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Generalized Separation of Variables in Nonlinear Heat and Mass Transfer Equations

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Abstract

We outline generalized separation of variables as applied to nonlinear second-order partial differential equations (PDEs). In this context, we suggest a method for constructing exact solutions of nonlinear PDEs. The method involves searching for transformations that "reduce the dimensionality" of the equation. New families of exact solutions of 3D nonlinear elliptic and parabolic equations that govern processes of heat and mass transfer in inhomogeneous anisotropic media are described. Moreover, the method makes it possible to construct exact solutions of nonlinear wave equations. We also present solutions for three families of equations with logarithmic heat sources; the solutions are obtained by nonlinear separation of variables.

Introduction

Heat and mass transfer phenomena in a medium (solid, liquid, or gas) at rest are governed by heat (diffusion) equations [1–4]. For a homogeneous and isotropic medium, the thermal diffusivity (diffusion coefficients) that occurs in these equations is constant in the entire domain under study [5–7] and the heat (diffusion) equation is a linear partial differential equation with constant coefficients. In anisotropic media, the thermal diffusivity (diffusion coefficient) depends on the heat (mass) transfer direction and, in inhomogeneous media, can depend on the coordinates and even on the temperature [8–11]. In the last case, the heat (diffusion) equation is nonlinear. Various authors suggested a lot of different relations to approximate the dependence of the transfer coefficients on the temperature or concentration, including linear, power-law, and exponential (e.g., see [8, 10, 12, 13]).

Heat (mass) transfer can be complicated by sources or sinks, which are associated with various physicochemical mechanisms of absorption and release of heat (substance). In combustion theory and nonisothermal macrokinetics of complex chemical reactions [4, 14], it is not infrequent that the power of heat sources/sinks depends on the temperature, often nonlinearly, e.g., exponentially [14] or in accordance with a power law [15]. In mass transfer theory, the rate of volumetric chemical reaction is widely approximated by power-law dependences on the concentration; at the same time, exponential, logarithmic, and other dependences are also used.

Exact solutions of heat and mass transfer equations play an important role in forming a proper understanding of qualitative features of various thermal and diffusion processes. Exact solutions of nonlinear equations make it possible to look into the mechanism of intricate phenomena such as spatial localization of heat transfer, peaking regimes, multiplicity and absence of steady states under certain conditions, etc. Even those particular exact solutions of PDEs which do not have a clear physical interpretation can be used as test problems for checking the correctness and accuracy of various numerical, asymptotic, and approximate analytical methods. In addition, model equations and problems that allow exact solutions serve as a basis for developing new numerical, asymptotic, and approximate methods. These, in turn, permit one to study more complicated problems that have no analytical solution.

Three basic approaches are traditionally used to seek exact solutions of nonlinear differential equations: (i) search for traveling-wave solutions, (ii) search for self-similar solutions, and (iii) application of groups to search for symmetries of the equations. The method of nonlinear separation of variables outlined below includes the first two approaches as its special cases and, quite often, allows finding exact solutions that cannot be obtained by application of groups. Except for special cases of partial differential equations, the precise connection between symmetries and separation of variables is not established at present; e.g., see [16, page xx].

1. Structure of Exact Solutions for Some Heat and Mass Transfer Equations

1.1. Self-similar solutions

For simplicity we consider the one-dimensional case. Self-similar solutions of one-dimensional heat equations are solutions of the form [17, 18]

$$T(x,t) = t^{\beta} f\left(\frac{x}{t^{\gamma}}\right), \tag{1}$$

where β and γ are some constants. The unknown function $f(x/t^{\gamma})$ is determined by an ordinary differential equation resulting from the substitution of solution (1) into the original PDE.

More generally, self-similar solutions are said to be solutions of the form

$$T(x,t) = \varphi(t) f\left(\frac{x}{\psi(t)}\right). \tag{2}$$

The functions $\varphi(t)$ and $\psi(t)$ are chosen for reasons of convenience in the specific problem.

For illustration, we consider a nonlinear problem describing unsteady heat transfer in a semiinfinite plate, $x \geq 0$, with thermal diffusivity depending on the temperature, $a = a(T)$. Initially, for $t \leq 0$, the plate has a uniform temperature T_i. For $t > 0$, a temperature T_s is maintained at the plate boundary $x = 0$. Thus, we have the following boundary value problem:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[a(T)\frac{\partial T}{\partial x}\right]; \quad T|_{t=0} = T_i, \ T|_{x=0} = T_s, \ T|_{x\to\infty} \to T_i. \tag{3}$$

This problem has been the subject of much investigation in heat conduction theory and seepage theory (e.g., see [4, 19]).

A solution of problem (3) is sought in the form $T = T(\omega), \omega = x/\sqrt{t}$, thus resulting in the ODE

$$[a(T)T'_\omega]'_\omega + \frac{1}{2}\omega T'_\omega = 0; \quad T|_{\omega=0} = T_s, \ T|_{\omega\to\infty} \to T_i. \tag{4}$$

Solutions of problem (4) have been obtained for linear dependence of a on T [2, 20, 21], hyperbolic approximation [3, 22], and power-law dependence [14, 23].

A detailed list of exact solutions to equations of the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[a(T)\frac{\partial T}{\partial x}\right] + \Phi(T)$$

can be found in [19] for $\Phi = 0$ and [24] for $\Phi \neq 0$.

1.2. Separation of variables in linear equations

For the sake of presentation of nonlinear separation of variables, we first briefly remind the procedure of separation of variables for linear equations. A lot of linear PDEs can be solved by separation of variables. For illustration, we consider a linear second order PDE of the from

$$F\left(x, t, T, \frac{\partial T}{\partial x}, \frac{\partial T}{\partial t}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial x \partial t}, \frac{\partial^2 T}{\partial t^2}\right) = 0, \tag{5}$$

with two independent variables, x and t, and the unknown function $T = T(x, t)$. The solution procedure involves several stages, which are outlined below.

1. At the first stage, one seeks a particular solution of the form

$$T(x, t) = \varphi(x)\psi(t). \tag{6}$$

After substituting solution (6) into equation (5), one rewrites, if possible, the equation so that its left-hand side depends only on x (involves x, φ, φ'_x, and φ''_{xx}) and the right-hand side depends only on t (involves t, ψ, ψ'_t, and ψ''_{tt}). The equality is possible only if both sides are equal to the same constant, k, called the separation constant. Thus, one obtains ODEs for $\varphi(x)$ and $\psi(t)$ which contain the parameter k.

This procedure is called separation of variables in linear equations.

2. At the second stage, the principle of linear superposition is used – a linear combination of exact solutions of a linear equation is also an exact solution of this equation.

The functions φ and ψ in solution (6) depend not only on x and t but also on the separation constant, $\varphi = \varphi(x;k)$ and $\psi = \psi(t;k)$. For various values $k_1, k_2, \ldots$ of k we obtain distinct particular solutions of equation (5),

$$T_1(x,t) = \varphi_1(x)\psi_1(t), \quad T_2(x,t) = \varphi_2(x)\psi_2(t), \ \ldots,$$

where $\varphi_i = \varphi(x;k_i)$ and $\psi_i = \psi(t;k_i), i = 1, 2, \ldots$ The spectrum of possible values of k can be established from the boundary conditions.

According to the principle of linear superposition, the sum

$$T(x,t) = A_1\varphi_1(x)\psi_1(t) + A_2\varphi_2(x)\psi_2(t) + \cdots, \tag{7}$$

where $A_1, A_2, \ldots$ are arbitrary constants, is an exact solution of the original equation. Formally, all A_i's can be set equal to 1, thus combining them with the φ_i's.

3. The third stage serves to determine the spectrum of k from the boundary conditions when solving specific problems. Here we arrive at the Sturm-Liouville eigenvalue problem for φ. The constants A_i can be determined from the initial conditions.

Remark. Note that a lot of linear equations of mathematical physics can also admit exact solutions of the form

$$T(x,t) = \vartheta(x) + \chi(t), \tag{8}$$

where $\vartheta(x)$ and $\chi(t)$ are determined by the corresponding ODEs after separating the variables.

Example. Consider the linear equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial C}{\partial x}\right] + U(x)\frac{\partial C}{\partial x} + KC + \Phi(t)$$

that governs a convective mass transfer at a velocity of $-U(x)$, provided that the diffusion coefficient $D(x)$ depends on the coordinate, a first order chemical reaction takes place, KC, and there is a volume absorption of substance with intensity depending on time, $\Phi(t)$. This equation admits solutions of the form (8) but does not have exact solutions of the form (6). However, the equation admits more complicated solutions of the form

$$C(x,t) = \vartheta(x)\chi_1(t) + \chi_2(t), \tag{9}$$

where $\chi_1(t) = \exp(Kt)$ and $\chi_2(t)$ is determined by the first order ODE $\chi_2' = K\chi_2 + \Phi(t)$.

1.3. Separation of variables in nonlinear equations

Just as linear PDEs, some nonlinear equations admit exact solutions of the form (6). In this case, the functions $\varphi(x)$ and $\psi(t)$ are determined by the ODEs obtained by substituting equation (6) into the original equation and followed by nonlinear separation of variables.

Example 1. The nonlinear heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha T^n \frac{\partial T}{\partial x}\right) \tag{10}$$

with the thermal diffusivity αT^n, where α and n are constants, admits exact solutions of the form (6), see [19].

There are also nonlinear PDEs that admit exact solutions of the form (8).

Example 2. The nonlinear heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\alpha e^{\beta T} \frac{\partial T}{\partial x}\right) \tag{11}$$

with the thermal diffusivity $\alpha e^{\beta T}$, where α and β are constants, admits exact solutions of the form (8), see [19].

Below we consider generalized separation of variables in nonlinear equations. Some aspects of this approach were considered in [25].

1. Suppose that a nonlinear equation for $T(x, t)$ is obtained from a linear equation for $u(x, t)$ admitting exact solutions of the form (6) or (8) by a nonlinear change of variable

$$T = F(u), \tag{12}$$

where $F(u)$ is some function. Then the nonlinear equation admits exact solutions of the form

$$T(x, t) = F(u), \quad u = \varphi(x)\psi(t), \tag{13}$$

$$T(x, t) = F(u), \quad u = \vartheta(x) + \chi(t). \tag{14}$$

For example, the above self-similar solution to the equation of (3) can be represented in the form (13) with $\varphi(x) = x$ and $\psi(t) = t^{-1/2}$.

Most commonly, solutions of nonlinear equations are sought in the form of traveling waves,

$$T(x, t) = F(u), \quad u = x + \lambda t. \tag{15}$$

Such solutions are special cases of equation (14) with $\vartheta(x) = x$ and $\chi(t) = \lambda t$. Note that solution (15) can also be represented in the form (13),

$$T(x,t) = F_1(v), \quad v = e^{x+\lambda t} = e^x e^{\lambda t}, \quad F_1(v) = F(\ln v).$$

Similarly, solution (14) can be represented in the form (13) by setting $u = \ln v$ and denoting $F(u) = F_1(v)$.

Usually, the functions φ and ψ or ϑ and χ, as well as the "temperature profile" $F = F(u)$, occurring in equations (13) and (14) can be determined in either of the following two ways:

– The profile $F = F(u)$ is determined by an ODE resulting from the original equation after appropriate φ and ψ (or ϑ and χ) have been chosen. The functions φ and ψ (or ϑ and χ) also are determined by ODEs. Self-similar solutions and some more complicated solutions can be found in this way.

– The profile $F = F(u)$ is prescribed a priori on the basis of some considerations (e.g., a solution of a simpler auxiliary equation can be used as the profile) so that the variables can be separated. This leads to ODEs for φ and ψ (or ϑ and χ).

Table 1 presents some specific nonlinear equations that admit exact solutions of the form (13) or (14). We do not consider here self-similar solutions with $\varphi(x) = x$ and traveling-wave solutions.

2. Suppose a nonlinear PDE for $T(x,t)$ is obtained from a linear PDE for $u(x,t)$ admitting exact solutions of the form (7) by a nonlinear change of variable $T = F(u)$. Then the nonlinear equation admits solutions of the form

$$T(x,t) = F(u), \quad u = \varphi_1(x)\psi_1(t) + \varphi_2(x)\psi_2(t) + \cdots. \tag{16}$$

The structural formula (16) can be used as a basis for seeking exact solutions to nonlinear equations that cannot be reduced to linear PDEs. The profile $F(u)$ and the functions $\varphi_1(x), \varphi_2(x), \ldots, \psi_1(t), \psi_2(t), \ldots$ are to be determined. It should be noted that generally solutions of this form cannot be obtained by group methods.

It is worth mentioning that in [28] exact solutions of the form (16) with $F(u) = u$, $\psi_2 = 1$, and $\psi_i = 0, i \geq 3$, were sought for PDEs with quadratic nonlinearities. In [25] a quite general procedure for seeking exact solutions of equations with quadratic nonlinearities for $F(u) = u$ is presented. Solutions of the form (16) are a natural extension of equations considered in the cited papers.

In the analysis of specific equations, it is useful to try the following special cases of formula (16):

$$T(x,t) = F(u), \quad u = \varphi_1(x)\psi_1(t) + \psi_2(t), \tag{17}$$

$$T(x,t) = F(u), \quad u = \varphi_1(x)\psi_1(t) + \varphi_2(x). \tag{18}$$

Table 2 presents some nonlinear equations admitting solutions of the form (16). One can see that solutions of the form (17) are most frequent.

It is important to note that in principle the representation (16) permits one to find exact solutions of nonlinear equations derived from a separable linear equation by a nonlinear transformation $T = F(u)$.

Table 1. Some nonlinear PDEs admitting (13) or (14) type solutions

Equation	Solution structure	References
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2$	$T = \varphi(x) + \psi(t)$;	[26]
	$T = \frac{a}{b}\ln u,\ u = \varphi(x) + \psi(t)$	
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^n \frac{\partial T}{\partial x}\right)$	$T = \varphi(x)\psi(t)$	[15, 17]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda T} \frac{\partial T}{\partial x}\right)$	$T = \varphi(x) + \psi(t)$	[15, 19, 27]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + aT\ln T$	$T = \varphi(x)\psi(t)$	[15, 24]
$\frac{\partial T}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial T}{\partial x}\right) + aT\ln T$	$T = \varphi(x)\psi(t)$	[15, 28]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = ae^T$	$T = -2\ln u,\ u = \varphi(x) + \psi(y)$	[14]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = a\sinh T$	$T = 2\ln\frac{1+u}{1-u},\ u = \varphi(x)\psi(y)$	[29]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = aT\ln T$	$T = e^u,\ u = \varphi(x) + \psi(y)$	[29]
$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = a\sin T$	$T = 4\arctan u,\ u = \varphi(x)\psi(y)$	[29]
$\frac{\partial}{\partial x}\left(ax^n \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial T}{\partial y}\right) = cT^k$	$T = F(u),\ u = \varphi(x) + \psi(y)$	[30]
$\frac{\partial}{\partial x}\left(ae^{\alpha x} \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y} \frac{\partial T}{\partial y}\right) = ce^{\gamma T}$	$T = F(u),\ u = \varphi(x) + \psi(y)$	[30]
$\frac{\partial}{\partial x}\left(ax^n \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y} \frac{\partial T}{\partial y}\right) = ce^{\gamma T}$	$T = F(u),\ u = \varphi(x) + \psi(y)$	[30]
$\frac{\partial}{\partial x}\left(aT^n \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(bT^m \frac{\partial T}{\partial y}\right) = 0$	$T = \varphi(x)\psi(y)$	[30]
$\frac{\partial}{\partial x}\left(ae^{\alpha T} \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta T} \frac{\partial T}{\partial y}\right) = 0$	$T = \varphi(x) + \psi(y)$	[26]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + ae^T$	$T = -2\ln u,\ u = \varphi(x) + \psi(t)$	[26]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + a\sinh T$	$T = 2\ln\frac{1+u}{1-u},\ u = \varphi(x)\psi(t)$	[29]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + aT\ln T$	$T = e^u,\ u = \varphi(x) + \psi(t)$	[29]
$\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + a\sin T$	$T = 4\arctan u,\ u = \varphi(x)\psi(t)$	[29]

Here $a,\ b,\ c,\ k,\ m,\ n,\ \alpha, \beta, \gamma$, and λ are constants.

3. Suppose now that a nonlinear equation for $T(x,t)$ is obtained from a linear equation for $u(x,t)$ by a more general nonlinear change of variable $T = g(x,t)F(u) + h(x,t)$. By narrowing the classes of the functions $g(x,t)$ and $h(x,t)$, one arrives at more simple dependences, which can be used as a basis for seeking exact solutions of nonlinear equations that cannot be reduced to linear equations.

We suggest below structural formulas that are generalizations of relations (17) and (18):

$$T(x,t) = g(t)F(u) + h(t), \quad u = \varphi_1(x)\psi_1(t) + \psi_2(t), \tag{19}$$

$$T(x,t) = g(x)F(u) + h(x), \quad u = \varphi_1(x)\psi_1(t) + \varphi_2(x). \tag{20}$$

In the special case $\varphi_1(x) = x$ and $\psi_2(t) = 0$, formula (19) corresponds to generalized self-similar solutions.

Table 2. Some nonlinear PDEs admitting solutions of the form (16)

Equation	Solution structure	References
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + bT\frac{\partial T}{\partial x}$	$T = 1/u,\ u = \varphi(x)\theta(t) + \psi(x)$	[27]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2 + c_1 T + c_0$	$T = \varphi(t)x^2 + \psi(t)x + \chi(t)$	[30]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + b\left(\frac{\partial T}{\partial x}\right)^2 + c_2 T^2 + c_1 T$	$T = \varphi(t)\theta(x) + \psi(t)$,	[28]
	$\theta(x) = e^{\lambda x},\ \theta(x) = \sin(\lambda x)$	
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right)$	$T = u^{1/m},\ u = \varphi(t)x^2 + \psi(t)$	[4, 19]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT$	$T = u^{1/m},\ u = \varphi(t)x^2 + \psi(t)$	[24–27]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT^{m+1}$	$T = u^{1/m},\ u = \varphi(t)\theta(x) + \psi(t)$	[15]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right) + bT^{1-m}$	$T = u^{1/m},\ u = \varphi(t)x^2 + \psi(t)$	[31]
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^T\frac{\partial T}{\partial x}\right) + be^T + c$	$T = \ln u,\ u = \varphi(t)\theta(x) + \psi(t)$,	[32]
	$\theta(x) = e^{\lambda x},\ \theta(x) = \sin(\lambda x)$	
$\frac{\partial T}{\partial t} = a\frac{\partial}{\partial x}\left(e^T\frac{\partial T}{\partial x}\right) + b + ce^{-T}$	$T = \ln u,\ u = \varphi(t)x^2 + \psi(t)x + \chi(t)$	[30]
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + aT\ln T + bT$	$T = e^u,\ u = \varphi(t)x + \psi(t)$;	[30]
	$T = e^u,\ u = \varphi(t)x^2 + \psi(t)$	
$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + T(a\ln^2 T + b\ln T + c)$	$T = e^u,\ u = \varphi(t)\theta(x) + \psi(t)$,	[28]
	$\theta(x) = e^{\lambda x},\ \theta(x) = \sin(\lambda x)$	
$\frac{\partial T}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial T}{\partial x}\right) + aT\ln T$	$T = e^u,\ u = \varphi(t)x^2 + \psi(t)$	[15]

Here a, b, c, c_0, c_1, c_2, m, n, and λ are constants.

2. Exact Solutions of 3D Nonlinear Heat and Mass Transfer Equations

2.1. Nonlinear separation of variables

Consider the following class of m-dimensional PDEs:

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left[p_i(x_i)\frac{\partial w}{\partial x_i}\right] = P[w], \tag{1}$$

where the $p_i(x_i)$ are some functions to be established below, $x_1, \ldots, x_m$ are independent variables ($m \geq 2$). In general, the right-hand side of equation (1) is assumed to be a given nonlinear differential operator that depends on w, its derivatives with respect to independent variables $x_{m+1}, \ldots, x_k$ that do not enter the left-hand side, and the variables $x_{m+1}, \ldots, x_k$ themselves. The unknown function w can play the role of temperature, concentration, or some other quantity.

We look for particular solutions of equation (1) of the form

$$w = w(r; \ldots), \quad r^2 = \sum_{i=1}^{m} \varphi_i(x_i), \tag{2}$$

in which the number of independent variables is reduced by $m-1$. The unknown functions $\varphi_i(x_i)$ and $p_i(x_i)$ will be determined in the course of the study.

Substituting solution (2) into equation (1), we arrive at the equation

$$\frac{1}{4r^3}\left(r\frac{\partial^2 w}{\partial r^2}-\frac{\partial w}{\partial r}\right)\sum_{i=1}^{m} p_i(\varphi_i')^2+\frac{1}{2r}\frac{\partial w}{\partial r}\sum_{i=1}^{m}(p_i\varphi_i')'=P[w], \tag{3}$$

where the primes denote the derivatives with respect to x_i.

The function of equation (2) is a solution of the original equation (1) only if the sums in equation (3) are constants or functions of r alone.

Generally, this is possible if

$$p_i(\varphi_i')^2=A\varphi_i+A_i, \quad (p_i\varphi_i')'=B\varphi_i+B_i, \tag{4}$$

where A, A_i, B, and B_i are some constants $(i=1,\ldots,m)$. In this case,

$$\sum_{i=1}^{m} p_i(\varphi_i')^2=Ar^2+A_\Sigma, \quad \sum_{i=1}^{m}(p_i\varphi_i')'=Br^2+B_\Sigma, \quad A_\Sigma=\sum_{i=1}^{m}A_i, \quad B_\Sigma=\sum_{i=1}^{m}B_i.$$

For each i we have a system of two ODEs (4) for $p_i(x_i)$ and $\varphi_i(x_i)$.

Express p_i from the first equation in (4) in terms of φ_i to obtain

$$p_i=\frac{A\varphi_i+A_i}{(\varphi_i')^2}. \tag{5}$$

Substituting this expression into the second equation in (4) yields the following autonomous equation for φ_i:

$$(A\varphi_i+A_i)\varphi_i''+(B\varphi_i+\mu_i)(\varphi_i')^2=0, \tag{6}$$

where $\mu_i=B_i-A$. This equation can be solved by the substitution $\varphi_i'=z_i(\varphi_i)$.

For $A\neq 0$ the general solution of equation (6) can be represented in the implicit form

$$\begin{aligned} x_i+C_2&=C_1\int\exp\left(\frac{B\varphi_i}{A}\right)|A\varphi_i+A_i|^{\frac{A\mu_i-BA_i}{A^2}}\,d\varphi_i,\\ \varphi_i'&=z_i(\varphi_i)=\frac{1}{C_1}\exp\left(-\frac{B\varphi_i}{A}\right)|A\varphi_i+A_i|^{\frac{BA_i-A\mu_i}{A^2}}, \end{aligned} \tag{7}$$

where C_1 and C_2 are arbitrary constants.

Table 3. Some cases where $p_i(x_i)$ and $\varphi_i(x_i)$ can be written out explicitly

#	$p_i(x_i)$	$\varphi_i(x_i)$	Relations for the parameters
1	$a_i\|x_i+s_i\|^{n_i}$	$b_i\|x_i+s_i\|^{2-n_i}+c_i$	$A_i=-Ac_i$, $B=0$, $B_i=\frac{A}{2-n_i}$, $b_i=\frac{A}{a_i(2-n_i)^2}$
2	$a_ie^{\lambda_ix_i}$	$b_ie^{-\lambda_ix_i}+c_i$	$A_i=-Ac_i$, $B=B_i=0$, $b_i=\frac{A}{a_i\lambda_i^2}$
3	$a_ix_i^2$	$b_i\ln\|x_i\|+c_i$	$A=0$, $A_i=a_ib_i^2, B=0$, $B_i=a_ib_i$
4	$(a\ln\|x_i\|+b_i)x_i^2$	$c\ln\|x_i\|+d_i$	$A=ac$, $A_i=(b_ic-ad_i)c$, $B=a$, $B_i=ac+(b_ic-ad_i)$

For $A=0$ and $A_i\neq 0$ the general solution of equation (6) is given by

$$x_i+C_2=C_1\int\exp\left(\frac{B\varphi_i^2+2B_i\varphi_i}{2A_i}\right)d\varphi_i,$$
$$\varphi_i'=z_i(\varphi_i)=\frac{1}{C_1}\exp\left(-\frac{B\varphi_i^2+2B_i\varphi_i}{2A_i}\right). \tag{8}$$

In some cases the dependences $p_i(x_i)$ and $\varphi_i(x_i)$ can be represented in explicit form. For example, if $A_i=B=B_i=0$, from (7) and (5) we obtain

$$x_i+C_2=\frac{C_1}{A}\ln|A\varphi_i|,\quad \varphi_i'=\frac{A}{C_1}\varphi_i,\quad p_i=\frac{A\varphi_i}{(\varphi_i')^2}.$$

Whence,

$$p_i(x_i)=a_ie^{\lambda_ix_i},\quad \varphi_i(x_i)=b_ie^{-\lambda_ix_i},$$

where $a_i=\pm C_1^2e^{-AC_2/C_1}$, $\lambda_i=-A/C_1$, and $b_i=\pm A^{-1}e^{AC_2/C_1}$.

Table 3 shows special cases where the $p_i(x_i)$ and $\varphi_i(x_i)$ can be represented in explicit form.

On the basis of the preceding, we can formulate results for specific equations. In this paper, we confine ourself to 3D equations and present exact solutions obtained using the above approach.

2.2. *Exact solutions of heat/mass transfer and wave equations*

Consider 3D equations corresponding to rows 1 and 2 in Table 3, which describe heat (mass) transfer or propagation of nonlinear waves in an anisotropic medium. In cases 1–4 below, we assume the operator $P[T]$ to be a nonlinear source function $\Phi(T)$.

1. The equation $(k,m,n\neq 2)$

$$\frac{\partial}{\partial x}\left(a|x|^k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(b|y|^m\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(c|z|^n\frac{\partial T}{\partial z}\right)=\Phi(T) \tag{9}$$

has exact solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{|x|^{2-k}}{a(2-k)^2} + \frac{|y|^{2-m}}{b(2-m)^2} + \frac{|z|^{2-n}}{c(2-n)^2}\right], \tag{10}$$

where A is an arbitrary constant. The function $T(r)$ is determined by the ODE

$$T''_{rr} + \frac{D}{r}T'_r = \frac{4}{A}\Phi(T), \quad D = 2\left(\frac{1}{2-k} + \frac{1}{2-m} + \frac{1}{2-n}\right) - 1. \tag{11}$$

This equation can be solved explicitly for $D = 1$ and $\Phi(T) = C\exp(\alpha T)$, where C and α are constants. For $D = 0$ and arbitrary $\Phi(T)$, equation (11) can be integrated in quadrature. For other exact solutions, see [33].

Note that $|x|, |y|$, and $|z|$ in equations (9) and (10) can be replaced by $x + s_1, y + s_2$, and $z + s_3$, respectively, where s_1, s_2, and s_3 are arbitrary constants.

For $k = m = n = 0$ and $a = b = c$, equation (9) becomes a classical equation of heat (mass) transfer in an isotropic medium with heat release (volume reaction). In this case, solution (10), (11) corresponds to a spherically symmetric case.

2. The steady-state heat equation $(\lambda\mu\nu \neq 0)$

$$\frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) = \Phi(T) \tag{12}$$

admits solutions of the form

$$T = T(r), \quad r^2 = A\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right),$$

where $T(r)$ is determined by the ODE

$$T''_{rr} - \frac{1}{r}T'_r = \frac{4}{A}\Phi(T).$$

3. The equation $(n, m \neq 2$ and $\nu \neq 0)$

$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) = \Phi(T) \tag{13}$$

admits solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where $T(r)$ is determined by the equation

$$T''_{rr} + \frac{D}{r} T'_r = \frac{4}{A} \Phi(T), \quad D = 2\left(\frac{1}{2-n} + \frac{1}{2-m}\right) - 1.$$

4. The equation ($n \neq 2$ and $\mu\nu \neq 0$)

$$\frac{\partial}{\partial x}\left(ax^n \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial T}{\partial z}\right) = \Phi(T) \tag{14}$$

has solutions of the form

$$T = T(r), \quad r^2 = A\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right].$$

The function $T(r)$ is determined by the ODE

$$T''_{rr} + \frac{D}{r} T'_r = \frac{4}{A} \Phi(T), \quad D = \frac{n}{2-n}.$$

For example, this equation is integrable in quadrature for $n = 0$ and arbitrary $\Phi(T)$ and explicitly for $n = 1$ and $\Phi(T) = Ce^{\alpha T}$, where C and α are constants.

5. Assume that $P[T] = \frac{\partial T}{\partial t} - \Phi(T)$. Consider the unsteady heat equation $(k, m, n \neq 2)$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(ax^k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^n \frac{\partial T}{\partial z}\right) + \Phi(T). \tag{15}$$

Following the approach of Subsection 2.1, we find that this equation has solutions of the form

$$T = T(t, r), \quad r^2 = 4A\left[\frac{x^{2-k}}{a(2-k)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-n}}{c(2-n)^2}\right].$$

The function $T(t, r)$ satisfies a simpler PDE with two independent variables, specifically,

$$\frac{\partial T}{\partial t} = A\left(\frac{\partial^2 T}{\partial r^2} + \frac{D}{r}\frac{\partial T}{\partial r}\right) + \Phi(T), \quad D = \frac{2}{2-k} + \frac{2}{2-m} + \frac{2}{2-n} - 1.$$

For exact solutions of this equation, see [26].

Remark 1. Solutions of unsteady equations corresponding to equations (12)–(14) can be constructed in a similar manner.

6. Assume that $P[T] = \partial^2 T/\partial t^2 - \Phi(T)$. Consider the following 3D equation describing the propagation of nonlinear waves in an inhomogeneous anisotropic medium ($\lambda\mu\nu \neq 0$):

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial T}{\partial z}\right) + \Phi(T). \tag{16}$$

It admits solutions of the form

$$T = T(r), \quad r^2 = A\left[-\frac{1}{4}(t+C)^2 + \frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where A and C are arbitrary constants and $T(r)$ is determined by the ODE

$$T''_{rr} + \frac{4}{A}\Phi(T) = 0,$$

which is integrable in quadrature for any $\Phi(T)$.

Remark 2. Solutions of wave analogues of the heat equations (9), (13), and (14) can be constructed using similar considerations.

Remark 3. Two-dimensional analogues of the 3D equations considered above can be treated in a similar manner.

3. Nonlinear Equations with a Logarithmic Source

Following the method of nonlinear separation of variables outlined in Subsection 1.3, we found solutions of a number of other nonlinear equations. We chose to present three families of equations.

3.1. A 2D steady-state heat equation

Consider the two-dimensional equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \alpha T \ln \beta T. \tag{1}$$

1. This equation can be treated as a 2D special case of equation (9) with $m = n = 0$ and $\Phi(T) = \alpha T \ln \beta T$. Thus, equation (1) has solutions of the form

$$T = T(r), \quad T''_{rr} + \frac{1}{r}T'_r = \frac{\alpha}{A}T\ln\beta T, \quad r^2 = A[(x+C_1)^2 + (y+C_2)^2],$$

where A, C_1 and C_2 are arbitrary constants.

2. Exact solutions of equation (1) can also be sought in the form $T = \frac{1}{\beta} e^{U(x,y)}$. With this change of variable, equation (1) becomes

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 = \alpha U. \tag{2}$$

Equation (2) admits traveling-wave solutions:

$$U(x,y) = F(u), \quad u = A_1 x + A_2 y + A_3, \tag{3}$$

where A_1, A_2, and A_3 are arbitrary constants. Substituting solution (3) into equation (2) and integrating the resulting equation, we obtain the dependence of $F(u)$ on u in the implicit form

$$u = C_1 \pm \int \left[C_2 e^{-2F} + \frac{\alpha}{A_1^2 + A_2^2} \left(F - \frac{1}{2} \right) \right]^{-1/2} dF,$$

where C_1 and C_2 are arbitrary constants.

3. In addition, equation (2) has solutions of the form

$$U(x,y) = \varphi(x) + \psi(y).$$

Substituting this expression into equation (2) yields

$$\varphi''_{xx} + \varphi'^2_x - \alpha\varphi = -\psi''_{yy} - \psi'^2_y + \alpha\psi.$$

It follows that the variables separate and both sides must be equal to the same constant, which here can be set equal to zero. Solving the resulting equations, we obtain

$$x = A_1 \pm \int \left(B_1 e^{-2\varphi} + \alpha\varphi - \tfrac{1}{2}\alpha \right)^{-1/2} d\varphi, \tag{4}$$

$$y = A_2 \pm \int \left(B_2 e^{-2\psi} + \alpha\psi - \tfrac{1}{2}\alpha \right)^{-1/2} d\psi, \tag{5}$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

4. Equation (2) admits also more complicated solutions of the form

$$U(x,y) = \varphi(\xi) + \psi(\eta), \quad \xi = x\cos\mu - y\sin\mu, \quad \eta = x\sin\mu + y\cos\mu,$$

where μ is an arbitrary constant and $\varphi(\xi)$ and $\psi(\eta)$ are determined by relations (4) and (5).

3.2. A 1D unsteady heat equation

Consider the one-dimensional equation

$$\frac{\partial T}{\partial t} = \frac{a}{x^k}\frac{\partial}{\partial x}\left(x^k \frac{\partial T}{\partial x}\right) + f(t)T\ln T, \tag{6}$$

where a and k are some constants and $f(t)$ is an arbitrary function. Note that the values $k = 0$, 1, and 2 correspond to the plane, cylindrical, and spherical cases. The variables separate with the transformation

$$T(x,t) = e^{U(x,t)}, \quad U(x,t) = \varphi(t)x^2 + \psi(t).$$

Analysis shows that $\varphi(t)$ and $\psi(t)$ are determined by the following system of first order ODEs:

$$\varphi'_t = f\varphi + 4a\varphi^2, \quad \psi'_t = f\psi + 2a(k+1)\varphi.$$

The first of these Bernoulli type equations is integrable in quadrature for any $f = f(t)$. Whenever $\varphi(t)$ is found, the second, linear equation can be easily solved.

3.3. A 2D unsteady heat equation

Consider the following two-dimensional heat equation:

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \alpha T\ln T.$$

We carry out the change of variable $T = e^{U(x,y,t)}$.

1. Exact solutions for U can be sought in the form $U(x,y,t) = \varphi(x,y) + \psi(t)$. The time-dependent term is expressed as $\psi(t) = Ae^{\alpha t}$, where A is an arbitrary constant. The function $\varphi(x,y)$ satisfies the stationary equation

$$a\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + a\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] - \alpha\varphi = 0,$$

which was considered in Subsection 3.1.

2. The equation for U admits other exact solutions, for example, $U(x,y,t) = \varphi(x,t) + \psi(y,t)$. The two unknown functions are determined by two independent one-dimensional nonlinear equation of the parabolic type,

$$\frac{\partial \varphi}{\partial t} = a\frac{\partial^2 \varphi}{\partial x^2} + a\left(\frac{\partial \varphi}{\partial x}\right)^2 - \alpha\varphi,$$

$$\frac{\partial \psi}{\partial t} = a\frac{\partial^2 \psi}{\partial y^2} + a\left(\frac{\partial \psi}{\partial y}\right)^2 - \alpha\psi.$$

3. The following more sophisticated solutions are also possible:

$$U(x,y,t) = \varphi(\xi,t) + \psi(\eta,t), \quad \xi = x + \beta t, \quad \eta = y + \gamma t.$$

Here, β and γ are arbitrary constants. The unknown functions $\varphi(\xi,t)$ and $\psi(\eta,t)$ are determined by two independent one-dimensional nonlinear equations of the parabolic type,

$$\frac{\partial\varphi}{\partial t} = a\frac{\partial^2\varphi}{\partial\xi^2} + a\left(\frac{\partial\varphi}{\partial\xi}\right)^2 - \beta\frac{\partial\varphi}{\partial\xi} - \alpha\varphi,$$

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial\eta^2} + a\left(\frac{\partial\psi}{\partial\eta}\right)^2 - \gamma\frac{\partial\psi}{\partial\eta} - \alpha\psi.$$

To the special case $\varphi(\xi,t) = \varphi(\xi), \psi(\eta,t) = \psi(\eta)$, there correspond autonomous ordinary differential equations.

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