frac{\phi_{m-1}^n-2\phi_m^n+\phi_{m+1}^n}{(\Delta x)^2}+\frac{\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}}{(\Delta y)^2},\quad \text{for } n=0\ldots N+1 \tag{25}$$

$$\frac{d\phi_0^n}{d\tau}=\frac{-\phi_2^n+4\phi_1^n-3\phi_0^n}{2\Delta x},\quad \text{for } n=0\ldots N+1 \tag{26}$$

$$\frac{d\phi_m^0}{d\tau}=\frac{-\phi_m^2+4\phi_m^1-3\phi_m^0}{2\Delta y},\quad \text{for } m=0\ldots M+1 \tag{27}$$

$$\frac{d\phi_{M+1}^n}{d\tau}=-\phi_{M+1}^n,\quad \text{for } n=0\ldots N+1 \tag{28}$$

$$\frac{d\phi_m^{N+1}}{d\tau} = 1 - \phi_m^{N+1}, \quad \text{for } m = 0\ldots M+1 \tag{29}$$

Note that the formulation of Eqs. (28) and (29) required a rearrangement of Eqs. (23) and (24) in order to develop stable ODEs which converge to the solution. In order to explain why such a rearrangement is necessary, recall that the following condition must be satisfied for convergence to occur:

$$\lim_{\tau\to\infty} \frac{d\phi_m^n}{d\tau} = 0 \tag{30}$$

Thus, Eqs. (25)–(29) reduce to Eqs. (20)–(24) at long psuedo times. However, if the method of false transients were applied directly to Eq. (23) to give:

$$\frac{d\phi_{M+1}^n}{d\tau} = \phi_{M+1}^n \tag{31}$$

The solution to the eigenfunction problem in Eq. (31) is an exponentially increasing function. Therefore, the resulting system of ODEs is unstable and Eq. (30) will not be satisfied. In this relatively simple example, the sign of Eq. (31) could simply be changed to ensure stability, as it can be determined to be unstable a priori. However, the instability may not be so obvious for more complicated problems, or the stability issue may not be fixed by simply changing the sign.

When the perturbation approach described above in Eq. (11) is applied to the system given in Eqs. (20)–(24) the following system of linearly coupled ODEs results

$$-\frac{\epsilon}{(\Delta y)^2}\left(\frac{d\phi_{m-1}^n}{d\tau} - 2\frac{d\phi_m^n}{d\tau} + \frac{d\phi_{m+1}^n}{d\tau}\right) - \frac{\epsilon}{(\Delta x)^2}\left(\frac{d\phi_m^{n-1}}{d\tau} - 2\frac{d\phi_m^n}{d\tau} + \frac{d\phi_m^{n+1}}{d\tau}\right)$$
$$= \frac{1}{(\Delta y)^2}\left(\phi_{m-1}^n - 2\phi_m^n + \phi_{m+1}^n\right) + \frac{1}{(\Delta x)^2}\left(\phi_m^{n-1} - 2\phi_m^n + \phi_m^{n+1}\right) \tag{32}$$

$$\frac{-\epsilon}{2\Delta x}\left(-\frac{d\phi_2^n}{d\tau} + 4\frac{d\phi_1^n}{d\tau} - 3\frac{d\phi_0^n}{d\tau}\right) = \frac{-\phi_2^n + 4\phi_1^n - 3\phi_0^n}{2\Delta x} \tag{33}$$

$$\frac{-\epsilon}{2\Delta y}\left(-\frac{d\phi_m^2}{d\tau} + 4\frac{d\phi_m^1}{d\tau} - 3\frac{d\phi_m^0}{d\tau}\right) = \frac{-\phi_m^2 + 4\phi_m^1 - 3\phi_m^0}{2\Delta y} \tag{34}$$

$$-\epsilon\frac{\phi_{M+1}^n}{d\tau} = \phi_{M+1}^n \tag{35}$$

$$-\epsilon\frac{\phi_m^{N+1}}{d\tau} = \phi_m^{N+1} - 1 \tag{36}$$

Eqs. (35) and (36) demonstrate the robustness of the perturbation method. Regardless of whether the boundary conditions are applied as Eqs. (23)and (24) or in the form required for the false transient solution, Eqs. (35) and (36) will converge to the expected solution. Considering Eqs. (32)–(36) in matrix form, as shown in Eq. (11) above, we have

$$-\epsilon\mathbf{J}\frac{d\mathbf{\Phi}}{d\tau} = \mathbf{J\Phi} + \mathbf{b} \tag{37}$$

for a linear system of equations. Eq. (37) can be explicitly solved for in the time derivatives to yield

$$\frac{d\mathbf{\Phi}}{d\tau} = -\epsilon^{-1}\mathbf{I\Phi} - \epsilon^{-1}\mathbf{J}^{-1}\mathbf{b} \tag{38}$$

which is unconditionally stable and will always converge to a solution.

Fig. 2 shows the converged 2-D numerical solution, as determined using 50 interior node points in x and y (for a total of 2704 points). Fig. 3 compares the solution found with a perturbation of $\epsilon = 0.001$ with the traditional method of false transients by showing the value of ϕ at $x=0$ and $y=0$ as a function of the pseudo time variable used in both methods. The proposed approach is superior because (1) steady state is achieved at shorter values of the dummy variable and (2) the method is robust, and is inherently stable as shown by Eq. (38).

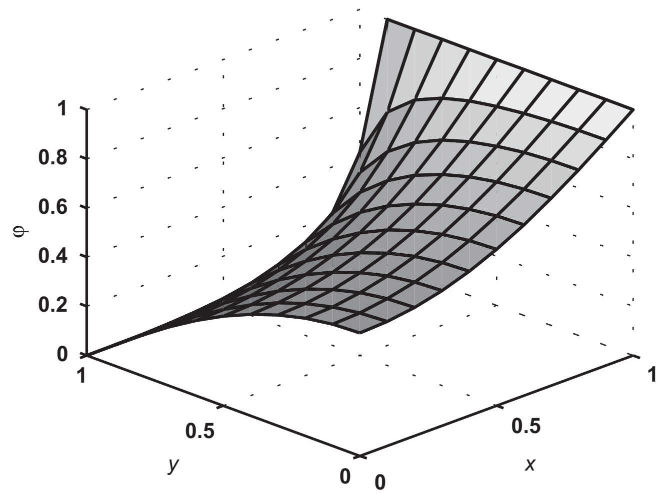

Fig. 2. Converged solution of Laplace's equation.

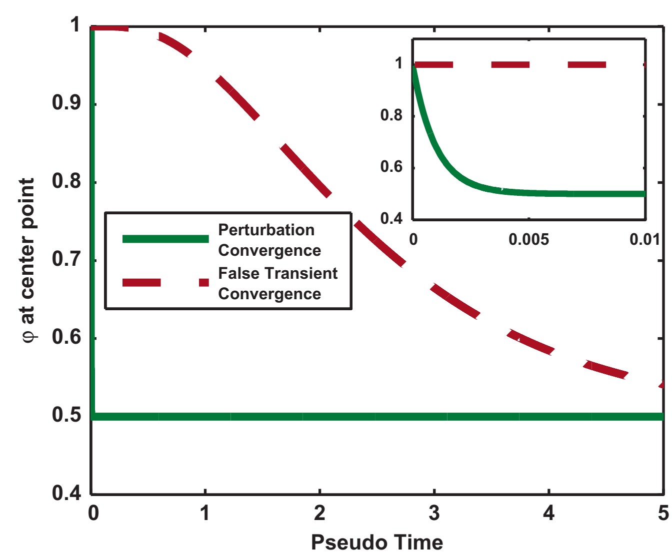

Fig. 3. Convergence of the perturbation method (solid line) and false transient method (dashed line) for Laplace's equation (Inset shows the graph at very short pseudo time).

3.2. Solving the Frank-Kamenetskii equation

The advantage of the proposed perturbation approach arises from its ability to handle nonlinearities and to solve problems with multiple steady stats. It is worth noting that this method can handle nonlinear source terms as well as nonlinearities in the state additive terms. However for demonstration purposes, only the Frank-Kamenetskii equation is considered, which has an exponential source term and exhibits multiple solutions. This is given by the following non-dimensional equation (Harley and Momoniat, 2008):

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \delta\,\exp(\phi) = 0 \tag{39}$$

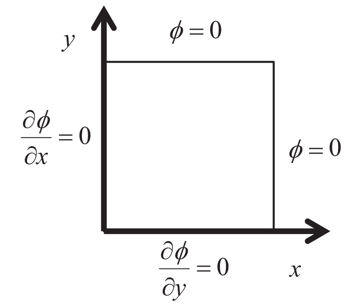

Fig. 4. Boundary conditions used for solving Example 2 (Frank-Kamenetskii equation).

where δ is referred to as the Frank-Kamenetskii parameter (Harley and Momoniat, 2008). This represents the dimensionless temperature when a zeroth order exothermic reaction occurs, while implicitly assuming that the reactant is being continuously fed. Note that the source term in Eq. (39) is derived from zeroth order Arrhenius kinetics for a reaction with sufficiently large activation energy such that some terms can be neglected. A more thorough derivation can be found in the literature (Harley and Momoniat, 2008). The following boundary conditions are used, and also shown in Fig. 4.

$$\frac{\partial\phi(0,y)}{\partial x}=0 \tag{40}$$

$$\frac{\partial\phi(x,0)}{\partial y}=0 \tag{41}$$

$$\phi(1,y)=0 \tag{42}$$

$$\phi(x,1)=0 \tag{43}$$

Note that it is not necessary to apply non-homogeneous boundary conditions for this case to analyze a non-trivial solution due to the nonlinear source term. Still, Eq. (39) cannot be solved analytically because of the nonlinearity. When the finite difference scheme used above is applied to this problem, the following system of non-linear algebraic equations is obtained:

$$\frac{\phi_{m-1}^{n}-2\phi_m^n+\phi_{m+1}^n}{(\Delta x)^2}+\frac{\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}}{(\Delta y)^2}+\delta\exp(\phi_m^n)=0,\quad \text{for } m=1...M,\quad n=1...N \tag{44}$$

$$\frac{\partial\phi(0,y)}{\partial x}\approx\frac{-\phi_2^n+4\phi_1^n-3\phi_0^n}{2\Delta x}=0,\quad \text{for } n=0...N+1 \tag{45}$$

$$\frac{\partial\phi(x,0)}{\partial y}\approx\frac{-\phi_m^2+4\phi_m^1-3\phi_m^0}{2\Delta y}=0,\quad \text{for } m=0...M+1 \tag{46}$$

$$\phi_{M+1}^n=0,\quad \text{for } n=0...N+1 \tag{47}$$

$$\phi_m^{N+1}=0,\quad \text{for } m=0...M+1 \tag{48}$$

Unlike the first two cases considered, this example results in a system of non-linear equations and cannot be solved using basic linear or non-linear solvers, such as Maple's built-in fsolve command. Standard numeric based solvers can also have trouble solving this system. Therefore, the method of false transients or the perturbation method is a good choice for finding the solution to this problem. Application of the false transient method gives the following system of ODEs:

$$\frac{d\phi_n^m}{d\tau}=\frac{\phi_{m-1}^n-2\phi_m^n+\phi_{m+1}^n}{(\Delta x)^2}+\frac{\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}}{(\Delta y)^2}+\delta\exp(\phi_n^m),\quad \text{for } m=1...M,\quad n=1...N \tag{49}$$

with the boundary conditions similar to in the previous cases. Conversely, the perturbation method gives:

$$-\frac{\epsilon}{(\Delta y)^2}\left(\frac{d\phi_{m-1}^n}{d\tau}-2\frac{d\phi_m^n}{d\tau}+\frac{d\phi_{m+1}^n}{d\tau}\right)-\frac{\epsilon}{(\Delta x)^2}\left(\frac{d\phi_m^{n-1}}{d\tau}-2\frac{d\phi_m^n}{d\tau}+\frac{d\phi_m^{n+1}}{d\tau}\right)$$
$$-\epsilon\delta\exp(\phi_m^n)\frac{d\phi_m^n}{d\tau}=\frac{1}{(\Delta y)^2}(\phi_{m-1}^n-2\phi_m^n+\phi_{m+1}^n)+\frac{1}{(\Delta x)^2}\left(\phi_m^{n-1}-2\phi_m^n+\phi_m^{n+1}\right)+\delta\exp(\phi_m^n),\quad \text{for } m=1...M,\quad n=1...N \tag{50}$$

This problem exhibits some interesting behavior. For example, for $\delta>\delta_{crit}$, there is no solution, while for $\delta<\delta_{crit}$ there exists two solutions. Fig. 5 shows the solution value(s) of ϕ at the origin for various values of δ as determined using the perturbation approach, demonstrating the multiple solutions of the problem. Note that the lower branch solution is a stable equilibrium point, while the upper branch solution is an unstable equilibrium point. When using the proposed approach, both stable and unstable solutions can be found depending upon the initial guesses used. However, it is not possible to find the upper branch solution using the method of false transients. If the initial guess provided is less than the upper branch solution, the false transient method will always converge to the stable lower branch solution. Conversely, if an initial guess is provided which is greater than the upper branch solution, the false transient method will diverge to infinity. This instability makes it impossible to track the upper branch solution by continuing from small values of δ using standard solving methods. An arc length approach can be used to trace the solution given in Fig. 5, by integrating all unknowns and all parameters across the arc length of the solution curve. However, that cannot be used to directly determine the solution profile for a given value of the parameter δ, as the parameter is solved as a function of arc length. Furthermore, such a method requires a two step predictor/corrector approach due to the nonlinearities, which increases complexity and computational cost.

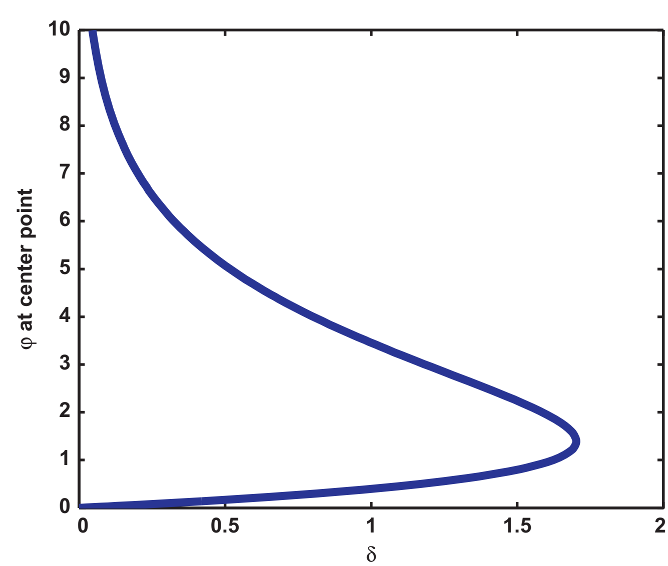

Fig. 5. Converged solution for ϕ located at the origin for various values of δ determined using the perturbation approach.

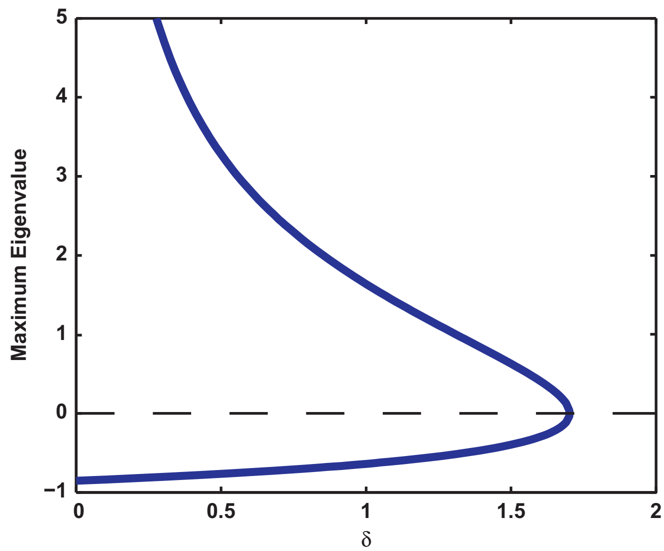

Fig. 6. Maximum eigenvalue of the Jacobian for the equilibrium solutions of the nonlinear problem. This indicates the stability of the lower branch solution, and the instability of the upper branch solution.

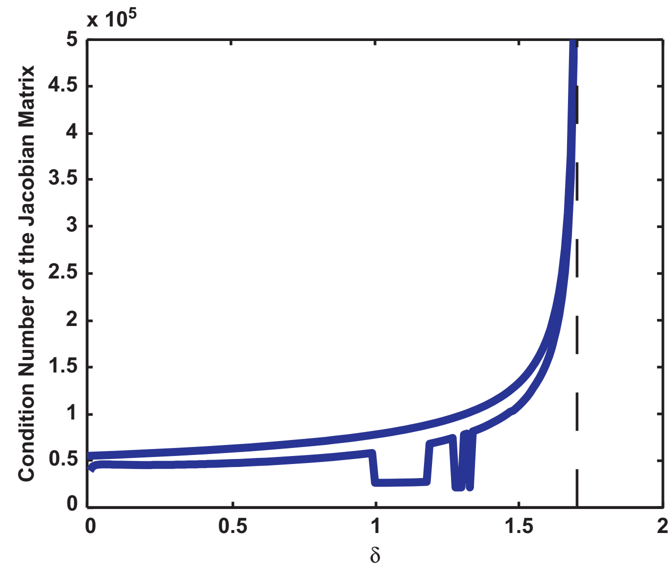

Fig. 7. Condition number of the Jacobian at various values of the parameter δ for the upper and lower branch solutions. The bifurcation point as δ approaches δ_{crit} is particularly ill-conditioned.

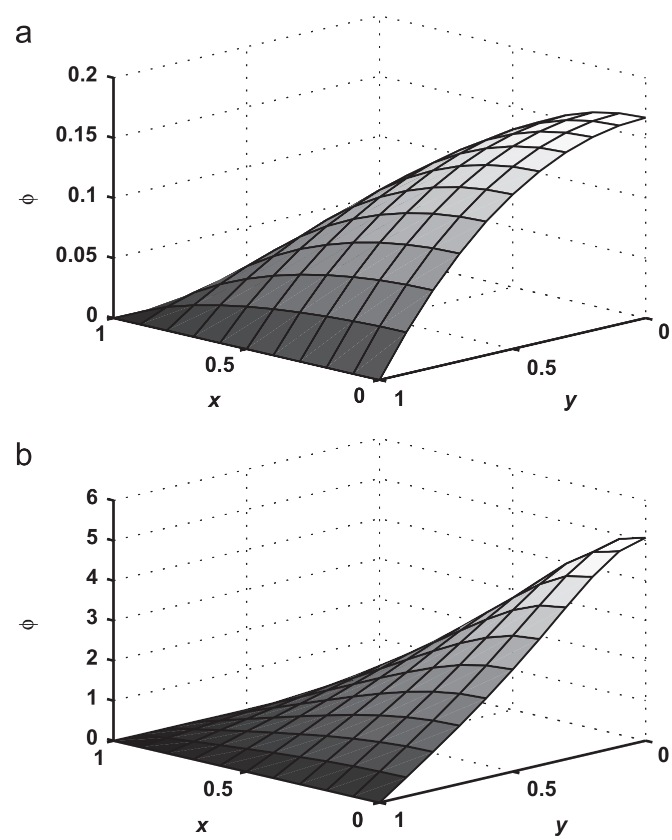

Fig. 8. Solution profiles when $\delta=0.5$ for (a) lower branch and (b) upper branch.

The difficulty observed in the convergence of the false transient method to the upper branch solution can be predicted by considering the Jacobian of the problem. For the lower branch solution, all the eigenvalues of the Jacobian are negative, indicating a stable equilibrium solution. In other words, the system of ODEs developed using the method of false transients will converge to the lower branch solution, even if the system is slightly perturbed from the steady state solution. In contrast, the upper branch solutions represent an unstable equilibrium point, as evidenced by the positive eigenvalues observed at those points. Graphically, this is shown in Fig. 6 which shows the maximum eigenvalue of the Jacobian for the various equilibrium points. Even though the upper branch solution does satisfy $d\mathbf{\Phi}/d\tau=0$, any deviation from equilibrium will cause the solution to diverge from the upper branch. If the deviation is above the upper branch solution, the instability will cause the solution to diverge to infinity. However, if the deviation is below the upper branch solution, the system will converge to the lower branch solution, a stable equilibrium point.

It is also worth noting the difficulty of finding the solution near the bifurcation point, when $\delta=\delta_{crit}$. At this point, the condition number of the Jacobian increases significantly at the solution points, indicating the system is particularly ill-conditioned as the parameter δ approaches its critical value. Interestingly, however, the Jacobian as computed from the upper branch solution is not significantly more ill-conditioned than the lower branch solution. This is shown graphically in Fig. 7. It is worth noting that other techniques, such as the arc-length tracking method can be used to better track the bifurcation of multiple steady states.

Fig. 8a shows the surface plot for the lower branch solution for the case that $\delta=0.5$, while Fig. 8b shows the profile for the upper branch solution. In order to show the importance of providing an initial guess as well as to compare convergence, both Fig. 9a and b shows the value of ϕ_0^0 as a function of pseudo-time for the false transient method and the perturbation approach when $\delta=0.5$. Fig. 9a uses an initial guess of $\phi_n^m=0$ for all m and n, while Fig. 9b uses the upper branch solution for $\delta=0.6$ as an initial guess for $\delta=0.5$. This is to show that the perturbation approach and the method of false transients will not necessarily converge to the same solution, even when identical initial conditions are applied. Furthermore, Fig. 9c shows the convergence when an initial guess of $\phi_n^m=3$ for all m and n. In this case, the perturbation approach converges to the upper branch, while the false transient method fails after a few pseudo-seconds of simulation. This further demonstrates the advantages of the purposed approach.

It also must be stated that the perturbation method will also not converge for certain initial conditions, such as for profiles significantly above the values in the upper branch solution. This is due to the presence of the exponential term which becomes unstable for large values of $\mathbf{\Phi}$. However, the proposed method is

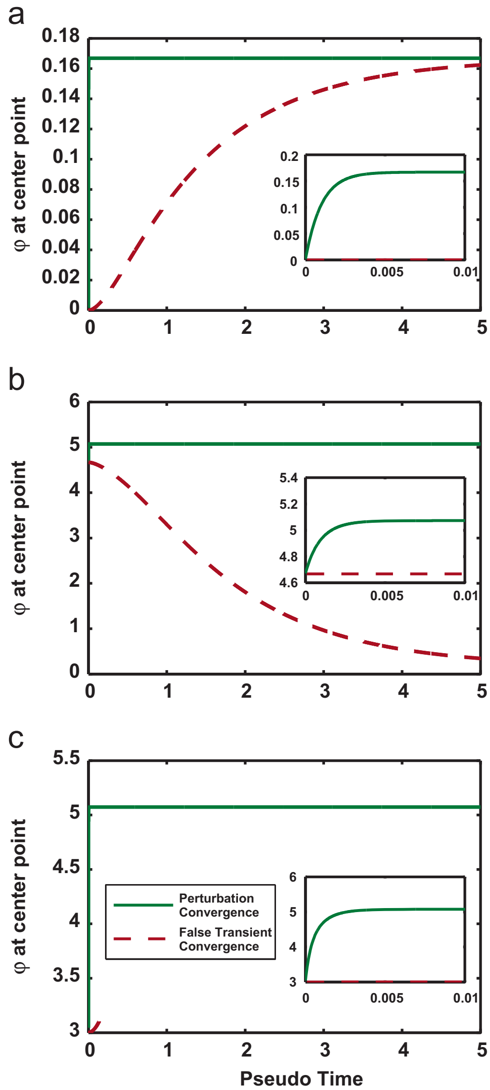

Fig. 9. (a) Convergence of Proposed method (solid line) and False Transient (dashed line) for a value of $\delta=0.5$, using $\phi=0$ as an initial condition (Inset shows the graph at very short pseudo time) (b) and (c) Same system with different initial guesses—the perturbation method converges to the upper branch solution while the false transient (b) converges to the lower branch solution (c) fails.

much more forgiving in that it will converge for a wider range of initial conditions than the false transient method.

4. Conclusions

A Jacobian-based perturbation approach was presented as an alternative to the method of false transients when solving elliptic PDEs. Both methods discretize the spatial variables using standard finite different schemes and introducing a pseudo time variable, although other discretization schemes, such as collocation, could be used. However, the perturbation approach is shown to converge to a meaningful solution for a wider range of problems and initial guesses than the method of false transients. Furthermore, when using the method of false transients, the equations must be carefully applied in such a way to ensure that the DAEs are stable and converge to the expected solution if possible. The proposed perturbation approach is much more robust and the equations can be applied in any logically consistent manner. Also, in cases where multiple solutions exist, the Jacobian-based perturbation approach is more capable of finding the multiple solutions, specifically those which represent unstable equilibrium points. In contrast, the false transient method may only converge to a stable solution regardless of the initial guesses used. It is important to note that there are many methods to solve elliptic PDEs. The objective of this paper is to make the false transient method more robust. Comparing other numerical approaches to solve such problems is beyond the scope of this paper.

The primary difficulty of the proposed approach arises from the calculation of the Jacobian of the system of equations. This requires symbolic calculations that are not trivial and require the use of a computer algebra system. In contrast, the method of false transients can be applied relatively easily to any system of equations. We believe that this has contributed to the popularity of the method of false transients in the past, despite some of the shortcomings of the method, some of which have been discussed above. Additionally, the resulting system of ODEs is not necessarily in an explicit form (one derivative in each ODE), which may be difficult for standard or library solvers to handle. As DAE solvers and computer algebra systems like Maple (Maplesoft, 2012) or Mathematica (Wolfram, 2012) are becoming more common and more efficient, the perturbation approach is a viable alternative for solving elliptic PDEs in a robust manner. Source codes for the problems discussed will be posted online in the corresponding author's website after the publication of the article.

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