Physics - Doklady, Vol. 39, No. 7, 1994, pp. 534 - 537. Translated from Doklady Akademii Nauk, Vol. 337, No. 2, 1994, pp. 196 - 199.

MECHANICS

The Algebraic Method for Integration of the Differential Equations of Nonlinear Mechanics

A. D. Polyanin and A. I. Zhurov

Presented by Academician D.M. Klimov on January 25, 1994

Received January 27, 1994

We propose an algebraic method of finding exact analytical solutions to nonlinear ordinary differential equations (sets of equations) and equations of nonlinear mechanics associated therewith. The method is based on direct specification of the structure of a solution in a parametric form, including its dependence on arbitrary constants and a number of indeterminate parameters and functions to be found later from other differential equations by methods of computer algebra [1]. We present some particular examples illustrating the potential scope of the proposed method. The approach indicated makes it possible to find new integrable equations unsolvable by other methods [2 - 6].

1. Preliminary remarks. There are relatively few ordinary differential equations known to admit exact analytical (general) solutions that can be written out in the explicit form

$$y = y(x; C_1, \dots, C_n) \tag{1}$$

or

$$x = x(y; C_1, \dots, C_n), \tag{2}$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants [2, 3]. For example, some fairly simple and well studied linear equations, Bernoulli equations, and others have solutions of the form (1).

Equations having exact analytical solutions that admit parametric representation of the form

$$x = x(t; C_1, \dots, C_n), \quad y = y(t; C_1, \dots, C_n) \tag{3}$$

are by far more numerous. For example, Abelian equations, Emden–Fowler equations, the equations of combustion theory and theory of chemical reactors, the equations of heat and mass transfer, the equations of boundary layers in non-Newtonian fluids, and others have such solutions in certain cases [5, 6].

2. Description of the algebraic method. We will seek a general solution to the family of differential equations of the polynomial type (with coefficients being polynomials in both dependent and independent variables) characterized by "free" parameters A_s. We specify that the solution will have a parametric form; it must contain a parameter t, arbitrary integration constants $C_1, C_2, \dots, C_n$ and "indeterminate" coefficients $a_m, b_m, m = 1, 2, \dots, M$. Examination of the particular results presented in the books [5, 6] shows that it is expedient to seek the solution in the form of polynomials (possibly containing fractional powers) or quotients of polynomials in the arbitrary constants $C_1, \dots, C_n$.

For clarity, we further restrict ourselves to the case of a first-order equation. We specify the dependence of its solution on the parameter t in the form

$$x = x(f_1, \dots, f_k; C), \quad y = y(f_1, \dots, f_k; C), \tag{4}$$

using a set of functions $f_i = f_i(t)$ to be determined in the course of analysis. For simplicity, we have not indicated the dependence of x and y on a_m and b_m. We *a priori* prescribe the structure of the right-hand sides of (4) in terms of f_i and C (e.g., x and y are linear in f_i and are functions of powers of C; see the example). Substituting solution (4) into the initial equation and collecting the terms with the same powers of constant C, we obtain

$$\sum_n C^n \left(\sum_l K_{nl} \Psi_{nl} \right) = 0, \tag{5}$$

where $K_{nl} = K_{nl}(A_s, a_m, b_m)$ are independent of f_i and C, and Ψ_{nl} depend on the functions f_i and their derivatives. We select the functions f_i subject to the condition that equation (5) must contain the least possible number of linearly independent Ψ_{nl}. If all Ψ_{nl} are linearly independent, then, to satisfy (5), we should solve the defining set of equations

$$K_{nl}(A_s, a_m, b_m) = 0. \tag{6}$$

We can readily use the methods of computer algebra [1] to solve set (6), which may contain a large number of unknowns. In this case, it is expedient to treat the quantities entering (6) linearly as the desired ones and express them in terms of the remaining ones (which may enter it nonlinearly).

3. Examples. To illustrate the efficiency of the outlined method, let us consider the following

Institute of Problems of Mechanics, Russian Academy of Sciences, pr. Vernadskogo 101, Moscow, 117526 Russia

10-parameter nonlinear ordinary differential equation of the first-order:

$$(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)\,y'_x$$
$$= B_{22}x^2 + B_{12}xy + B_{11}y^2 + B_2x + B_1y, \qquad (7)$$

frequently encountered in the theory of dynamical systems of the second order [7 - 9]. It is also important to note that we can reduce equation (7) to the Abelian equations of the second kind and equations of combustion theory, theory of chemical reactors, and theory of nonlinear oscillations associated therewith [5, 6].

We seek a solution in the parametric form

$$x = a_1C^mf(t) + a_2Cg(t), \quad y = b_1C^mf(t) + b_2Cg(t), \quad (8)$$

where C is an arbitrary constant, and functions f and g, the values of m, and the "indeterminate" coefficients a_1, a_2, b_1, and b_2 are to be determined in course of solving the problem.

We substitute expressions (8) into equation (7), using the identity $y'_x = y'_t/x'_t$. After collecting terms with the same powers of integration constants C, we obtain

$$K_1C^{3m}\varphi_1 + C^{2m+1}(K_2\varphi_2 + K_3\varphi_3) + K_4C^{2m}\varphi_4$$
$$+ C^{m+2}(K_5\varphi_5 + K_6\varphi_6) + C^{m+1}(K_7\varphi_7 + K_8\varphi_8)$$
$$+ K_9C^3\varphi_9 + K_{10}C^2\varphi_{10} = 0. \qquad (9)$$

Here, coefficients $K_i = K_i(A_{ij}, B_{ij}; A_i, B_i; a_1, a_2; b_1, b_2)$ are independent of C and t and linear in the parameters $A_{ij}, B_{ij}; A_i$, and B_i of equation (7), and $\varphi_i = \varphi_i(t)$ depend on f and g as follows (the prime indicates differentiation with respect to t):

$$\varphi_1 = f^2f', \quad \varphi_2 = f^2g', \quad \varphi_3 = gff', \quad \varphi_4 = ff',$$
$$\varphi_5 = fgg', \quad \varphi_6 = g^2f', \quad \varphi_7 = fg', \quad \varphi_8 = gf', \quad (10)$$
$$\varphi_9 = g^2g', \quad \varphi_{10} = gg'.$$

If all φ_i are linearly independent, then we satisfy (9) by setting $K_i = 0, i = 1, 2, \ldots, 10$. As a result, we obtain a set of 10 linear homogeneous equations for the 10 unknowns A_{ij}, B_{ij}, A_i, and B_i, which admits only the trivial (zero) solution in the nondegenerate case. Equation (9) admits a nontrivial solution (for $A_{ij}, B_{ij}; A_i$, and B_i), if at least two functions, φ_i and φ_j entering it as the coefficients at the same powers of C are linearly dependent (in this case, the number of equations is smaller than the number of unknowns).

For example, in the case of $m = 3/2$, C^3 enters (9) with functions φ_4 and φ_9. Requiring that $\varphi_4 = \text{const} \cdot \varphi_9$ and using (10), we obtain an equation for f and g: $ff' = \text{const} \cdot g^2g'$. After integration, we have $f^2 = \alpha g^3 + \beta$, where α and β are arbitrary constants. We can set $f = \sqrt{\alpha g^3 + \beta}$ and $g = t$ without loss of generality (it is always possible to set either f or g equal to t by reparametrization, if $f'g' \neq 0$). Searching through the values of parameter m and treating other functions φ_i at the same powers of C as linearly dependent (proportional), we obtain alternative equations for f and g. Table 1 summarizes the results of the analysis (we have omitted the most cumbersome formulas).

Let us consider in some detail Case 1 (see the table), which corresponds, by virtue of (7), to a solution of the form

$$x = a_nt^n + a_1Ct, \quad y = b_nt^n + b_1Ct, \qquad (11)$$

where n, a_n, a_1, b_n, and b_1 are some "indeterminate" coefficients. We substitute expressions (11) into equation (7). After collecting terms with the same powers of t and C, we obtain

$$\Lambda_{3n-1}t^{3n-1} + \Lambda_{2n}Ct^{2n} + \Lambda_{2n-1}t^{2n-1} + \Lambda_{n+1}C^2t^{n+1}$$
$$+ \Lambda_nCt^n + \Lambda_2C^3t^2 + \Lambda_1C^2t = 0, \qquad (12)$$

where the coefficients $\Lambda_k = \Lambda_k(A_{ij}, B_{ij}; A_i, B_i; a_n, a_1; b_n, b_1; n)$ are of the following form:

$$\Lambda_{3n-1} = n(b_n^3A_{22} + b_n^2a_nA_{12} + b_na_n^2A_{11}$$
$$- a_n^3B_{22} - a_n^2b_nB_{12} - a_nb_n^2B_{11}),$$
$$\Lambda_{2n} = b_n^2b_1(2n+1)A_{22} + a_n(a_nb_1 + 2na_1b_n)A_{11}$$
$$+ b_n[(n+1)a_nb_1 + na_1b_n]A_{12} - a_n^2a_1(2n+1)B_{22}$$
$$- b_n(b_na_1 + 2b_1a_nn)B_{11}$$
$$- a_n[(n+1)b_na_1 + nb_1a_n]B_{12}, \qquad (13)$$
$$\Lambda_{2n-1} = n(b_n^2A_2 + b_na_nA_1 - a_n^2B_2 - a_nb_nB_1),$$
$$\Lambda_{n+1} = b_nb_1^2(n+2)A_{22} + a_1(na_1b_n + 2a_nb_1)A_{11}$$
$$+ b_1[a_nb_1 + (n+1)a_1b_n]A_{12} - a_1^2a_n(n+2)B_{22}$$
$$- b_1(nb_1a_n + 2b_na_1)B_{11} - a_1[b_na_1 + (n+1)b_1a_n]B_{12},$$
$$\Lambda_n = (n+1)b_nb_1A_2 + (a_nb_1 + na_1b_n)A_1$$
$$- (n+1)a_na_1B_2 - (b_na_1 + nb_1a_n)B_1,$$
$$\Lambda_2 = b_1^3A_{22} + b_1^2a_1A_{12} + b_1a_1^2A_{11}$$
$$- a_1^3B_{22} - a_1^2b_1B_{12} - a_1b_1^2B_{11},$$
$$\Lambda_1 = b_1^2A_2 + b_1a_1A_1 - a_1^2B_2 - a_1b_1B_1.$$

In this case, the defining set consists of seven equations

$$\Lambda_k(A_{ij}, B_{ij}; A_i, B_i; a_n, a_1; \; b_n, b_1; \; n) = 0, \qquad (14)$$

which are linear in the coefficients A_{ij}, B_{ij}, A_i, and B_i of equation (7) [in view of expressions (13)].

Assuming parameters $A_{22}, B_{22}, A_2, a_n, a_1, b_n, b_1$, and n to be arbitrary, we solve (14) for the remaining parameters, A_{12}, A_{11}, B_{12}, B_{11}, A_1, B_1, and B_2, by the methods of computer algebra, using the symbolic calculus implemented in the *Reduce* package [1]. We introduce parameters p, q, and r, instead of A_{22}, B_{22} and

Table 1. Exponents m and functions f and g, for which equation (7) admits a solution of the form (8)

No.	m	f	g
1	0	t^n	t
2	0	$\ln\|t\|$	t
3	0	$1/\ln\|t\|$	t
4	0	$t^n + \beta$	t
5	0	t	$\|t\|^n\|\alpha t + \beta\|^k$
6	1	t^n	t
7	1	$tP_\alpha^n P_\beta^k$	$P_\alpha^n P_\beta^k$
8	1	$tQ_2^n \psi(t)$	$Q_2^n \psi(t)$
9	3/2	t	$(\alpha t^3 + \beta)^{1/2}$
10	2	t	$(\alpha t^n + \beta t)^{1/2}$
11	2	$t^n + \beta t^2$	t

Note: $P_\alpha = |\alpha_1 t + \alpha_0|$, $P_\beta = |\beta_1 t + \beta_0|$, $Q_2 = \alpha t^2 + \beta t + \gamma$,

$\psi(t) = \exp\left(k \arctan \dfrac{2\alpha t + \beta}{\Delta^{1/2}}\right)$, $\Delta = 4\alpha\gamma - \beta^2 > 0$.

A_2, to obtain the solution in a simpler form. As a result, we have

$$\begin{gathered}
A_{22} = (n-1)\, a_1 a_n p, \\
A_{12} = (a_1 b_n - n a_n b_1)\, p - (n-1)\, a_1 a_n q, \\
A_{11} = (n a_n b_1 - a_1 b_n)\, q, \quad A_2 = (n-1)\, a_1 a_n r, \\
A_1 = (a_1 b_n - n a_n b_1)\, r, \quad B_{22} = (n-1)\, b_1 b_n q, \\
B_{12} = -(n-1)\, b_1 b_n p + (b_1 a_n - n b_n a_1) q, \\
B_{11} = (n b_n a_1 - b_1 a_n)\, p, \quad B_2 = -(n-1)\, b_1 b_n r, \\
B_1 = -(b_1 a_n - n b_n a_1) r.
\end{gathered} \tag{15}$$

Formulas (15) define an 8-parameter family of nonlinear differential equations (6) with arbitrary parameters p, q, r, a_n, a_1, b_n, b_1, and n, which admits an analytical solution of the form (11), where C is an arbitrary constant.

As an additional illustration of the method, we present the final results for Cases 2, 3, 9, and 11 listed in Table 1.

With

$$\begin{gathered}
A_{22} = a_1 a_2 p, \quad A_{12} = -b_1 a_2 p - a_1 a_2 q, \quad A_{11} = b_1 a_2 q, \\
A_2 = a_1 (a_1 b_2 - a_2 b_1)\, p, \quad A_1 = -a_1 (a_1 b_2 - a_2 b_1)\, q, \\
B_{22} = b_1 b_2 q, \quad B_{12} = -b_1 b_2 p - a_1 b_2 q, \quad B_{11} = a_1 b_2 p, \\
B_2 = -b_1 (a_1 b_2 - a_2 b_1)\, q, \quad B_1 = b_1 (a_1 b_2 - a_2 b_1)\, p
\end{gathered} \tag{16}$$

the solution to equation (7) has the form

$$x = a_1 \ln|t| + a_2 C t, \quad y = b_1 \ln|t| + b_2 C t, \tag{17}$$

where a_1, a_2, b_1, b_2, p, and q are arbitrary parameters.

With

$$\begin{gathered}
A_{22} = a_1 a_2^2, \quad A_{12} = -2a_1 a_2 b_2, \quad A_{11} = a_1 a_2^2, \\
A_2 = -a_1 a_2 (a_1 b_2 - a_2 b_1), \quad A_1 = -b_1 a_2 (a_1 b_2 - a_2 b_1), \\
B_{22} = b_1 b_2^2, \quad B_{12} = -2b_1 b_2 a_2, \quad B_{11} = b_1 a_2^2, \\
B_2 = b_1 b_2 (a_1 b_2 - a_2 b_1), \quad B_1 = a_1 b_2 (a_1 b_2 - a_2 b_1)
\end{gathered} \tag{18}$$

the solution to equation (7) has the form

$$x = a_1 \frac{1}{\ln|t|} + a_2 C t, \quad y = b_1 \frac{1}{\ln|t|} + b_2 C t, \tag{19}$$

where a_1, a_2, b_1, and b_2 are arbitrary parameters.

With

$$\begin{gathered}
A_{22} = 3\alpha a_1^3, \quad A_{12} = -6\alpha a_1^2 b_1, \quad A_{11} = 3\alpha a_1 b_1^2, \\
A_2 = -2a_2^2 (a_1 b_2 - a_2 b_1), \quad A_1 = 2a_2 b_2 (a_1 b_2 - a_2 b_1), \\
B_{22} = 3\alpha b_1^3, \quad B_{12} = -6\alpha b_1^2 a_1, \quad B_{11} = 3\alpha b_1 a_1^2, \\
B_2 = -2b_2^2 (b_1 a_2 - b_2 a_1), \quad B_1 = 2b_2 a_2 (b_1 a_2 - b_2 a_1),
\end{gathered} \tag{20}$$

the solution to equation (7) has the form

$$\begin{gathered}
x = a_1 C^3 t + a_2 C^2 \sqrt{\alpha t^3 + \beta}, \\
y = b_1 C^3 t + b_2 C^2 \sqrt{\alpha t^3 + \beta},
\end{gathered} \tag{21}$$

where a_1, a_2, b_1, b_2, α, and β are arbitrary parameters. For convenience, we have replaced C by C^2 in (21).

With

$$\begin{gathered}
A_{22} = (n-2)\, a_1^3 \beta, \quad A_{12} = -2(n-2)\, a_1^2 b_1 \beta, \\
A_{11} = (n-2)\, a_1 b_1^2 \beta, \\
A_2 = (n-1)\, (a_1 b_2 - a_2 b_1)\, a_1 a_2, \\
A_1 = (n a_1 b_2 - a_2 b_1)\, (a_1 b_2 - a_2 b_1), \\
B_{22} = (n-2)\, b_1^3 \beta, \quad B_{12} = -2(n-2)\, b_1^2 a_1 \beta, \\
B_{11} = (n-2)\, b_1 a_1^2 \beta, \\
B_2 = (n-1)\, (b_1 a_2 - b_2 a_1)\, b_1 b_2, \\
B_1 = (n b_1 a_2 - b_2 a_1)\, (b_1 a_2 - b_2 a_1)
\end{gathered} \tag{22}$$

the solution to equation (7) has the form

$$\begin{gathered}
x = a_1 C^2 (t^n + \beta t^2) + a_2 C t, \\
y = b_1 C^2 (t^n + \beta t^2) + b_2 C t,
\end{gathered} \tag{23}$$

where a_1, a_2, b_1, b_2, n, and β are arbitrary parameters.

Note that, apart from solutions of the form (8), equation (7) has other solutions corresponding to the appropriate values of coefficients A_{ij}, B_{ij}, A_i, and B_i. In particular, with

$$\begin{aligned} &A_{22} = 0, \quad A_{12} = -na_1c_n, \quad A_{11} = nb_1c_n, \\ &A_2 = (n-1)\,a_na_1, \quad A_1 = -na_nb_1 + a_1b_n, \\ &B_{22} = 0, \quad B_{12} = nb_1c_n, \quad B_{11} = -na_1c_n, \\ &B_2 = -(n-1)\,b_nb_1, \quad B_1 = nb_na_1 - b_1a_n \end{aligned} \tag{24}$$

the solution to equation (7) has the form

$$x = \frac{a_nt^n + a_1Ct}{c_nt^n + c_0C}, \quad y = \frac{b_nt^n + b_1Ct}{c_nt^n + c_0C}, \tag{25}$$

where n, a_n, a_1, b_n, b_1, c_n, and c_0 are arbitrary parameters.

REFERENCES

1. Klimov, D.M. and Rudenko, V.M., *Metody Komp'yuternoi Algebry v Zadachakh Mekhaniki* (Methods of Computer Algebra for Problems in Mechanics), Moscow: Nauka, 1989.
2. von Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen*, I. Gewöhnliche Differentialgleichungen, Leipzig: 1959.
3. Murphy, G.M., *Ordinary Differential Equations and Their Solutions*, New York: D. Van Norstrand, 1960.
4. Zwillinger, D., *Handbook of Differential Equations*, San Diego: Academic, 1989.
5. Zaitsev, V.F. and Polyanin, A.D., *Spravochnik po Nelineinym Differentsial'nym Uravneniyam: Prilozheniya v Mekhanike, Tochnye Resheniya* (Handbook of Nonlinear Differential Equations: Applications in Mechanics, Exact Solutions), Moscow: Nauka, 1993.
6. Zaitsev, V.F. and Polyanin, A.D., *Discrete Group Methods for Integrating Equations of Nonlinear Mechanics*, Boca Raton: CRC, 1993.
7. Andronov, A.A., Leontovich, E.A., Gordon, I.I., and Maier, A.G., *Kachestvennaya Teoriya Dinamicheskikh Sistem Vtorogo Poryadka* (The Qualitative Theory of Dynamical Systems of the Second Order), Moscow: Nauka, 1966.
8. Bautin, N.N. and Leontovich, E.A., *Metody i Priemy Kachestvennogo Issledovaniya Dinamicheskikh Sistem na Ploskosti* (Methods and Techniques of Qualitative Analysis of Dynamical Systems of the Second Order in a Plane), Moscow: Nauka, 1976.
9. Andronov, A.A., Vitt, A.A., and Khaikin, S.E., *Teoriya Kolebanii* (The Theory of Oscillations), Moscow: Nauka, 1981.