ORDINARY DIFFERENTIAL EQUATIONS

Handbook of Exact Solutions

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1.4.3. Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x$*

Preliminary comments.

1. For $A_{22} = 0$, this is the Abel equation (see Subsection 1.3.4). For $B_{11} = 0$ this is the Abel equation with respect to $x = x(y)$.

2. The transformation $\xi = y/x$, $w = 1/x$ leads to the Abel equation of the second kind:

$$\{[A_2\xi^2 + (A_1 - B_2)\xi - B_1]w + A_{22}\xi^3 + (A_{12} - B_{22})\xi^2 + (A_{11} - B_{12})\xi - B_{11}\}w'_\xi = (A_2\xi + A_1)w^2 + (A_{22}\xi^2 + A_{12}\xi + A_{11})w.$$

3. In Paragraph 3 of Subsection 1.4.4, another transformation is given which reduces the original equation to the Abel equation of the second kind.

4. Dynamical systems of the second order

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y) \tag{1}$$

which describe the behavior of the simplest Lagrangian and Hamiltonian systems in mechanics are often reduced to equations of the considered type when

$$\begin{aligned} P(x,y) &= f(x,y)(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x), \\ Q(x,y) &= f(x,y)(B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x), \end{aligned} \tag{2}$$

where $f = f(x,y)$ is an arbitrary function.

In particular, dynamical systems (1) with functions (2) and $f \equiv 1$ are met with in analyzing complex equilibrium states. In this case, functions P and Q are substituted by their Taylor-series expansions in the vicinity of the equilibrium state $x = y = 0$ with the first and second order terms retained.

When obtained the solution of the ordinary differential equation

$$(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x$$

in the parametric form $x = x(u, C_1)$, $y = y(u, C_1)$, the solution of the system (1), (2) is determined by the formulae

$$x = x(u, C_1), \quad y = y(u, C_1), \quad t = \int \frac{x'_u\,du}{P\bigl(x(u,C_1),\, y(u,C_1)\bigr)} + C_2.$$

The latter relation defines the implicit dependence of parameter u on t: $u = u(t, C_1, C_2)$, and makes it possible to find, with the aid of two former formulae, the dependence of x and y on t.

1. $(y^2 - x^2 + ay)y'_x = y^2 - x^2 + ax$.

Solution in the parametric form:

$$x = at + C|t|^{-1}e^{4t}, \quad y = -at + C|t|^{-1}e^{4t}.$$

* This section was written with A.I. Zhurov

2. $(y^2 - x^2 + ay)y'_x = 2y^2 - 2xy + ay$.

Solution in the parametric form:

$$x = t + Ct^2 e^{a/t}, \quad y = Ct^2 e^{a/t}.$$

3. $(y^2 - x^2 + ay - ax)y'_x = y^2 - x^2 - ay + ax$.

Solution in the parametric form:

$$x = at + Ce^{2t}, \quad y = -at + Ce^{2t}.$$

4. $(y^2 - x^2 + ay + 2ax)y'_x = y^2 - x^2 + 2ay + ax$.

Solution in the parametric form:

$$x = -at + C|t|^3 e^{4t}, \quad y = at + C|t|^3 e^{4t}.$$

5. $(y^2 - x^2 + ay + 2ax)y'_x = 2xy - 2x^2 + ay + 2ax$.

Solution in the parametric form:

$$x = t + Ct^{-2} e^{-a/t}, \quad y = -2t + Ct^{-2} e^{-a/t}.$$

6. $(y^2 - x^2 + ay - 2ax)y'_x = 4y^2 - 6xy + 2x^2 + ay - 2ax$.

Solution in the parametric form:

$$x = \tfrac{1}{3}t + C|t|^{2/3} e^{a/t}, \quad y = \tfrac{2}{3}t + C|t|^{2/3} e^{a/t}.$$

7. $(y^2 - x^2 + ay + 3ax)y'_x = -y^2 + 4xy - 3x^2 + ay + 3ax$.

Solution in the parametric form:

$$x = \tfrac{1}{2}t + C|t|^{-1} e^{-a/t}, \quad y = -\tfrac{3}{2}t + C|t|^{-1} e^{-a/t}.$$

8. $(y^2 - xy + ay + ax)y'_x = xy - x^2 + ay + ax$.

Solution in the parametric form:

$$x = -t + C|t|^{-1} e^{a/t}, \quad y = t + C|t|^{-1} e^{a/t}.$$

9. $(y^2 - xy + ay + ax)y'_x = y^2 - xy + 2ay$.

Solution in the parametric form:

$$x = -at + Ct^2 e^t, \quad y = Ct^2 e^t.$$

10. $(y^2 - xy + ay - 2ax)y'_x = 3y^2 - 5xy + 2x^2 + ay - 2ax$.

Solution in the parametric form:

$$x = \tfrac{1}{2}t + C|t|^{1/2} e^{a/t}, \quad y = t + C|t|^{1/2} e^{a/t}.$$

11. $(y^2 + xy - 2x^2 + ay + ax)y'_x = y^2 + xy - 2x^2 + 2ax.$

Solution in the parametric form:

$$x = at + Ct^{-2}e^{9t}, \quad y = -2at + Ct^{-2}e^{9t}.$$

12. $(y^2 + xy - 2x^2 + ay + ax)y'_x = 2y^2 - xy - x^2 + ay + ax.$

Solution in the parametric form:

$$x = t + C|t|^3 e^{a/t}, \quad y = -t + C|t|^3 e^{a/t}.$$

13. $(y^2 + xy - 2x^2 + ay - ax)y'_x = y^2 + xy - 2x^2 - 2ay + 2ax.$

Solution in the parametric form:

$$x = at + Ce^{3t}, \quad y = -2at + Ce^{3t}.$$

14. $(y^2 + xy - 2x^2 + ay - 2ax)y'_x = 5y^2 - 7xy + 2x^2 + ay - 2ax.$

Solution in the parametric form:

$$x = \tfrac{1}{4}t + C|t|^{3/4}e^{a/t}, \quad y = \tfrac{1}{2}t + C|t|^{3/4}e^{a/t}.$$

15. $(y^2 - 2xy + x^2 + ay)y'_x = ay.$

Solution: $x = y + \dfrac{a}{C - \ln|y|}.$

16. $(y^2 - 2xy + x^2 + ay + ax)y'_x = -y^2 + 2xy - x^2 + ay + ax.$

Solution in the parametric form:

$$x = -\frac{a}{2\ln|t|} + Ct, \quad y = \frac{a}{2\ln|t|} + Ct.$$

17. $(y^2 - 2xy + x^2 + ay + 2ax)y'_x = -2(y^2 - 2xy + x^2) + ay + 2ax.$

Solution in the parametric form:

$$x = -\frac{a}{3\ln|t|} + Ct, \quad y = \frac{2a}{3\ln|t|} + Ct.$$

18. $(y^2 - 2xy + x^2 + ay - 2ax)y'_x = 2(y^2 - 2xy + x^2) + ay - 2ax.$

Solution in the parametric form:

$$x = \frac{a}{\ln|t|} + Ct, \quad y = \frac{2a}{\ln|t|} + Ct.$$

19. $(y^2 + 2xy + x^2 + ay + 2ax)y'_x = -y^2 - 2xy - x^2 + 2ay + ax.$

Solution in the parametric form:

$$x = C^2\Big(t^{1/3} + \frac{4t^2}{5a}\Big) + Ct, \quad y = -C^2\Big(t^{1/3} + \frac{4t^2}{5a}\Big) + Ct, \qquad a \neq 0.$$

20. $(y^2 + 2xy + x^2 + ay - ax)y'_x = -y^2 - 2xy - x^2 + ay - ax.$

Solution in the parametric form:

$$x = C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \quad y = -C^3\sqrt{1 - \frac{4t^3}{3a}} + C^2 t, \qquad a \neq 0.$$

21. $(y^2 + 2xy + x^2 + ay - 2ax)y'_x = -y^2 - 2xy - x^2 - 2ay + ax.$

Solution in the parametric form:

$$x = C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \quad y = -C^2\left(t^3 + \frac{4t^2}{a}\right) + Ct, \qquad a \neq 0.$$

22. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = 3y^2 - 2xy - 1x^2 + ay + ax.$

Solution in the parametric form:

$$x = \tfrac{1}{2}t + Ct^2 e^{a/t}, \quad x = -\tfrac{1}{2}t + Ct^2 e^{a/t}.$$

23. $(y^2 + 2xy - 3x^2 + ay + ax)y'_x = y^2 + 2xy - 3x^2 - ay + 3ax.$

Solution in the parametric form:

$$x = at + C|t|^{-1}e^{8t}, \quad y = -3at + C|t|^{-1}e^{8t}.$$

24. $(y^2 + 2xy - 3x^2 + ay + 2ax)y'_x = y^2 + 2xy - 3x^2 + 3ax.$

Solution in the parametric form:

$$x = at + C|t|^{-3}e^{16t}, \quad y = -3at + C|t|^{-3}e^{16t}.$$

25. $(y^2 - x^2 + ay + bx)y'_x = y^2 - x^2 + by + ax.$

Solution in the parametric form:

$$x = (a-b)t + C|t|^{-\frac{a+b}{a-b}}e^{4t}, \quad y = (b-a)t + C|t|^{-\frac{a+b}{a-b}}e^{4t}, \qquad a \neq b.$$

26. $(y^2 - xy + ay + bx)y'_x = y^2 - xy + (a+b)y.$

Solution in the parametric form:

$$x = -bt + C|t|^{\frac{a+b}{b}}e^{t}, \quad y = C|t|^{\frac{a+b}{b}}e^{t}, \qquad b \neq 0.$$

27. $(y^2 + xy - 2x^2 + ay + bx)y'_x = y^2 + xy - 2x^2 + (b-a)y + 2ax.$

Solution in the parametric form:

$$x = (2a-b)t + C|t|^{-\frac{a+b}{2a-b}}e^{9t}, \quad y = 2(b-2a)t + C|t|^{-\frac{a+b}{2a-b}}e^{9t}, \qquad b \neq 2a.$$

28. $(y^2 - 2xy + x^2 + ay - abx)y'_x = b(y^2 - 2xy + x^2) + ay - abx.$

Solution in the parametric form:

$$x = \frac{a}{b-1}\frac{1}{\ln|t|} + Ct, \quad y = \frac{ab}{b-1}\frac{1}{\ln|t|} + Ct, \qquad b \neq 1.$$

29. $(y^2 + 2xy - 3x^2 + ay + bx)y'_x = y^2 + 2xy - 3x^2 + (b - 2a)y + 3ax.$

Solution in the parametric form:

$$x = (3a - b)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \quad y = 3(b - 3a)t + C|t|^{-\frac{a+b}{3a-b}} e^{16t}, \qquad b \neq 3a.$$

30. $(y^2 - 3xy + 2x^2 + ay + bx)y'_x = y^2 - 3xy + 2x^2 + (3a + b)y - 2ax.$

Solution in the parametric form:

$$x = (2a + b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \quad y = 2(2a + b)t + C|t|^{\frac{a+b}{2a+b}} e^{-t}, \qquad b \neq -2a.$$

31. $(y^2 + 3xy - 4x^2 + ay + bx)y'_x = y^2 + 3xy - 4x^2 + (b - 3a)y + 4ax.$

Solution in the parametric form:

$$x = (4a - b)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \quad y = 4(b - 4a)t + C|t|^{-\frac{a+b}{4a-b}} e^{25t}, \qquad b \neq 4a.$$

32. $[y^2 + Axy - (A+1)x^2 + by - 2bx]y'_x = (A+4)y^2 - (A+6)xy + 2x^2 + by - 2bx.$

Solution in the parametric form:

$$x = \frac{t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \quad y = \frac{2t}{A+3} + C|t|^{\frac{A+2}{A+3}} e^{b/t}, \qquad A \neq -3.$$

33. $(y^2 - 2Axy + A^2x^2 + by - bx)y'_x = Ay^2 - 2A^2xy + A^3x^2 + by - bx.$

Solution in the parametric form:

$$x = C^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \quad y = AC^3\sqrt{1 + \frac{2(A-1)}{3b}t^3} + C^2 t, \qquad b \neq 0.$$

34. $[y^2 - 2Axy + (2A-1)x^2 + by - Abx]y'_x = (2 - A)y^2 - 2xy + Ax^2 + by - Abx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + Ct^2 e^{b/t}, \quad y = \frac{At}{1-A} + Ct^2 e^{b/t}, \qquad A \neq 1.$$

35. $(y^2 - 2Axy + A^2x^2 + ay + bx)y'_x = A(y^2 - 2Axy + A^2x^2) + (aA + a + b)y - aAx.$

Solution in the parametric form:

$$x = C^2\left[t^{\frac{aA+b}{a+b}} + \frac{(1-A)^2}{(2-A)a+b}t^2\right] + Ct, \quad y = AC^2\left[t^{\frac{aA+b}{a+b}} + \frac{(1-A)^2}{(2-A)a+b}t^2\right] + Ct,$$

where $a + b \neq 0$ and $(2 - A)a + b \neq 0$.

36. $[y^2 - (A+2)xy + (A+1)x^2 + by - Abx]y'_x = -Axy + Ax^2 + by - Abx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^A e^{(A-1)b/t}, \quad y = \frac{At}{1-A} + C|t|^A e^{(A-1)b/t}, \qquad A \neq 1.$$

37. $[Ay^2 + xy - (A+1)x^2 + by + bx]y'_x = (A+1)y^2 - xy - Ax^2 + by + bx.$

Solution in the parametric form:

$$x = t + C|t|^{2A+1}e^{b/t}, \quad y = -t + C|t|^{2A+1}e^{b/t}.$$

38. $(Ay^2 + Bxy + Cx^2 + kx)y'_x = Dy^2 + Exy + Fx^2 + ky.$

The substitution $y = xz$ leads to a linear equation with respect to $x = x(z)$:

$$[-Az^3 + (D-B)z^2 + (E-C)z + F]x'_z = (Az^2 + Bz + C)x + k.$$

39. $(Ay^2 + Bxy + Cx^2 - \alpha By - \alpha Cx)y'_x = Dy^2 + Exy + \alpha(C-E)y.$

The transformation $x = w + \alpha$, $y = \xi w$ leads to a linear equation:

$$[-A\xi^3 + (D-B)\xi^2 + (E-C)\xi]w'_\xi = (A\xi^2 + B\xi + C)w + \alpha C.$$

40. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.57 with $C = Ak^2$.

41. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $C = Ak^2$.

42. $(Ay^2 + 2Bxy + Ak^2x^2 + ay - akx)y'_x = By^2 + 2Ak^2xy + Bk^2x^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $C = Ak^2$.

43. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.58 with $m = b$.

44. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $C = -Bk$.

45. $(Ay^2 + 2Bxy - Bkx^2 + ay - akx)y'_x = By^2 + 2Ak^2xy - Ak^3x^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $C = -Bk$.

46. $(Ay^2 + 2Akxy + Cx^2 + ay + bx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + by + ak^2x.$

This is a special case of equation 1.4.3.57 with $B = Ak$.

47. $(Ay^2 + 2Akxy + Cx^2 + ay + bx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + aky + bkx.$

This is a special case of equation 1.4.3.62 with $B = Ak$.

48. $(Ay^2 + 2Akxy + Cx^2 + ay - akx)y'_x = Aky^2 + 2Ak^2xy + Ckx^2 + my - mkx.$

This is a special case of equation 1.4.3.61 with $B = Ak$.

49. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + by + ak^2x.$

This is a special case of equation 1.4.3.59 with $m = b$.

50. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x = -By^2 + 2Bkxy - Ak^3x^2 + aky + bkx.$

This is a special case of equation 1.4.3.59 with $m = ak$.

51. $[y^2 + 2Axy + A^2x^2 + (A-1)By - 2ABx]y'_x$
$= -A(y^2 + 2Axy + A^2x^2) - (A^2+1)By + A(A-1)Bx.$

Solution in the parametric form:

$$x = C^2\left[t^A + \frac{A+1}{(A-2)B}t^2\right] + Ct, \quad y = -AC^2\left[t^A + \frac{A+1}{(A-2)B}t^2\right] + Ct, \qquad A \neq 2,\ B \neq 0.$$

52. $[y^2 - 2Axy + A^2x^2 + (B-1)ky + (A-B)kx]y'_x$
$= A(y^2 - 2Axy + A^2x^2) + (AB-1)ky - A(B-1)kx.$

Solution in the parametric form:

$$x = C^2\left[t^B - \frac{A-1}{(B-2)k}t^2\right] + Ct, \quad y = AC^2\left[t^B - \frac{A-1}{(B-2)k}t^2\right] + Ct, \qquad B \neq 2,\ k \neq 0.$$

53. $[2y^2 - (A+3)xy + (A+1)x^2 + By - ABx]y'_x$
$= (A+1)y^2 - (3A+1)xy + 2Ax^2 + By - ABx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^{-1}e^{-B/t}, \quad y = \frac{At}{1-A} + C|t|^{-1}e^{-B/t}, \qquad A \neq 1.$$

54. $[2y^2 - (3A+1)xy + (3A-1)x^2 + By - ABx]y'_x$
$= (3-A)y^2 - (A+3)xy + 2Ax^2 + By - ABx.$

Solution in the parametric form:

$$x = \frac{t}{1-A} + C|t|^3e^{B/t}, \quad y = \frac{At}{1-A} + C|t|^3e^{B/t}, \qquad A \neq 1.$$

55. $[A(y^2 - 2xy + x^2) - A(A-B)y + B(A-B)x]y'_x$
$= B(y^2 - 2xy + x^2) - A(A-B)y + B(A-B)x.$

Solution in the parametric form:

$$x = \frac{A}{\ln|t|} + Ct, \quad y = \frac{B}{\ln|t|} + Ct.$$

56. $(Ay^2 + Bxy + Cx^2 + ay + bx)y'_x = Aky^2 + Bkxy + Ckx^2 + ny + (ak + b - n)x.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(n - ak)zx'_z = (Ak^2 + Bk + C)x^2 + [(2Ak + B)z + ak + b]x + Az^2 + az.$$

57. $(Ay^2 + 2Bxy + Cx^2 + ay + bx)y'_x$
$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + by + ak^2x.$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + b - ak]zx'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

58. $(Ay^2 + 2Bxy - Bkx^2 + ay + bx)y'_x$
$$= By^2 + 2Ak^2xy - Ak^3x^2 + my + k(ak + b - m)x.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + m - ak]zx'_z = (Ak^2 + Bk)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

59. $(Ay^2 - 2Akxy + Bkx^2 + ay + bx)y'_x$
$$= -By^2 + 2Bkxy - Ak^3x^2 + my + k(ak + b - m)x.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[-(Ak + B)z + m - ak]zx'_z = k(B - Ak)x^2 + (ak + b)x + Az^2 + az.$$

60. $(Ay^2 + 2Bxy + Ak^2x^2 + ay + bx)y'_x$
$$= By^2 + 2Ak^2xy + Bk^2x^2 + my + k(ak + b - m)x.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z + m - ak]zx'_z = 2k(Ak + B)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

61. $(Ay^2 + 2Bxy + Cx^2 + ay - akx)y'_x$
$$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + my - mkx.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B - Ak)z^2 + m - ak]zx'_z = (Ak^2 + 2Bk + C)x^2 + 2(Ak + B)zx + Az^2 + az.$$

62. $(Ay^2 + 2Bxy + Cx^2 + ay + bx)y'_x$
$$= By^2 + 2Ak^2xy + k(-Ak^2 + Bk + C)x^2 + aky + bkx.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$(B - Ak)z^2x'_z = (Ak^2 + 2Bk + C)x^2 + [2(Ak + B)z + ak + b]x + Az^2 + az.$$

63. $\{(A - 1)y^2 + [2 - A(k + 1)]xy + (Ak - 1)x^2 + By - Bkx\}y'_x$
$$= (A - k)y^2 + [2k - A(k + 1)]xy + (A - 1)kx^2 + By - Bkx.$$

Solution in the parametric form:

$$x = \frac{t}{1 - k} + C|t|^A e^{B/t}, \quad y = \frac{kt}{1 - k} + C|t|^A e^{B/t}, \qquad k \neq 1.$$

64. $[A(\alpha y^2 + \beta xy + \gamma x^2) + (2\alpha - A^2\sigma)y + (\beta - AB\sigma)x]y'_x$
$$+ B(\alpha y^2 + \beta xy + \gamma x^2) + (\beta - AB\sigma)y + (2\gamma - B^2\sigma)x = 0.$$

Solution: $\alpha y^2 + \beta xy + \gamma x^2 - A\sigma y - B\sigma x + \sigma = C\exp(-Ay - Bx)$.

65. $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x)y'_x$
$$= B_{22}y^2 + k(2A_{22}k + A_{12} - 2B_{22})xy + k(-A_{22}k^2 + B_{22}k + A_{11})x^2$$
$$+ B_2y + k(A_2k + A_1 - B_2)x.$$

The substitution $y = z + kx$ leads to the Riccati equation with respect to $x = x(z)$:

$$[(B_{22} - A_{22}k)z + B_2 - A_2k]zx'_z$$
$$= (A_{22}k^2 + A_{12}k + A_{11})x^2 + [(2A_{22}k + A_{12})z + A_2k + A_1]x + A_{22}z^2 + A_2z.$$

▶ *In equations 66–70, the following notation is used:*

$$\Delta = Ab - aB, \quad \delta = Ab + aB.$$

66. $(Aa^2y^2 - 2Aabxy + Ab^2x^2 - \Delta Aay + \Delta aBx)y'_x$
$= a^2By^2 - 2aBbxy + Bb^2x^2 - \Delta Aby + \Delta Bbx.$

Solution in the parametric form:

$$x = \frac{A}{\ln|t|} + aCt, \qquad y = \frac{B}{\ln|t|} + bCt.$$

67. $[kAa^2y^2 - k\delta axy + kaBbx^2 - m\Delta Aay + (maB - \Delta)\Delta x]y'_x$
$= kAaby^2 - k\delta bxy + kBb^2x^2 - (mAb + \Delta)\Delta y + m\Delta Bbx.$

Solution in the parametric form:

$$x = At + aC|t|^{m+1}e^{kt}, \qquad y = Bt + bC|t|^{m+1}e^{kt}.$$

68. $[mAa^2y^2 - a(m\delta - \Delta)xy + b(maB - \Delta)x^2 + k\Delta Aay - k\Delta aBx]y'_x$
$= a(mBb + \Delta)y^2 - b(m\delta + \Delta)xy + mBb^2x^2 + k\Delta Aby - k\Delta Bbx.$

Solution in the parametric form:

$$x = At + aC|t|^{m+1}e^{k/t}, \qquad y = Bt + bC|t|^{m+1}e^{k/t}.$$

69. $(kA^3y^2 - 2kA^2Bxy + kAB^2x^2 - 2\Delta a^2y + 2\Delta abx)y'_x$
$= kA^2By^2 - 2kAB^2xy + kB^3x^2 - 2\Delta aby + 2\Delta b^2x.$

Solution in the parametric form:

$$x = AC^3\sqrt{\tfrac{1}{3}kt^3 + 1} + aC^2t, \qquad y = BC^3\sqrt{\tfrac{1}{3}kt^3 + 1} + bC^2t.$$

70. $[kA^3y^2 - 2kA^2Bxy + kAB^2x^2 + m\Delta Aay - (mAb + \Delta)\Delta x]y'_x$
$= kA^2By^2 - 2kAB^2xy + kB^3x^2 + (maB - \Delta)\Delta y - m\Delta Bbx.$

Solution in the parametric form:

$$x = AC^2\Big(t^{m+1} + \frac{k}{m-1}t^2\Big) + aCt, \quad y = BC^2\Big(t^{m+1} + \frac{k}{m-1}t^2\Big) + bCt, \qquad m \neq 1.$$

1.4.4. Equations of the Form $(A_{22}y^2 + A_{12}xy + A_{11}x^2 + A_2y + A_1x + A_0)y'_x = B_{22}y^2 + B_{12}xy + B_{11}x^2 + B_2y + B_1x + B_0$

Preliminary comments.

1. With $A_{22} = 0$, this is the Abel equation (see Subsection 1.3.4). With $B_{11} = 0$, this is the Abel equation with respect to $x = x(y)$.

See Subsection 1.4.2 for the case $A_2 = A_1 = B_2 = B_1 = 0$.

See Subsection 1.4.3 for the case $A_0 = B_0 = 0$.