Theoretical Foundations of Chemical Engineering, Vol. 34, No. 5, 2000, pp. 403–415.
Translated from Teoreticheskie Osnovy Khimicheskoi Tekhnologii, Vol. 34, No. 5, 2000, pp. 451–464.

Exact Solutions of Nonlinear Heat- and Mass-Transfer Equations

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Received December 8, 1999

Abstract—The method of generalized separation of variables for solving nonlinear steady and unsteady heat- and mass-transfer equations is outlined. New exact solutions of one-, two-, and three-dimensional heat equations are obtained. Anisotropic media with a nonlinear heat source of general form are considered for the case in which the main thermal diffusivities show a power or an exponential dependence on the spatial coordinates. Equations with a logarithmic heat source are analyzed in detail. The results obtained are applied to the problem of thermal explosion in an anisotropic medium.

Heat (mass) transfer in a stagnant medium (solid, liquid, or gas) is described by a heat (diffusion) equation [1–4]. In a homogeneous and isotropic medium, the thermal diffusivity (diffusion coefficient) appearing in the equation remains constant throughout the range under examination [5–7], and the heat (diffusion) equation is linear and has constant coefficients. In anisotropic media, the thermal diffusivity (diffusion coefficient) depends on the direction of heat (mass) transfer; in inhomogeneous media, it may depend on coordinates and even on temperature [8–11]. In the latter case, the heat (diffusion) equation is nonlinear. There are numerous approximation formulas (among them linear, power-low, and exponential) describing the dependence of the transfer coefficients on temperature or concentration (see, e.g., [8, 10, 12, 13]).

Heat (mass) transfer in a stagnant medium may be complicated by the presence of bulk sources or sinks, which emerge through various physicochemical mechanisms of absorption and release of heat (matter). In combustion theory and the nonisothermal macrokinetics of complex chemical reactions [4, 14], the power of heat sources (sinks) often depends on temperature, and it often does so nonlinearly, being an exponential [14] or power [15] function. In the mass-transfer theory, the concentration dependence of the rate of a bulk chemical reaction is commonly described by a power-low function, whereas for complex reactions, other (exponential or logarithmic) functions are used.

Exact solutions of heat- and mass-transfer equations play a significant role in gaining correct insight into various thermal and diffusion processes. Exact solutions of nonlinear equations enable one to look into the mechanisms of important and complex physical phenomena, such as spatial localization of heat-transfer processes, peaking processes, and the multiplicity or absence of a steady state. Even if particular exact solutions of differential equations have no clear physical meaning, they can be used in test problems for checking the correctness and estimating the accuracy of various numerical, asymptotic, and approximate analytical methods. Moreover, model equations and problems admitting exact solutions serve as a basis for developing new numerical, asymptotic, and approximate methods, which, in turn, enable one to study more complex heat- and mass-transfer problems that have no exact analytical solution.

STRUCTURE OF EXACT SOLUTIONS OF SOME HEAT- AND MASS-TRANSFER EQUATIONS

Self-similar solutions of nonlinear heat- and mass-transfer equations. For simplicity, let us consider the one-dimensional case. Self-similar solutions of one-dimensional heat-transfer equations are usually represented in the following form [16, 17]:

$$T(x, t) = t^{\beta} f(x/t^{\gamma}), \qquad (1)$$

where β and γ are some constants. The sought function $f(x/t^{\gamma})$ is found from the ordinary differential equation that is obtained by substituting solution (1) into the original partial differential equation.

In a more general case, the term *self-similar solution* is used for solutions of the form

$$T(x, t) = \varphi(t) f(x/\psi(t)), \qquad (2)$$

where the form of the functions $\varphi(t)$ and $\psi(t)$ is chosen with consideration for convenience of solving a specific problem.

Let us give an example of the simplest self-similar solution of a nonlinear heat-transfer equation in the case for which the nonlinearity is caused by temperature variation of the thermal diffusivity. Let us consider a one-dimensional problem of unsteady heat transfer in a semi-infinite plate whose initial ($t \le 0$) temperature is T_i. For $t > 0$, the temperature of the $x = 0$ plate boundary is maintained at T_s. It is required to find the temperature distribution $T(x, t)$ for $t > 0$. The corresponding boundary-value problem has the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[a(T)\frac{\partial T}{\partial x}\right], \tag{3}$$
$$T|_{t=0} = T_i, \quad T|_{x=0} = T_s, \quad T|_{x\to\infty} \longrightarrow T_i.$$

This problem has been an object of numerous investigations in the nonlinear heat-conduction and filtration theories (see, e.g., [4, 18]). The solution to problem (3) is sought for in the form

$$T = T(\omega), \quad \omega = x/\sqrt{t},$$

which leads to the following ordinary differential equation and boundary conditions:

$$[a(T)T'_\omega]'_\omega + \frac{1}{2}\omega T'_\omega = 0,$$
$$T|_{\omega=0} = T_s, \quad T|_{\omega\to\infty} \longrightarrow T_i.$$

An analytical solution to this problem has been obtained for linear [2, 19, 20], hyperbolic [3, 21], and power-low [15, 22] functions $a(T)$.

A comprehensive list of exact solutions to equations of the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[a(T)\frac{\partial T}{\partial x}\right] + \Phi(T)$$

is available for both $\Phi = 0$ [18] and $\Phi \neq 0$ [23].

Separation of variables in linear equations. Many linear equations of mathematical physics (partial differential equations) are solvable by separation of variables. For definiteness, let us further examine second-order linear equations

$$\mathfrak{L}\left(x, t, T, \frac{\partial T}{\partial x}, \frac{\partial T}{\partial t}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial t^2}\right) = 0 \tag{4}$$

in two independent variables x and t (the sought function $T(x, t)$ is a function of these variables). Below, we briefly describe the procedure of solving Eq. (4) and problems involving this equation.

At the first step, a particular solution is sought for in the form

$$T(x, t) = \varphi(x)\psi(t). \tag{5}$$

Expression (5) is substituted into Eq. (4), and the latter is then represented as the equality whose left-hand side depends only on x (and contains x, φ, φ'_x, and φ''_{xx}) and whose right side depends only on t (and contains t, ψ, ψ'_t, and ψ''_{tt}). Two expressions with different variables can be equal only if both of them are equal to a certain constant, which is termed the separation constant. Therefore, when finding functions φ and ψ, we arrive at ordinary differential equations with a free parameter k.

This procedure is called the separation of variables in linear equations (from which the name of the method arises).

At the second step, the principle of linear superposition is used: a linear combination of particular solutions of a linear equation is also a solution of this equation.

The functions φ and ψ in solution (5) depend not only on the variables x and t but also on the separation constant:

$$\varphi = \varphi(x, k), \quad \psi = \psi(t, k).$$

For different values $k_1, k_2, \ldots$ of the parameter k, different particular solutions of the original equation are obtained:

$$T_1(x, t) = \varphi_1(x)\psi_1(t), \quad T_2(x, t) = \varphi_2(x)\psi_2(t), \ \ldots,$$

where

$$\varphi_i = \varphi(x, k_i), \quad \psi_i = \psi(t, k_i), \quad i = 1, 2, 3, \ldots.$$

According to the principle of linear superposition, the set of particular solutions

$$T(x, t) = \varphi_1(x)\psi_1(t) + \varphi_2(x)\psi_2(t) + \ldots \tag{6}$$

is also a solution of the original equation.

Note that relation (6) is usually written as

$$T(x, t) = A_1\varphi_1(x)\psi_1(t) + A_2\varphi_2(x)\psi_2(t) + \ldots,$$

where $A_1, A_2, \ldots$ are arbitrary constants. In relation (6), they are combined, for convenience, with the functions $\varphi_i(x)$, which are determined up to a constant factor.

At the third step, which is executed in solving specific problems, the spectrum of the separation parameter values $\{k_1, k_2, \ldots\}$ is found from the boundary conditions, which lead to the Sturm–Liouville eigenvalue problem for the function φ. The arbitrary constants appearing as normalization factors in the products $\varphi_i(x)\psi_i(t)$ are determined from the initial conditions.

Remark. Many linear equations of mathematical physics also admit exact solutions in the form of the sum of functions of different arguments:

$$T(x, t) = \theta(x) + \chi(t), \tag{7}$$

where the functions $\theta(x)$ and $\chi(t)$ are found, after separation of variables, from the corresponding ordinary differential equations.

Example. Let us consider the linear equation that describes convective mass transfer at the rate $-U(x)$ under the following conditions: the diffusion coefficient D is coordinate-dependent ($D = D(x)$); a first-order chemical reaction (K_0C) occurs in the system; and bulk absorption (sink) of matter, whose rate $\Phi(t)$ is time-dependent ($\Phi = \Phi(t)$) there takes place:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial C}{\partial x}\right] + U(x)\frac{\partial C}{\partial x} + K_0 C + \Phi(t).$$

This equation admits exact solutions of form (7) but has no exact solutions of form (5). Furthermore, this equation has more complex solutions of the form

$$C(x, t) = \theta(x)\chi_1(t) + \chi_2(t), \tag{8}$$

where $\chi_1(t) = \exp(K_0 t)$ and the function $\chi_2(t)$ is determined from the first-order ordinary differential equation $\chi_2' = K_0\chi_2 + \Phi(t)$.

Separation of variables in nonlinear equations. Some nonlinear equations, like linear ones, admit exact solutions in the form of the product of functions of different arguments (see Eq. (5)). The functions are found from ordinary differential equations that are obtained, after substituting Eq. (5) into the original equation, by nonlinear separation of variables.

Example 1. The nonlinear heat equation in which the thermal diffusivity is a power-low function of temperature,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha T^n \frac{\partial T}{\partial x}\right),$$

where α and n are constants, admits exact solutions of form (5) [18].

There also exist nonlinear equations that admit exact solutions in the form of the sum of functions of different arguments (form (7)).

Example 2. The nonlinear heat equation in which the thermal diffusivity is an exponential function of temperature,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha e^{\beta T} \frac{\partial T}{\partial x}\right),$$

where α and β are constants, admits exact solutions of form (7) [18].

Let us outline the generalized separation of variables in nonlinear equations.

1. Suppose that a nonlinear equation in T is obtained from a linear equation in u by the following nonlinear change of the dependent variable:

$$T = F(u), \tag{9}$$

where F is some function. Let the linear equation admit solutions in the form of the product or sum of functions of different variables (forms (5) and (7), respectively). The exact solutions of the nonlinear partial differential equation will then be expressed as

$$T(x, t) = F(u), \quad u = \varphi(x)\psi(t); \tag{10}$$

$$T(x, t) = F(u), \quad u = \theta(x) + \chi(t). \tag{11}$$

For example, the above simplest self-similar solution of Eq. (3) is representable in form (10).

Nonlinear equations most often admit traveling-wave solutions

$$T(x, t) = F(u), \quad u = x + \lambda t, \tag{12}$$

which are the special case of Eq. (11) with $\theta(x) = x$ and $\chi(t) = \lambda t$. Note that solution (12) can also be represented in form (10):

$$T(x, t) = F_1(v), \quad v = e^{x+\lambda t} = e^x e^{\lambda t},$$
$$F_1(v) = F(\ln v).$$

Similarly, solution (11) can be represented in form (10) by performing the change of variables $u = \ln v$ and designating $F(u) = F_1(v)$.

The functions φ and ψ (or θ and χ) and the temperature profile $F = F(u)$ in formulas (10) and (11) are found in one of the following ways:

(1) The profile is determined from an ordinary differential equation that is obtained after choosing suitable functions φ and ψ (or θ and χ). The functions φ and ψ (or θ and χ) are also given by ordinary differential equations. Using this method, one can find self-similar and some more complex solutions.

(2) The profile $F = F(u)$ is specified *a priori* from various considerations (as this profile, one can use, for exemple, a solution of a simpler auxiliary equation). Separation of variables in the equations obtained, if possible, yields differential equations for the functions φ and ψ (or θ and χ).

Table 1 lists some specific nonlinear equations admitting exact solutions of forms (10) and (11) (self-similar solutions at $\varphi(x) = x$ and traveling-wave solutions are not considered here).

2. Suppose that a nonlinear equation is derived from a linear equation by the change of the dependent variable according to formula (9). Then, the exact solutions of the resulting nonlinear partial differential equation, which correspond to the exact solutions of the linear equation of form (6), are given by

$$T(x, t) = F(u),$$
$$u = \varphi_1(x)\psi_1(t) + \varphi_2(x)\psi_2(t) + \dots . \tag{13}$$

Structural formula (13) can serve as the basis for seeking exact solutions of various nonlinear partial differential equations of mathematical physics that are irreducible to linear equations. The profile $F = F(u)$ and the functions $\varphi_1(x)$, $\varphi_2(x)$, …, $\psi_1(t)$, $\psi_2(t)$, … are to be found. Solutions of form (13) generally cannot be obtained by the methods of group analysis.

Exact solutions of partial differential equations with quadratic nonlinearity have been sought for in form (13) with $F(u) = u$, $\psi_2 = 1$, and $\psi_i = 0$ for $i \geq 3$ [26]. Quite a general scheme has been proposed [29] for searching for exact solutions of differential equations with quadratic nonlinearity at $F(u) = u$ (under the assumption that the equations considered are not explicitly dependent on spatial variables and time). Solutions of form (13) are a natural generalization of the solutions examined in [26, 29].

Table 1. Some nonlinear equations having exact solutions of forms (10) and (11)

Equation	Solution	References
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2$	$T = \varphi(x) + \psi(t)$; $T = (a/b)\ln u$, $u = \varphi(x) + \psi(t)$	[24]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right)$	$T = \varphi(x)\psi(t)$	[15, 16]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda T}\frac{\partial T}{\partial x}\right)$	$T = \varphi(x) + \psi(t)$	[15, 18, 25]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + aT\ln T$	$T = \varphi(x)\psi(t)$	[15, 23]
$\frac{\partial T}{\partial t} = ax^{-n}\frac{\partial}{\partial x}\left(x^n\frac{\partial T}{\partial x}\right) + bT\ln T$	$T = \varphi(x)\psi(t)$	[15, 26]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = ae^T$	$T = -2\ln u$, $u = \varphi(x) + \psi(y)$	[14]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = a\,\mathrm{sh}\,T$	$T = 2\ln\frac{1+u}{1-u}$, $u = \varphi(x)\psi(y)$	[27]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = aT\ln T$	$T = e^u$, $u = \varphi(x) + \psi(y)$	[27]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = a\sin T$	$T = 4\,\mathrm{atan}\,u$, $u = \varphi(x)\psi(y)$	[27]
$\frac{\partial}{\partial x}\left(ax^n\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial T}{\partial y}\right) = cT^k$	$T = F(u)$, $u = \varphi(x) + \psi(y)$	[28]
$\frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y}\frac{\partial T}{\partial y}\right) = ce^{\gamma T}$	$T = F(u)$, $u = \varphi(x) + \psi(y)$	[28]
$\frac{\partial}{\partial x}\left(ax^n\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y}\frac{\partial T}{\partial y}\right) = ce^{\gamma T}$	$T = F(u)$, $u = \varphi(x) + \psi(y)$	[28]
$\frac{\partial}{\partial x}\left(aT^n\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(bT^m\frac{\partial T}{\partial y}\right) = 0$	$T = \varphi(x)\psi(y)$	[28]
$\frac{\partial}{\partial x}\left(ae^{\lambda T}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta T}\frac{\partial T}{\partial y}\right) = 0$	$T = \varphi(x) + \psi(y)$	[24]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + ae^T$	$T = -2\ln u$, $u = \varphi(x) + \psi(t)$	[24]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + a\,\mathrm{sh}\,T$	$T = 2\ln\frac{1+u}{1-u}$, $u = \varphi(x)\psi(t)$	[27]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + aT\ln T$	$T = e^u$, $u = \varphi(x) + \psi(t)$	[27]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + a\sin T$	$T = 4\,\mathrm{atan}\,u$, $u = \varphi(x)\psi(t)$	[27]

Note: a, b, c, k, m, n, β, γ, and λ are parameters.

Table 2. Some nonlinear equations having exact solutions of form (13)

Equation	Solution	References
$\frac{\partial T}{\partial t} = a\frac{\partial T^2}{\partial x^2} + bT\frac{\partial T}{\partial x}$	$T = 1/u$, $u = \varphi(x)\theta(t) + \psi(x)$	[25]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2 + c_1 T + c_0$	$T = \varphi(t)x^2 + \psi(t)x + \chi(t)$	[28]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2 + c_2 T^2 + c_1 T$	$T = \varphi(t)\theta(x) + \psi(t)$ $\theta(x) = e^{\lambda x}$, $\theta(x) = \sin(\lambda x)$	[26]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right)$	$T = u^{1/m}$, $u = \varphi(t)x^2 + \psi(t)$	[4, 18]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT$	$T = u^{1/m}$, $u = \varphi(t)x^2 + \psi(t)$	[23–25, 29]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT^{m+1}$	$T = u^{1/m}$, $u = \varphi(t)\theta(x) + \psi(t)$	[15]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT^{1-m}$	$T = u^{1/m}$, $u = \varphi(t)x^2 + \psi(t)$	[30]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^T\frac{\partial T}{\partial x}\right) + be^T + c$	$T = \ln u$, $u = \varphi(t)\theta(x) + \psi(t)$ $\theta(x) = e^{\lambda x}$, $\theta(x) = \sin(\lambda x)$	[31]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^T\frac{\partial T}{\partial x}\right) + b + ce^{-T}$	$T = \ln u$, $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$	[26. 28]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + aT\ln T + bT$	$T = e^u$, $u = \varphi(t)x + \psi(t)$ $T = e^u$, $u = \varphi(t)x^2 + \psi(t)$	[15, 23, 28]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + T(a\ln T + b\ln T + c)$	$T = e^u$, $u = \varphi(t)\theta(x) + \psi(t)$ $\theta(x) = e^{\lambda x}$, $\theta(x) = \sin(\lambda x)$	[26]
$\frac{\partial T}{\partial t} = ax^{-n}\frac{\partial}{\partial x}\left(x^n\frac{\partial T}{\partial x}\right) + aT\ln T$	$T = e^u$, $u = \varphi(t)x^2 + \psi(t)$	[15]

Note: $a, b, c, c_0, c_1, c_2, m, n$, and λ are parameters.

In the analysis of special equations, it is convenient to use particular cases of formula (13):

$$T(x, t) = F(u), \quad u = \varphi_1(x)\psi_1(t) + \psi_2(t); \tag{14}$$

$$T(x, t) = F(u), \quad u = \varphi_1(x)\psi_1(t) + \varphi_2(x). \tag{15}$$

Table 2 presents some special nonlinear equations admitting exact solutions in form (13). One can see that most of the solutions are representable by formula (14).

It is important that, in principle, representation (13) enables one to find exact solutions to nonlinear equations that are obtained by the change of variables of form (9) from the linear equation with separable variables.

3. Suppose that, in the initial equation, one can perform a more general change of the dependent variable:

$$T = g(x, t)F(u) + h(x, t).$$

Narrowing down the classes of the functions $g(x, t)$ and $h(x, t)$, one can arrive at simpler dependences, and on their basis, search for exact solutions of nonlinear equations (which are already irreducible to linear equations).

Below, we present structural formulas that are the respective generalizations of formulas (14) and (15):

$$T(x, t) = g(t)F(u) + h(t), \quad u = \varphi_1(x)\psi_1(t) + \psi_2(t); \tag{16}$$

$$T(x, t) = g(x)F(u) + h(x), \quad u = \varphi_1(x)\psi_1(t) + \varphi_2(x). \tag{17}$$

They can be used in searching for exact solutions of nonlinear equations.

In the special case of $\varphi_1(x) = x$ and $\psi_2(t) = 0$, formula (16) corresponds to generalized self-similar solutions.

Further, we use the above approach to obtain new exact solutions to various classes of nonlinear equations.

NONLINEAR EQUATIONS OF HEAT AND MASS TRANSFER IN ANISOTROPIC MEDIA

Separation of variables. Let us consider a class of partial differential equations of the form

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left[p_i(x_i)\frac{\partial w}{\partial x_i}\right] = \mathfrak{F}[w], \qquad (18)$$

where $p_i(x_i)$ are some functions, whose form will be determined below, and $x_1, \ldots, x_m$ are independent variables ($m \geq 2$). The right-hand side of this equation is generally a prescribed nonlinear differential operator. This operator depends on w, on a number of independent variables $x_{m+1}, \ldots, x_k$, which do not appear on the left-hand side of the equation, and on the derivatives of w with respect to these variables.

We will search for particular solutions to Eq. (18) in the form

$$w = w(r; x_{m+1}, \ldots, x_k), \quad r^2 = \sum_{i=1}^{m} \varphi_i(x_i), \qquad (19)$$

which, owing to fewer independent variables, are given by a simpler equation. The unknown functions $\varphi_i(x_i)$ and $p_i(x_i)$ will be found in the course of the investigation.

Substituting expression (19) into Eq. (18) and taking into account

$$\frac{\partial w}{\partial x_i} = \frac{\partial w}{\partial r}\frac{\varphi_i'}{2r},$$

$$\frac{\partial^2 w}{\partial x_i^2} = \left(r\frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{\partial r}\right)\frac{(\varphi_i')^2}{4r^3} + \frac{\partial w}{\partial r}\frac{\varphi_i''}{2r},$$

we obtain

$$\frac{1}{4r^3}\left(r\frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{\partial r}\right)\sum_{i=1}^{m} p_i(\varphi_i')^2 + \frac{1}{2r}\frac{\partial w}{\partial r}\sum_{i=1}^{m}(p_i\varphi_i')' = \mathfrak{F}[w], \qquad (20)$$

where the primes at the functions φ_i mean differentiation with respect to x_i.

Function (19) is a solution of the original Eq. (18) if the sums in expression (20) are constants or functions only of a new variable r.

In the general case, this is possible if

$$p_i(\varphi_i')^2 = A\varphi_i + A_i, \quad (p_i\varphi_i')' = B\varphi_i + B_i, \qquad (21)$$

where A, A_i, B, and B_i are some constants ($i = 1, \ldots, m$). In this case, in Eq. (20), one should take

$$\sum_{i=1}^{m} p_i(\varphi_i')^2 = Ar^2 + A_\Sigma,$$

$$\sum_{i=1}^{m} (p_i\varphi_i')' = Br^2 + B_\Sigma,$$

where

$$A_\Sigma = \sum_{i=1}^{m} A_i, \quad B_\Sigma = \sum_{i=1}^{m} B_i.$$

At any i, we have two ordinary differential Eqs. (21) in $p_i(x_i)$ and $\varphi_i(x_i)$.

Let us express the function p_i in terms of φ_i from the first of Eqs. (21):

$$p_i = \frac{A\varphi_i + A_i}{(\varphi_i')^2}. \qquad (22)$$

Substituting this expression into the second of Eqs. (21), we obtain the following autonomous equation for the function φ_i:

$$(A\varphi_i + A_i)\varphi_i'' + (B\varphi_i + \mu_i)(\varphi_i')^2 = 0, \qquad (23)$$

where $\mu_i = B_i - A$. This equation is solved through the change $\varphi_i' = z_i(\varphi_i)$.

At $A \neq 0$, the general solution of Eq. (23) can be written in the implicit form

$$x_i + C_2 = C_1\int \exp(B\varphi_i/A)|A\varphi_i + A_i|^{\frac{A\mu_i - BA_i}{A^2}} d\varphi_i,$$
$$\varphi_i' = z_i(\varphi_i) = \frac{1}{C_1}\exp\left(-\frac{B\varphi_i}{A}\right)|A\varphi_i + A_i|^{\frac{BA_i - A\mu_i}{A^2}}, \qquad (24)$$

where C_1 and C_2 are arbitrary constants.

At $A = 0$ and $A_i \neq 0$, the general solution of Eq. (23) can be written in the implicit form

$$x_i + C_2 = C_1\int \exp\left(\frac{B\varphi_i^2 + 2B_i\varphi_i}{2A_i}\right)d\varphi_i,$$
$$\varphi_i' = z_i(\varphi_i) = \frac{1}{C_1}\exp\left(-\frac{B\varphi_i^2 + 2B_i\varphi_i}{2A_i}\right). \qquad (25)$$

Table 3. Cases when the functions $_i(x_i)$ and $\varphi_i(x_i)$ are representable in an explicit form

No.	$p_i(x_i)$	$\varphi_i(x_i)$	Relations
1	$a_i\|x_i+s_i\|^{n_i}$	$b_i\|x_i+s_i\|^{2-n_i}+c_i$	$A_i=-Ac_i$, $B=0$, $B_i=A/(2-n_i)$, $b_i=A/(a_i(2-n_i)^2)$
2	$a_i e^{\lambda_i x_i}$	$b_i e^{-\lambda_i x_i}+c_i$	$A_i=-Ac_i$, $B=B_i=0$, $b_i=A/(a_i\lambda_i^2)$
3	$a_i x_i^2$	$b_i\ln\|x_i\|+c_i$	$A=0$, $A_i=a_ib_i^2$, $B=0$, $B_i=a_ib_i$
4	$(a\ln\|x_i\|+b_i)x_i^2$	$c\ln\|x_i\|+d_i$	$A=ac$, $A_i=(b_ic-ad_i)c$, $B=a$, $B_i=ac+(b_ic-ad_i)$

In some cases, the functions $p_i(x_i)$ and $\varphi_i(x_i)$ are representable explicitly. For example, at $A_i = B = B_i = 0$, from Eqs. (24) and (22), we have

$$x_i+C_2 = \frac{C_1}{A}\ln|A\varphi_i|, \quad \varphi_i' = \frac{A}{C_1}\varphi_i, \quad p_i = \frac{A\varphi_i}{(\varphi_i')^2},$$

whence

$$p_i(x_i) = a_i e^{\lambda_i x_i}, \quad \varphi_i(x_i) = b_i e^{-\lambda_i x_i},$$

where

$$a_i = \pm C_1^2 e^{-AC_2/C_1}, \quad \lambda_i = -A/C_1, \quad b_i = \pm A^{-1}e^{AC_2/C_1}.$$

Table 3 covers the special cases when the functions $p_i(x_i)$ and $\varphi_i(x_i)$ can be represented in an explicit form.

On the basis of the results obtained, one can construct exact solutions to specific equations.

Exact solutions of three-dimensional heat- and mass-transfer equations. To illustrate the approach described, let us examine some families of nonlinear three-dimensional heat- and mass-transfer equations and obtain their exact solutions.

Let us consider the equations corresponding to cases 1 and 2 in Table 3, which are of most interest. In the equations analyzed below (cases 1–4), the operator $\mathfrak{F}[T]$ is assumed to be a nonlinear function of the source $\Phi(T)$.

1. The equation $(k, m, n \neq 2)$

$$\frac{\partial}{\partial x}\left(a|x|^k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(b|y|^m\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(c|z|^n\frac{\partial T}{\partial z}\right) = \Phi(T), \qquad (26)$$

which describes steady-state heat or mass transfer in an inhomogeneous anisotropic medium with heat release (bulk reaction), has exact solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{|x|^{2-k}}{a(2-k)^2}+\frac{|y|^{2-m}}{b(2-m)^2}+\frac{|z|^{2-n}}{c(2-n)^2}\right]. \qquad (27)$$

The function $T(r)$ is determined from the ordinary differential equation

$$T''_{rr}+\frac{D}{r}T'_r = \frac{4}{A}\Phi(T), \quad D = 2\left(\frac{1}{2-k}+\frac{1}{2-m}+\frac{1}{2-n}\right)-1. \qquad (28)$$

This equation is solvable in an explicit form at $D = 1$ and $\Phi(T) = C\exp(\alpha T)$, where C and α are constants. At $D = 0$ and an arbitrary function $\Phi(T)$, Eq. (28) is integrable in quadrature (there exist other exact solutions [32]).

Note that, instead of $|x|$, $|y|$, and $|z|$, Eqs. (26) and (27) can contain $|x + s_1|$, $|y + s_2|$, and $|z + s_3|$, respectively, where s_1, s_2, and s_3 are some constant.

At $k = m = n = 0$ and $a = b = c$, Eq. (26) appears as the classical equation of heat and mass transfer in an isotropic medium with heat release (bulk reaction). Solution (27), (28) then corresponds to the spherically symmetric case.

2. The equation $(\lambda\mu\nu \neq 0)$

$$\frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) = \Phi(T) \qquad (29)$$

admits exact solutions of the form

$$T = T(r), \quad r^2 = A\left(\frac{e^{-\lambda x}}{a\lambda^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right).$$

The function $T(r)$ is found from the ordinary differential equation

$$T''_{rr}-\frac{1}{r}T'_r = \frac{4}{A}\Phi(T).$$

3. The equation ($n, m \neq 2$; $\nu \neq 0$)

$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^m\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) = \Phi(T) \quad (30)$$

admits exact solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{|x|^{2-n}}{a(2-n)^2}+\frac{|y|^{2-m}}{b(2-m)^2}+\frac{e^{-\nu z}}{c\nu^2}\right].$$

The function $T(r)$ is obtained from the ordinary differential equation

$$T''_{rr}+\frac{D}{r}T'_r = \frac{4}{A}\Phi(T), \quad D = 2\left(\frac{1}{2-n}+\frac{1}{2-m}\right)-1.$$

4. The equation ($n \neq 2$, $\mu\nu \neq 0$)

$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) = \Phi(T) \quad (31)$$

has exact solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{|x|^{2-n}}{a(2-n)^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right].$$

The function $T(r)$ is determined from the ordinary differential equation

$$T''_{rr}+\frac{D}{r}T'_r = \frac{4}{A}\Phi(T), \quad D = n/(2-n).$$

This equation is integrable in quadrature, e.g., at $n = 0$ and an arbitrary function $\Phi(T)$. At $n = 1$ and $\Phi(T) = Ce^{\alpha T}$, where α and C are constants, this equation is integrable in an explicit form.

5. Let

$$\mathfrak{F}[T] = \frac{\partial T}{\partial t}-\Phi(T).$$

Consider the unsteady-state heat equation ($k, m, n \neq 2$)

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(ax^k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^m\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^n\frac{\partial T}{\partial z}\right)+\Phi(T). \quad (32)$$

In terms of the approach presented above, this equation has exact solutions of the form

$$T = T(t, r),$$
$$r^2 = 4A\left[\frac{|x|^{2-k}}{a(2-k)^2}+\frac{|y|^{2-m}}{b(2-m)^2}+\frac{|z|^{2-n}}{c(2-n)^2}\right].$$

The function $T(t, r)$ satisfies a simpler partial differential equation in two independent variables:

$$\frac{\partial T}{\partial t} = A\left(\frac{\partial^2 T}{\partial r^2}+\frac{D}{r}\frac{\partial T}{\partial r}\right)+\Phi(T),$$
$$D = \frac{2}{2-k}+\frac{2}{2-m}+\frac{2}{2-n}-1.$$

Exact solutions of this equation are described in the literature [24, 32].

Remark 1. Solutions of unsteady-state equations corresponding to Eqs. (29)–(31) can be constructed in a similar manner.

The approach proposed is applicable not only to elliptic and parabolic equations but also to hyperbolic ones.

6. Let

$$\mathfrak{F}[T] = \frac{\partial^2 T}{\partial t^2}-\Phi(T).$$

Consider the equation ($\lambda\mu\nu \neq 0$)

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right)+\Phi(T), \quad (33)$$

which describes the propagation of nonlinear waves through an inhomogeneous anisotropic medium. This equation admits exact solutions of the form

$$T = T(r), \quad r^2 = A\left[-\frac{1}{4}(t+C)^2+\frac{e^{-\lambda x}}{a\lambda^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right],$$

where A and C are arbitrary constants. The function $T(r)$ is found from the ordinary differential equation

$$T''_{rr}+\frac{4}{A}\Phi(T) = 0,$$

which is integrable in quadrature for any $\Phi(T)$ function:

$$r = C_1 \pm \int\left[C_2-\frac{8}{A}\int\Phi(T)dT\right]^{-1/2}dT.$$

Remark 2. Exact solutions of wave analogs of heat Eqs. (26), (30), and (31) are built similarly.

Remark 3. Exact solutions of two-dimensional analogs of the above three-dimensional equations can be obtained in a similar way.

NONLINEAR EQUATIONS WITH A HEAT SOURCE OF THE LOGARITHMIC TYPE

Further, we will present some other exact solutions of nonlinear heat- and mass-transfer equations, which are obtained by the method of generalized separation of variables.

Steady-state equation. Let us examine the nonlinear equation of heat transfer in an isotropic medium with a heat source of the logarithmic type:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} = \alpha T \ln \beta T. \quad (34)$$

This equation can be regarded as a two-dimensional special case of Eq. (26) at $k = m = c = 0$, $a = b$, and $\Phi(T) = \alpha T \ln \beta T$. Therefore, Eq. (34) has exact solutions of the form

$$T = T(r), \quad r^2 = A[(x+C_1)^2+(y+C_2)^2],$$

where A, C_1, and C_2 are arbitrary constants and the unknown functions are determined from the ordinary differential equation

$$T''_{rr}+\frac{1}{r}T'_r = \frac{\alpha}{A}T\ln\beta T.$$

Exact solutions of Eq. (34) can also be sought for in the form

$$\beta T = e^U.$$

After substitution, we have

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2 = \alpha U. \quad (35)$$

Equation (35) has traveling-wave solutions

$$U(x, y) = F(u), \quad u = A_1x+A_2y+A_3, \quad (36)$$

where A_1, A_2, and A_3 are arbitrary constants. Substituting expression (36) into Eq. (35), we arrive at the equation

$$(A_1^2+A_2^2)(F''_{uu}+F'^2_u) = \alpha F,$$

whose solution is implicitly expressed as

$$u = C_1 \pm \int\left[C_2 e^{-2F}+\frac{\alpha}{A_1^2+A_2^2}\left(F-\frac{1}{2}\right)\right]^{-1/2} dF,$$

where C_1 and C_2 are arbitrary constants.

In addition, Eq. (35) has exact solutions in the form of the sum of functions of different arguments:

$$U(x, y) = \varphi(x)+\psi(y).$$

Substituting this expression into Eq. (35) yields

$$\varphi''_{xx}+\varphi'^2_x-\alpha\varphi = -\psi''_{yy}-\psi'^2_y+\alpha\psi.$$

The left and right of this equation are expressions in independent variables. Hence, the variables in this equation are separable, and the left and right sides of this equation should be equal to a constant, which, in this case, can be taken to be zero. Solving these equations, one can implicitly express the unknown functions by the formulas

$$x = A_1 \pm \int\left(B_1 e^{-2\varphi}+\alpha\varphi-\frac{1}{2}\alpha\right)^{-1/2} d\varphi, \quad (37)$$

$$y = A_2 \pm \int\left(B_2 e^{-2\psi}+\alpha\psi-\frac{1}{2}\alpha\right)^{-1/2} d\psi, \quad (38)$$

where A_1, A_2, B_1, B_2 are arbitrary constants.

Equation (35) admits more complex exact solutions in the form of the sum of functions of different arguments:

$$U(x, y) = \varphi(\xi)+\psi(\eta),$$

$$\xi = \cos(\lambda x)-\sin(\lambda y), \quad \eta = \sin(\lambda x)+\cos(\lambda y),$$

where λ is an arbitrary constant and the functions $\varphi(\xi)$ and $\psi(\eta)$ are found from relations (37) and (38), respectively.

Unsteady-state equations in one spatial variable.

1. Let us consider the one-dimensional unsteady-state equation of heat transfer in an isotropic medium:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}+\alpha T\ln T. \quad (39)$$

Exact solutions are sought for in the form

$$T = e^{U(x,t)}.$$

There are several ways of representation of $U(x, t)$ that are suitable for finding exact solutions.

Let

$$U(x, t) = \varphi(x)+\psi(t).$$

In this case, the variables in the equation are separable and equations integrable in quadrature are obtained for the unknown functions:

$$\pm\int\left(A_1 e^{-2\varphi}-\alpha\varphi+\frac{1}{2}\alpha\right)^{-1/2} d\varphi = x+A_2,$$

$$\psi(t) = A_3 e^{\alpha t},$$

where A_1, A_2, and A_3 are arbitrary constants.

Let

$$U(x, t) = \varphi(x+\beta t)+\psi(t).$$

In this instance, the variables are separable as well. The equation for ψ is easily integrable, and φ is determined from the ordinary differential equation

$$\varphi''_{\xi\xi}+\varphi'^2_\xi-\beta\varphi'_\xi+\alpha\varphi = 0, \quad \psi(t) = Ae^{\alpha t},$$

where $\xi = x+\beta t$.

2. Let us examine the one-dimensional equation

$$\frac{\partial T}{\partial t} = ax^{-k}\frac{\partial}{\partial x}\left(x^k\frac{\partial T}{\partial x}\right)+f(t)T\ln T,$$

where a and k are some constants and $f(t)$ is an arbitrary function. Note that the values $k = 0$, 1, and 2 correspond to the plane, cylindrical, and spherical cases, respectively. The variables are separable by the transformation

$$T(x, t) = e^{U(x,t)}, \quad U(x, t) = \varphi(t)x^2 + \psi(t).$$

An analysis shows that the functions $\varphi(t)$ and $\psi(t)$ can be found from the set of first-order ordinary differential equations

$$\varphi'_t = f\varphi + 4a\varphi^2, \quad \psi'_t = f\psi + 2a(k+1)\varphi.$$

The first of these equations is the Bernoulli equation; it is integrable in quadrature at an arbitrary function $f = f(t)$. If the $\varphi(t)$ dependence is known, the second equation, which is linear in ψ, is readily integrable as well. As a result, we obtain

$$\varphi(t) = e^F\left(A - 4a\int e^F dt\right)^{-1}, \quad F = \int f(t)dt,$$

$$\psi(t) = Be^F + 2a(k+1)e^F\int\varphi(t)e^{-F}dt,$$

where A and B are arbitrary constants.

3. The equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial T}{\partial x}\right] + aT\ln T + [g(x) + h(t)]T,$$

where $f(x)$, $g(x)$, and $h(t)$ are arbitrary functions, has exact solutions of the form

$$T(x, t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}h(t)dt\right]\varphi(x).$$

Here, C is an arbitrary constant and the function $\varphi(x)$ is found from the ordinary differential equation

$$(f\varphi'_x)'_x + a\varphi\ln\varphi + g(x)\varphi = 0.$$

4. A more general equation,

$$\frac{\partial T}{\partial t} = f(x)\frac{\partial^2 T}{\partial x^2} + g(x)\frac{\partial T}{\partial x} + aT\ln T + [h(x) + s(t)]T$$

has exact solutions of the form

$$T(x, t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}s(t)dt\right]\varphi(x),$$

where C is an arbitrary constant and the function $\varphi(x)$ is the solution of the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + a\varphi\ln\varphi + h(x)\varphi = 0.$$

Unsteady-state equation in two spatial variables for an isotropic medium. Let us consider the following two-dimensional equation of heat transfer in an isotropic medium:

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \alpha T\ln T. \quad (40)$$

Let us make the change $T = e^{U(x, y, t)}$. Exact solutions for the function U can be sought for in the form

$$U(x, y, t) = \varphi(x, y) + \psi(t).$$

The time dependence is described by the expression $\psi(t) = Ae^{\alpha t}$, where A is an arbitrary constant and the function $\varphi(x, y)$ obeys the steady-steady equation

$$a\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + a\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - \alpha\varphi = 0.$$

The equation for U have other exact solutions, e.g., solutions of the form

$$U(x, y, t) = \varphi(x, t) + \psi(y, t).$$

Here, the unknown functions are found from two independent one-dimensional nonlinear parabolic differential equations:

$$\frac{\partial\varphi}{\partial t} = a\frac{\partial^2\varphi}{\partial x^2} + a\left(\frac{\partial\varphi}{\partial x}\right)^2 - \alpha\varphi,$$

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial y^2} + a\left(\frac{\partial\psi}{\partial y}\right)^2 - \alpha\psi.$$

There also exist more complex exact solutions of the form

$$U(x, y, t) = \varphi(\xi, t) + \psi(\eta, t),$$

$$\xi = x + \beta t, \quad \eta = y + \gamma t.$$

Here, β and γ are arbitrary constants and the unknown functions $\varphi(\xi, t)$ and $\psi(\eta, t)$ are determined from two independent one-dimensional nonlinear parabolic differential equations

$$\frac{\partial\varphi}{\partial t} = a\frac{\partial^2\varphi}{\partial\xi^2} + a\left(\frac{\partial\varphi}{\partial\xi}\right)^2 - \beta\frac{\partial\varphi}{\partial\xi} - \alpha\varphi,$$

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial\eta^2} + a\left(\frac{\partial\psi}{\partial\eta}\right)^2 - \gamma\frac{\partial\psi}{\partial\eta} - \alpha\psi.$$

In the special case of $\varphi(\xi, t) = \varphi(\xi)$ and $\psi(\eta, t) = \psi(\eta)$, we deal with autonomous ordinary differential equations.

Unsteady-state equation in two spatial variables for an anisotropic medium. The nonlinear unsteady-state equation of heat and mass transfer and combustion in an anisotropic medium at an arbitrary concentration dependence of the main thermal diffusivities has the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[f(x, y)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x, y)\frac{\partial T}{\partial y}\right] + kT\ln T.$$

There exist exact solutions

$$T(x, y, t) = \exp(Ae^{kt})\Theta(x, y),$$

where A is an arbitrary constant and the function $\Theta(x, y)$ satisfies the steady-state equation

$$\frac{\partial}{\partial x}\left[f(x, y)\frac{\partial \Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x, y)\frac{\partial \Theta}{\partial y}\right] + k\Theta\ln\Theta = 0.$$

In the particular case of

$$f(x, y) = f(x), \quad g(x, y) = g(y)$$

there are exact solutions of the form

$$T(x, y, t) = \varphi(x, t)\psi(y, t).$$

Here, the functions $\varphi(x, t)$ and $\psi(y, t)$ are found from two independent one-dimensional nonlinear parabolic differential equations

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial \varphi}{\partial x}\right] + k\varphi\ln\varphi + C(t)\varphi,$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial y}\left[g(y)\frac{\partial \psi}{\partial y}\right] + k\psi\ln\psi - C(t)\psi,$$

where $C(t)$ is an arbitrary function.

Unsteady-state equation in three spatial variables for an isotropic medium. Let us consider the equation

$$\frac{\partial T}{\partial t} = a\Delta T + f(t)T\ln T + g(t)T,$$

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

The change $T = e^U$ leads to the equation with quadratic nonlinearity:

$$\frac{\partial U}{\partial t} = a\Delta U + a|\nabla U|^2 + f(t)U + g(t).$$

Therefore, the original equation has exact solutions

$$T(x_1, x_2, x_3, t) = \exp\left[\sum_{n, m = 1}^{3} \varphi_{nm}(t)x_n x_m + \sum_{n = 1}^{3} \psi_n(t)x_n + \chi(t)\right].$$

At $f(t) = b$ (b = const), there are also exact solutions of the form

$$T(x_1, x_2, x_3, t) = \exp[\varphi(t) + \Theta(x_1, x_2, x_3)].$$

Here, $\varphi(t)$ is given by the formula

$$\varphi(t) = Ae^{bt} + e^{bt}\int e^{-bt}g(t)dt,$$

where A is an arbitrary constant and $\Theta(x_1, x_2, x_3)$ is any solution of the steady-state equation

$$a\Delta\Theta + a|\nabla\Theta|^2 + b\Theta = 0.$$

EXACT SOLUTIONS OF SOME PROBLEMS OF THERMAL EXPLOSION IN INHOMOGENEOUS MEDIA

Let us use the above results to obtain an exact solution of the nonlinear unsteady-state problem of heat transfer with the kinetic function having an exponential form (thermal-explosion problem).

Classical theory of thermal explosion. The nonlinearity of this problem is due to the presence of distributed heat sources.

In the Frank-Kamenetskii steady-state thermal-explosion theory [14], equations are written under the assumption of uniformity and constancy of the thermal conductivities λ. The main equation that describes the temperature distribution in a spatial region with boundary S in the presence of distributed heat sources with density $Q\Phi(T)$ and the boundary conditions have the form

$$\lambda\Delta T = -Q\Phi(T), \tag{41}$$

$$r \in S, \quad T = T_s. \tag{42}$$

We assume that the temperature variation of the reaction rate is described by the Arrhenius law.

After introducing the dimensionless temperature θ and the parameter $\delta = Q/(\lambda L^2)$ (L is the characteristic length), under the assumption that the preexplosion temperature is low in comparison with the absolute temperature of the walls, the problem is substantially simplified and is described by the following ordinary differential equation and boundary conditions:

$$r\theta''_{rr} + \gamma\theta'_r + \delta re^{\theta} = 0, \tag{43}$$

$$r = 1, \quad \theta = 0; \qquad r = 0, \quad \theta'_r = 0. \tag{44}$$

Here, $\gamma = 0$, 1, and 2 correspond to the plane, cylindrical, and spherical cases, respectively. For a circular tube and sphere, r is the dimensionless radial coordinate (related to the radius).

For a plane-parallel strip ($\gamma = 0$), Eq. (43) is independent of r and easily integrable. The solution obeying the symmetry condition at the center of half-strip (44) has the form [14]

$$\theta = \ln\left[\frac{2b}{\cosh^2(\sqrt{b\delta}r)}\right]. \tag{45}$$

The constant b is found by solving the transcendental equation

$$2b = \cosh^2(\sqrt{b\delta}), \tag{46}$$

which follows from the first of the boundary conditions (44). It has been shown [14] that, at $0 \le \delta < \delta_*$, $\delta_* = 0.88$, Eq. (46) has two unequal roots. The smaller root corresponds to an unsteady temperature distribution; the larger root, to a steady one. At $\delta = \delta_*$, the roots coincide and are equal to 1.64. At $\delta > \delta_*$, the transcendental equation (46) has no roots. The critical value δ_* specifies the ignition condition for a gas mixture. The maximum preexplosion temperature is calculated by formula (45), where one should set $r = 0$ and $b = b(\delta_*)$. As a result, we obtain $\theta = 1.2$.

For a circular tube ($\gamma = 1$), Eq. (43) is also integrable in quadrature. Introducing the new variables

$$\xi = r, \quad \psi = \theta + 2\ln r,$$

one can bring this equation to the form that coincides with Eq. (43) for the plane case. Integration with the boundary conditions yields a solution [14] in the form

$$\theta = \ln(8/\delta) - 2\ln(e^{-b}r^2 + e^b), \tag{47}$$

where b is found from the transcendental equation

$$\delta\cosh^2 b = 2. \tag{48}$$

Since $\cosh^2 b \ge 1$, then, from Eq. (48), we obtain the critical value $\delta_* = 2$, which corresponds to $b_* = 0$. According to (47), the maximum preexplosion temperature at the tube axis is $\theta_* = \ln 4 \approx 1.38$.

A qualitative analysis of Eq. (43) at arbitrary γ with boundary conditions of form (44) was performed earlier [32].

Thermal explosion in inhomogeneous media. Let us demonstrate how one can extend the classical theory of thermal explosion to the case of inhomogeneous media with the use of the results obtained.

1. Let us consider a medium that is isotropic in one direction and anisotropic in the other. Assume that the anisotropy is described by an exponential function. In the dimensionless coordinates, the corresponding heat equation has the form

$$\frac{\partial}{\partial x}\left(ae^{\mu x}\frac{\partial\theta}{\partial x}\right) + \frac{\partial^2\theta}{\partial y^2} = \Phi(\theta). \tag{49}$$

This equation is a special case of Eq. (31) and has exact solutions of the form

$$\theta = \theta(r), \quad r^2 = ae^{-\mu x} + \frac{1}{4}y^2. \tag{50}$$

Here, the function is given by the ordinary differential equation

$$\theta''_{rr} = 4\Phi(\theta),$$

whose general solution is

$$r + A = \int\frac{d\theta}{\sqrt{B + 8\int\Phi(\theta)d\theta}},$$

where A and B are constants of integration.

Suppose that the kinetic function satisfies the Frank-Kamenetskii law [14]

$$\Phi(\theta) = \delta e^{\theta}.$$

Let us consider a spatial region with the boundary specified by the condition $r = 1$. Let the temperature at this boundary be constant and equal to T_s; then

$$\theta = 0 \text{ at } r = 1. \tag{51}$$

The problem we arrived at completely coincides with the above classical thermal-explosion problem; hence, its solution is given by expressions (45) and (46), where r is determined from formula (50). In this case, all the above values corresponding to the critical ignition conditions are valid.

2. Let us consider a problem of steady-state thermal explosion in a medium that is linearly anisotropic along one axis and isotropic along the other. As above, the kinetic function is taken to be exponential. The heat equation in this medium is written as

$$\frac{\partial}{\partial x}\left[a_1(|x| + a_2)\frac{\partial\theta}{\partial x}\right] + \frac{\partial^2\theta}{\partial y^2} = \delta e^{\theta}, \tag{52}$$

and condition (51) is met on the surface of the body.

Equation (52) has exact solutions of the form

$$\theta = \theta(r), \quad r^2 = \frac{|x| + a_2}{a_1} + \frac{y^2}{4}, \tag{53}$$

where $\theta(r)$ is determined from the ordinary differential equation

$$\theta''_{rr} + \frac{1}{r}\theta'_r = 4\delta e^{\theta}.$$

Note that this problem coincides with the above problem of thermal explosion in a tube. Therefore, the solution within the region under examination can be represented in form (47), with the constant b determined from the transcendental equation

$$\delta\cosh^2 b = 0.5,$$

whence the parameters corresponding to the critical ignition conditions are found to be $\delta_* = 0.5$ and $b_* = 0$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, projects no. 00-02-18033, no. 00-03-32055, and no. 99-02-17546.

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