# On "new travelling wave solutions" of the KdV and the KdV-Burgers equations

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**Abstract**

The Korteweg-de Vries and the Korteweg-de Vries-Burgers equations are considered. Using the travelling wave the general solutions of these equations are presented. "New travelling wave solutions" of the KdV and the KdV-Burgers equations by Wazzan [Wazzan L., Commun. Nonllinear Sci Numer Simulat, 2009; 14: 443 - 50] are analyzed. We demonstrate that all his solutions are not new and are transformed to known exact solutions.

*Keywords:* Nonlinear evolution equation; Korteweg-de Vries equation; Korteweg-de Vries-Burgers equation; General solution; Travelling wave; Exact solution

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## 1 Introduction

The Korteweg-de Vries equation takes the form

$$u_t + 6\,u\,u_x + u_{xxx} = 0. \tag{1.1}$$

Eq. (1.1) was discovered in 1895 in [2] by Korteweg and de Vries but this equation was forgotten during a long time. When great Martin D. Kruskal obtained Eq. (1.1) from the Fermi-Pasta-Ulam model [3], at the beginning he thought that he found a new nonlinear partial differential equation. He was glad but he decided to ask specialists from the department of hydrodynamics about this equation [3] and they told him about the work by Korteweg and de Vries. Now Eq. (1.1) is the most known nonlinear partial differential equation.

Eq. (1.1) is integrable and the Cauchy problem for this equation can be solved by the inverse scattering transform [4–6]. Certainly there are many different exact solutions of Eq. (1.1). This equation has soliton, rational and elliptic solutions [7–11].

Recently Wazzan [1] made an effort to obtain some new exact solutions of the Korteweg-de Vries equation. He used "a modified tanh-coth method to solve the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations". In fact, the author [1] considered these equations taking the travelling wave into account. He found 22 solitary wave solutions of the KdV equation and he believed that 14 of them were new solutions.

The aim of this paper is to analyze these "new travelling wave solutions" of the Korteweg-de Vries (KdV) and the Korteweg-de Vries-Burgers equations and show that these solutions are not new.

The paper is organized as follows. In section 2 we give the examples of nonlinear partial differential equations with solutions expressed via the general solution of the KdV equation. We also present the general solution of the KdV equation in the travelling wave and show that all other exact solutions can be found from this general solution. In section 3 we analyze the solitary wave solutions of the KdV equation by Wazzan and demonstrate that his solutions can be transformed to more simple forms. We observe that all solutions from the list by Wazzan can be obtained from the general solution of the KdV equation in the travelling wave. In section 4 we apply the simplest equation method to the KdV equation and obtain all solitary wave solutions from the list by Wazzan as well. In section 5 we give the general solution of the KdV-Burgers equation using the travelling wave and obtain all known solitary wave solutions of this equation from the general solution.

# 2 General solutions of the Korteweg-de Vries equation in the travelling wave

Let us present the general solution of the KdV equation. Using the travelling wave in Eq. (1.1)

$$u(x,t) = y(\xi), \qquad \xi = x - \omega\, t. \tag{2.1}$$

and integrating with respect to $\xi$, we have the nonlinear ordinary differential equation

$$y_{\xi\xi} + 3\,y^2 - \omega\,y + C_1 = 0, \tag{2.2}$$

where $C_1$ is a constant of integration.

Eq. (2.2) is very important and we can observe this equation in studying of many nonlinear partial differential equations. Let us give some examples of nonlinear partial differential equations where Eq. (2.2) arises.

*Example 1. The Boussinesq equation* [10, 11]

$$u_{tt} + \alpha\,u\,u_{xx} + \alpha\,u_x^2 + \beta\,u_{xxxx} = 0, \tag{2.3}$$

where $\alpha$ and $\beta$ are constants. Eq. (2.3) was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water. The Boussinesq equation also arises in many other physical applications including nonlinear lattice waves, vibrations on a nonlinear string and ion sound waves in plasma.

Taking the travelling wave $u(x,t) = U(\xi)$, $\xi = x - C_0\,t$ into account we obtain from Eq. (2.3)

$$C_0^2\,U_{\xi\xi} + \alpha\,U\,U_{\xi\xi} + \alpha\,U_\xi^2 + \beta\,U_{\xi\xi\xi\xi} = 0. \tag{2.4}$$

Integrating Eq. (2.4) with respect to $\xi$ two times we have the second order ordinary differential equation in the form

$$U_{\xi\xi} + \frac{\alpha}{2\,\beta}\,U^2 + \frac{C_0^2}{\beta}\,U + C_2\,\xi + C_3 = 0, \tag{2.5}$$

where $C_2$ and $C_3$ are constants of integration.

Assuming in Eq. (2.5)

$$U(\xi) = \frac{6\,\beta}{\alpha}\,y(\xi), \tag{2.6}$$

we have equation

$$y_{\xi\xi} + 3\,y^2 + \frac{C_0^2}{\beta}\,y + \frac{C_2\,\alpha}{6\,\beta}\,\xi + \frac{C_3\,\alpha}{6\,\beta} = 0. \tag{2.7}$$

At $C_2 \neq 0$ Eq. (2.7) can be reduced to the first Painleve equation [10, 11]

$$w_{zz} = 3\,w^2 + z. \tag{2.8}$$

The Cauchy problem for Eq. (2.8) can be solved by the inverse monodromy transform method [10] but this equation does not have solutions in the form of classical functions [10–12].

If we take

$$C_2 = 0, \qquad \omega = -\frac{C_0^2}{\beta}, \qquad C_1 = \frac{C_3\,\alpha}{6\,\beta}, \tag{2.9}$$

we have Eq. (2.2). So, in the case (2.9) we obtain the exact solutions of the Boussinesq equation (2.3) expressed via solutions of Eq. (2.2) by the formula (2.6).

*Example 2. The improved Boussinesq equation* [13]

$$u_{tt} - u_{xx} - u\,u_{xx} - u_x^2 - u_{xxtt} = 0. \tag{2.10}$$

The solitary wave solutions of Eq. (2.10) were looked for by the Exp-function method [13]. The formulae by the authors [13] are very cumbersome and we could not check them.

Let us demonstrate that the solitary wave solutions of this equation can be found via solutions of Eq. (2.2). Using the travelling wave $u(x,t) = U(\xi)$ and $\xi = x - C_0\,t$ we have from Eq. (2.10)

$$(C_0^2 - 1)U_{\xi\xi} - \frac{1}{2}\,(U^2)_{\xi\xi} - C_0^2\,U_{\xi\xi\xi\xi} = 0. \tag{2.11}$$

Integrating Eq. (2.11) with respect to $\xi$ two times we obtain

$$U_{\xi\xi} + \frac{1}{2\,C_0^2}\,U^2 - \frac{(C_0^2 - 1)}{C_0^2}\,U + \frac{C_2}{C_0^2}\,\xi + \frac{C_3}{C_0^2} = 0 \tag{2.12}$$

Eq. (2.12) at $C_2 \neq 0$ can be transformed to the first Painleve equation (2.8). Assuming $U(\xi) = 6\,C_0^2\,y(\xi)$, $\omega = \frac{C_0^2-1}{C_0^2}$ and $C_1 = \frac{C_3}{6\,C_0^4}$ at $C_2 = 0$ we get Eq. (2.2). The solitary wave solutions of Eq.(2.10) is found by the formula

$$U(\xi) = 6\,C_0^2\,y(\xi) \tag{2.13}$$

where $y(\xi)$ is a solution of Eq.(2.2).

*Example 3. The symmetric regular long wave equation* [14]

$$u_{tt} + u_{xx} + u\,u_{xt} + u_x\,u_t + u_{xxtt} = 0. \tag{2.14}$$

Xu [14] searched for the solitary wave solutions of Eq. (2.14) using the Exp-function method. We have not checked solutions by Xu because they are cumbersome as well but let us show that solutions of Eq.(2.14) can be found via solutions of Eq. (2.2).

Taking into consideration $u(x,t) = U(\xi)$, where $\xi = x - C_0\,t$ we find from Eq. (2.14)

$$(C_0^2 + 1)\,U_{\xi\xi} - C_0\,U\,U_{\xi\xi} - C_0\,U_\xi^2 + C_0^2\,U_{\xi\xi\xi\xi} = 0. \tag{2.15}$$

Eq. (2.15) can be integrated with respect to $\xi$. We obtain

$$U_{\xi\xi} - \frac{1}{2\,C_0}\,U^2 + \frac{(C_0^2+1)}{C_0^2}\,U + C_2\,\xi + C_3 = 0, \tag{2.16}$$

where $C_2$ and $C_3$ are arbitrary constants. Assuming $C_2 = 0$ and using the variable and the parameters

$$U = -6\,C_0\,y(\xi), \qquad \omega = -\frac{(1+C_0^2)}{C_0^2}, \qquad C_1 = -\frac{C_3}{6\,C_0}. \tag{2.17}$$

we have Eq. (2.2). So, the solutions of Eq. (2.15) can be obtained by

$$U(\xi) = -6\,C_0\,y(\xi), \tag{2.18}$$

where $y(\xi)$ is a solution of Eq.(2.2).

*Example 4. The generalized shallow water wave equation* [15]

$$u_{xxtt} + \alpha\,u_x\,u_{xt} + \beta\,u_t\,u_{xx} - u_{xt} - u_{xx} = 0. \tag{2.19}$$

The solitary wave solutions of Eq.(2.19) were considered taking the Exp-function method in [15]. Let us demonstrate that solutions of this equation can be found via the general solution of Eq.(2.2).

Using the travelling wave $u(x,t) = U(\xi)$ and $\xi = x - C_0\,t$ we have from Eq. (2.19) the nonlinear ordinary differential equation in the form

$$C_0^2\,U_{\xi\xi\xi\xi} - C_0\,(\alpha\,+\beta)U_\xi\,U_{\xi\xi} - (1-C_0)\,U_{\xi\xi} = 0. \tag{2.20}$$

After integration of Eq. (2.20) with respect to $\xi$ we get the equation

$$C_0^2\,U_{\xi\xi\xi} - \frac{1}{2}\,C_0\,(\alpha\,+\beta)(U_\xi)^2 - (1-C_0)\,U_\xi + C_2 = 0. \tag{2.21}$$

From Eq. (2.21) we have

$$U_{\xi\xi\xi} - \frac{(\alpha+\beta)}{2\,C_0}\,(U_\xi)^2 - \frac{(1-C_0)}{C_0^2}\,U_\xi + \frac{C_2}{C_0^2} = 0. \tag{2.22}$$

Taking the new variable and the parameters into account

$$U_\xi = -\frac{6\,C_0}{(\alpha+\beta)}\,y(\xi), \qquad \omega = \frac{(1-C_0)}{C_0^2}, \qquad C_1 = -\frac{C_2\,(\alpha+\beta)}{6\,C_0^3} \tag{2.23}$$

we have Eq. (2.2) again. Solution of Eq. (2.20) can be found by formula

$$U(\xi) = -\frac{6\,C_0}{(\alpha+\beta)}\int y(\xi)d\xi, \tag{2.24}$$

where $y(\xi)$ is solution of Eq. (2.2).

*Example 5. The Klein-Gordon equation with quadratic nonlinearity* [16]

$$u_{tt} - \alpha^2\, u_{xx} + \beta\, u - \gamma\, u^2 = 0. \tag{2.25}$$

Zhang [16] has found the solitary wave solutions by means of the Exp-function method. Let us show that these solutions are expressed via solutions of Eq.(2.2).

Taking the travelling wave $u(x,t) = U(\xi)$ and $\xi = x - C_0\, t$ again we obtain from Eq. (2.25)

$$(C_0^2 - \alpha^2)\, U_{\xi\xi} + \beta\, U - \gamma\, U^2 = 0. \tag{2.26}$$

At $C_0^2 \neq \alpha^2$ Eq. (2.26) can be written in the form

$$U_{\xi\xi} - \frac{\gamma}{(C_0^2 - \alpha^2)} U^2 + \frac{\beta}{(C_0^2 - \alpha^2)} U = 0. \tag{2.27}$$

Using new variable and parameter in Eq. (2.27)

$$U(\xi) = -\frac{3\,(C_0^2 - \alpha^2)}{\gamma}\, y(\xi), \qquad \omega = \frac{\beta}{(\alpha^2 - C_0^2)} \tag{2.28}$$

we obtain all solutions of Eq. (2.26) expressed via solutions of Eq. (2.2).

The list of equations with solutions expressed via solutions of Eq. (2.2) can be continued but we hope this list is enough to understand that Eq. (2.2) is important.

Now let us present the general solution of Eq. (2.2). Multiplying Eq. (2.2) by $y_\xi$ and integrating this equation with respect to $\xi$, we have the nonlinear differential equation in the form

$$y_\xi^2 + 2\, y^3 - \omega\, y^2 + 2\, C_1\, y + 2\, C_4 = 0, \tag{2.29}$$

where $C_4$ is the second constant of integration.

Assuming that $\alpha$, $\beta$ and $\gamma$ ($\alpha \geq \beta \geq \gamma$) are roots of the algebraic equation

$$y^3 - \frac{1}{2}\,\omega\, y^2 + C_1\, y + C_4 = 0, \tag{2.30}$$

we can write Eq. (2.29) in the form

$$y_\xi^2 = -2\,(y - \alpha)\,(y - \beta)\,(y - \gamma). \tag{2.31}$$

The general solution of Eq. (2.29) is expressed via the Jacobi elliptic function [11, 17, 18]

$$y(\xi) = \beta + (\alpha - \beta)\, cn^2\left\{\sqrt{\frac{\alpha-\gamma}{2}}\,\xi,\ S^2\right\}, \quad S^2 = \frac{\alpha-\beta}{\alpha-\gamma}, \tag{2.32}$$

where $cn(\xi)$ is the elliptic cosine. The general solution of equation (2.32) were first found by Korteweg and de Vries [2].

Comparison of Eq. (2.30) and Eq. (2.31) allows us to find relations between the roots $\alpha$, $\beta$, $\gamma$ and the constants $\omega$, $C_1$, $C_4$ in the form

$$\alpha\,\beta\,\gamma = -C_4, \quad \alpha\,\beta + \alpha\,\gamma + \beta\,\gamma = C_1, \quad \alpha + \beta + \gamma = \frac{\omega}{2}. \tag{2.33}$$

We have obtained that there are solutions of the Eq.(2.4), Eq.(2.11), Eq.(2.15), Eq.(2.20) and Eq.(2.26) which are expressed via the general solution (2.32).

The solitary wave solutions of Eq. (2.2) arise when Eq. (2.30) has two equal roots. Wazzan took $C_1 = 0$ in Eq.(2.2). Assuming $\alpha = \beta$ we have two cases of Eq. (2.31) for the solitary waves

$$y_\xi^2 = -2\,y^2\,(y - \frac{1}{2}\,\omega), \tag{2.34}$$

and

$$y_\xi^2 = -2\,\left(y - \frac{\omega}{3}\right)^2\,(y + \frac{\omega}{6}). \tag{2.35}$$

So, to find the solitary wave solutions of Eq. (2.29) at $C_1 = 0$ we need to have two known integrals

$$\int \frac{dy}{\sqrt{2\,y^2\,(y - \frac{1}{2}\,\omega)}} = -\xi, \qquad \int \frac{dy}{\sqrt{2\,\left(y - \frac{\omega}{3}\right)^2\,(y + \frac{\omega}{6})}} = -\xi. \tag{2.36}$$

Calculating these integrals we obtain the following solitary waves solutions of Eq. (2.29) at $C_1 = 0$

$$y_1 = -\frac{2\,\omega\,C_5\,e^{(\xi\sqrt{\omega})}}{\left(1 - C_5\,e^{(\xi\sqrt{\omega})}\right)^2}, \qquad \omega > 0, \tag{2.37}$$

$$y_2 = -\frac{\omega\,(1 + C_5^2)}{2\,\cos^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\,\left(1 - C_5\,\tan\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right)^2}, \qquad \omega < 0, \tag{2.38}$$

$$y_3 = \frac{\omega}{3} - \frac{\omega\,(1+C_5^2)}{2\,\cos^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\,\left(1 - C_5\,\tan\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right)^2}, \qquad \omega > 0, \tag{2.39}$$

$$y_4 = \frac{\omega}{3} + \frac{2\,\omega\,C_5\,e^{(\xi\sqrt{-\omega})}}{\left(1 - C_5\,e^{(\xi\sqrt{-\omega})}\right)^2}, \qquad \omega < 0, \tag{2.40}$$

where $C_5$ is an arbitrary constant. All solutions by Wazzan can be obtained from the solutions Eq. (2.37) - Eq. (2.40) if we use different values of the constant $C_5$ and some additional transformations.

# 3 "New travelling wave solutions" of the Korteweg-de Vries equation by Wazzan

Wazzan in [1] have found 22 solitary waves of Eq. (2.2). He have asserted that his 14 solutions are "new travelling wave solutions". Below we present these solutions ( the formulae after the first sign of equality) by Wazzan. We also give these solutions transformed by us (the formulae after the second sign of equality). We hope these formulae can be useful as the identities for the hyperbolic and the trigonometric functions. These identities are the following

$$\begin{gathered} u_1 = \frac{\omega}{3} + \frac{2\,\omega}{\exp\,\xi\,\sqrt{-\omega} - 1} + \frac{2\,\omega}{\left(\exp\,\xi\,\sqrt{-\omega} - 1\right)^2} = \\ = \frac{\omega}{3} + \frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\,t; \end{gathered} \tag{3.1}$$

$$\begin{gathered} u_2 = -\frac{2\,\omega}{\exp\,\xi\,\sqrt{-\omega} - 1} - \frac{2\,\omega}{\left(\exp\,\xi\,\sqrt{-\omega} - 1\right)^2} = \\ = -\frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\,\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\,t; \end{gathered} \tag{3.2}$$

$$\begin{gathered} u_3 = -\frac{\omega}{6} + \frac{\omega}{2}\,\left(\coth\left\{\xi\,\sqrt{-\omega}\right\} - \mathrm{csch}\left\{\xi\,\sqrt{-\omega}\right\}\right)^2 = \\ = \frac{\omega}{3} - \frac{\omega}{2\,\cosh^2\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\,t; \end{gathered} \tag{3.3}$$

$$u_4 = -\frac{\omega}{6} - \frac{\omega}{2}\,\left(\cot\{\xi\sqrt{\omega}\} - \csc\{\xi\sqrt{\omega}\}\right)^2 =$$
$$= \frac{\omega}{3} - \frac{\omega}{2\,\cos^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.4}$$

$$u_5 = -\frac{\omega}{6} - \frac{\omega}{2}\,\left(\tan\{\xi\sqrt{\omega}\} - \sec\{\xi\sqrt{\omega}\}\right)^2 =$$
$$= \frac{\omega}{3} - \frac{\omega}{\left(\cos\left\{\frac{\xi}{2}\sqrt{\omega}\right\} + \sin\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right)^2}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.5}$$

$$u_6 = \frac{\omega}{\cosh\{\xi\sqrt{\omega}\} + 1} =$$
$$= \frac{\omega}{2\,\cosh^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.6}$$

$$u_7 = \frac{\omega}{\cos\{\xi\sqrt{-\omega}\} + 1} =$$
$$= \frac{\omega}{2\,\cos^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\, t; \tag{3.7}$$

$$u_8 = \frac{\omega}{2} + \frac{\omega}{2}\,\left(\tan\{\xi\sqrt{-\omega}\} - \sec\{\xi\sqrt{-\omega}\}\right)^2 =$$
$$= \frac{\omega}{\left(\cos\left\{\frac{\xi}{2}\sqrt{-\omega}\right\} + \sin\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right)^2}, \qquad \omega < 0, \qquad \xi = x - \omega\, t; \tag{3.8}$$

$$u_9 = -\frac{\omega}{6} + \frac{\omega}{2}\,\left(\coth\{\xi\sqrt{-\omega}\} - \operatorname{csch}\{\xi\sqrt{-\omega}\}\right)^{-2} =$$
$$= \frac{\omega}{3} + \frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\, t; \tag{3.9}$$

$$u_{10} = -\frac{\omega}{6} - \frac{\omega}{2}\,\left(\cot\{\xi\,\sqrt{\omega}\} - \csc\{\xi\,\sqrt{\omega}\}\right)^{-2} =$$

$$= \frac{\omega}{3} - \frac{\omega}{2\,\sin^2\left\{\frac{\xi}{2}\,\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\,t; \tag{3.10}$$

$$u_{11} = -\frac{\omega}{6} - \frac{\omega}{2}\,\left(\tan\{\xi\,\sqrt{\omega}\} - \sec\{\xi\,\sqrt{\omega}\}\right)^{-2} =$$

$$= \frac{\omega}{3} - \frac{\omega}{\left(\cos\left\{\frac{\xi}{2}\,\sqrt{\omega}\right\} - \sin\left\{\frac{\xi}{2}\,\sqrt{\omega}\right\}\right)^2}, \qquad \omega > 0, \qquad \xi = x - \omega\,t; \tag{3.11}$$

$$u_{12} = -\frac{\omega}{\cosh\{\xi\,\sqrt{\omega}\} - 1} =$$

$$= -\frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\,\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\,t; \tag{3.12}$$

$$u_{13} = -\frac{\omega}{\cosh\{\xi\,\sqrt{-\omega}\} - 1} =$$

$$= \frac{\omega}{2\,\sin^2\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\,t; \tag{3.13}$$

$$u_{14} = \frac{\omega}{2} + \frac{\omega}{2}\,\left(\tan\{\xi\,\sqrt{\omega}\} - \sec\{\xi\,\sqrt{\omega}\}\right)^{-2} =$$

$$= \frac{\omega}{\left(\cos\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\} + \sin\left(\frac{\xi}{2}\,\sqrt{-\omega}\right)\right)^2}, \qquad \omega < 0, \qquad \xi = x - \omega\,t; \tag{3.14}$$

$$u_{15} = -\frac{\omega}{6}\left(1 - 3\,\coth^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right) =$$

$$= \frac{\omega}{3} + \frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\,t; \tag{3.15}$$

$$u_{16} = -\frac{\omega}{6}\left(1 - 3\,\tanh^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right) =$$

$$= \frac{\omega}{3} - \frac{\omega}{2\,\cosh^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\, t; \tag{3.16}$$

$$u_{17} = -\frac{\omega}{6}\left(1 + 3\,\cot^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right) =$$

$$= \frac{\omega}{3} - \frac{\omega}{2\,\sin^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.17}$$

$$u_{18} = -\frac{\omega}{6}\left(1 + 3\,\tan^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right) =$$

$$= \frac{\omega}{3} - \frac{\omega}{2\,\cos^2\left\{\frac{\xi}{2}\sqrt{\omega}z\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.18}$$

$$u_{19} = -\frac{\omega}{2}\left(\operatorname{csch}^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right) =$$

$$= -\frac{\omega}{2\,\sinh^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.19}$$

$$u_{20} = \frac{\omega}{2}\left(\operatorname{sech}^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}\right) =$$

$$= \frac{\omega}{2\,\cosh^2\left\{\frac{\xi}{2}\sqrt{\omega}\right\}}, \qquad \omega > 0, \qquad \xi = x - \omega\, t; \tag{3.20}$$

$$u_{21} = \frac{\omega}{2}\left(\csc^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right) =$$

$$= \frac{\omega}{2\,\sin^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\, t; \tag{3.21}$$

$$u_{22} = \frac{\omega}{2}\left(\sec^2\left\{\frac{\xi}{2}\sqrt{-\omega}\right\}\right) =$$
$$= \frac{\omega}{2\,\cos^2\left\{\frac{\xi}{2}\,\sqrt{-\omega}\right\}}, \qquad \omega < 0, \qquad \xi = x - \omega\,t. \tag{3.22}$$

Studying the formulae (3.1) - (3.22) we can observe that some of these solutions coincide: $u_9 = u_1$, $u_{12} = u_2$, $u_{14} = u_8$, $u_{15} = u_1$, $u_{16} = u_3$, $u_{17} = u_{10}$, $u_{18} = u_4$, $u_{19} = u_2$, $u_{20} = u_6$, $u_{21} = u_{13}$ and $u_{22} = u_7$. The solutions $u_1$, $u_2$, $u_3$, $u_4$, $u_5$, $u_6$, $u_7$, $u_8$, $u_{10}$, $u_{11}$ and $u_{13}$ are differed but these solutions can be found from the solutions (2.37) - (2.40) if we use different values of the constant $C_5$. Assuming $C_5 = 1$ in (2.40), $C_5 = 1$ in (2.37), $C_5 = -1$ in (2.40), $C_5 = 0$ and $C_5 = -1$ in (2.39), $C_5 = -1$ in (2.37), $C_5 = 0$ and $C_5 = -1$ in (2.38), $C_5 \to \infty$ and $C_5 = 1$ in (2.40), $C_5 \to \infty$ in (2.39) we accordingly have the solutions $u_1$, $u_2$, $u_3$, $u_4$, $u_5$, $u_6$, $u_7$, $u_8$, $u_{10}$, $u_{11}$ and $u_{13}$.

The results of our analysis are the following. Wazzan [1] did not find new solitary travelling wave solutions of the KdV equation. His statement that his solutions $u_1$, $u_2$, ..., $u_{14}$ are new is not correct.

# 4 Application of the simplest equation method to the Korteweg-de Vries equation

In [1] Wazzan used the complicated variant of the simplest equation method to look for the solitary wave solutions of nonlinear differential equations. He have looked for solutions of the KdV equation using the transformation in the form

$$y(\xi) = a_0 + a_1\,Y + a_2\,Y^2 + b_1\,Y^1 + b_2\,Y^{-2}, \tag{4.1}$$

where $Y \equiv Y(\xi)$ satisfies the Riccati equation

$$Y_\xi = A + B\,Y + C\,Y^2. \tag{4.2}$$

Sometimes using (4.1) and Eq. (4.2) we can find new solutions of nonlinear differential equations. This fact depends on the behaviour of the poles of solutions for the nonlinear differential equations. However the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations are not these cases.

When we started the study of paper [1] we were very surprised why author [1] had used formula (4.1) but he did not take more simple formula [19–23]

$$y(\xi) = a_0 + a_1\,Y(\xi) + a_2\,Y(\xi)^2. \tag{4.3}$$

Let us demonstrate that taking into account only transformation (4.3), where $Y(\xi)$ satisfies the Riccati equation in the form

$$Y_\xi = -Y^2 + \beta, \tag{4.4}$$

we can find solutions (2.37) - (2.40) and consequently all solutions which were found by Wazzan.

Taking $Y = \frac{\psi_\xi}{\psi}$ into account we have from (4.3)

$$y(\xi) = a_0 + a_1\,\frac{\psi_\xi}{\psi} + a_2\left(\frac{\psi_\xi}{\psi}\right)^2. \tag{4.5}$$

In this case Eq. (4.4) reduced to the second order linear equation

$$\psi_{\xi\xi} - \beta\,\psi = 0. \tag{4.6}$$

Substituting (4.3) (or (4.5)) into Eq. (2.2) and taking Eq. (4.4) (or (4.6)) into account we have the coefficients $a_0$, $a_1$, $a_2$ and the parameter $\beta$ as following

$$a_2 = -2, \qquad a_1 = 0, \qquad a_0 = \frac{4\,\beta}{3} + \frac{\omega}{6}, \qquad \beta_{1,2} = \mp\,\frac{\omega}{4}. \tag{4.7}$$

Using these coefficients and the value of the parameter $\beta$ we have two solutions of Eq. (2.2) at $\omega > 0$ in the form

$$U_1 = \frac{\omega}{3} - \frac{\omega\,({C_1}^2 + {C_2}^2)}{2\,\left(C_1\sin\left(\frac{\xi}{2}\sqrt{\omega}\right) + C_2\cos\left(\frac{\xi}{2}\sqrt{\omega}\right)\right)^2}, \qquad \omega > 0, \tag{4.8}$$

$$U_2 = \frac{2\,\omega\,C_1C_2}{\left(C_1e^{\frac{\xi}{2}\sqrt{\omega}} + C_2e^{-\frac{\xi}{2}\sqrt{\omega}}\right)^2}, \qquad \omega > 0. \tag{4.9}$$

In the case $\omega < 0$ we have the following values of the coefficients $a_0$, $a_1$, $a_2$ and the parameter $\beta$

$$a_2 = -2, \qquad a_1 = 0, \qquad a_0 = \frac{4\,\beta}{3} + \frac{\omega}{6}, \qquad \beta_{1,2} = \pm\,\frac{\omega}{4}. \tag{4.10}$$

Taking (4.10) and (4.6) we have two other solutions of Eq. (2.2)

$$U_3 = \frac{\omega\,{C_1}^2 + \omega\,{C_2}^2}{2\,\left(C_1\sin\left(\frac{\xi}{2}\sqrt{-\omega}\right) + C_2\cos\left(\frac{\xi}{2}\sqrt{-\omega}\right)\right)^2}, \qquad \omega < 0, \tag{4.11}$$

$$U_4 = \frac{\omega}{3} - \frac{2\,\omega\,C_1 C_2}{\left(C_1 e^{\frac{\xi}{2}\sqrt{-\omega}} + C_2 e^{-\frac{\xi}{2}\sqrt{-\omega}}\right)^2}, \qquad \omega < 0. \tag{4.12}$$

Assuming $C_2 = -C_5\,C_1$ in (4.9), (4.10), (4.11) and (4.12) we have $U_1 = y_3$, $U_2 = y_1$, $U_3 = y_2$ and $U_4 = y_4$ and consequently we obtain the solutions (2.37) - (2.40). We get all "new travelling wave solutions" by Wazzan using more simple method. The results of the application of the formula (4.3) (or (4.5)) show that a modified tanh-coth method did not give any new travelling wave solutions for the KdV equation.

# 5 Exact solutions of the KdV-Burgers equation

Consider the Korteweg-de Vries-Burgers equation

$$u_t + u\,u_x + \beta\,u_{xxx} - \alpha\,u_{xx} = 0 \tag{5.1}$$

Using the travelling wave $u(x,t) = y(\xi)$, $\xi = x - \omega\,t$ we have form Eq. (5.1) the nonlinear ordinary differential equation

$$\beta\,y_{\xi\xi\xi} - \alpha\,y_{\xi\xi} + y\,y_\xi - \omega\,y_\xi = 0, \tag{5.2}$$

Integrating Eq.(5.2) with respect to $\xi$ we obtain the second-order equation

$$\beta\,y_{\xi\xi} - \alpha\,y_\xi + \frac{1}{2}\,y^2 - \omega\,y + C_6 = 0, \tag{5.3}$$

where $C_6$ is a constant of integration.

We can meet Eq. (5.3) in studying of other mathematical models. For a example if we consider the Fisher equation [24–27]

$$u_t = u_{xx} + \gamma\,u\,(1 - \delta\,u). \tag{5.4}$$

then using the travelling wave $u(x,t) = U(\xi)$ and $\xi = x - C_0\,t$ we have the equation from Eq. (5.4)

$$U_{\xi\xi} + C_0\,U_\xi - \gamma\,\delta\,U^2 + \gamma\,U = 0. \tag{5.5}$$

Comparison of Eq. (5.3) and Eq. (5.5) points out that these equations coincide at $\beta = 1$, $\alpha = -C_0$, $\gamma = -\omega$, $\delta = -\frac{1}{2\,\omega}$ and $C_6 = 0$. Therefore solutions of Eq. (5.3) and Eq. (5.5) are similar.

Let us look for solution of Eq.(5.3) in the form

$$y(\xi) = b - v(\xi), \tag{5.6}$$

where b is constant which will be found. Substituting (5.6) into (5.3) we have equation

$$\beta\, v_{\xi\xi} - \alpha\, v_\xi - \frac{1}{2}\, v^2 + (b-\omega)\, v. + b\,\omega - \frac{b^2}{2} - C_6 = 0 \tag{5.7}$$

Assuming in (5.7)

$$C_6 = b\,\omega - \frac{b^2}{2} \tag{5.8}$$

we get equation

$$v_{\xi\xi} - \frac{\alpha}{\beta}\, v_\xi - \frac{1}{2\,\beta}\, v^2 + \frac{(b-\omega)}{\beta}\, v. \tag{5.9}$$

Let us search for solution of Eq.(5.9) using the new variable

$$v(\xi) = e^{-m\,\xi}\, W(\xi), \tag{5.10}$$

where $m$ is unknown parameter which will be found.

Taking (5.10) into account we have

$$v_\xi = (W_\xi - m\,W)\, e^{-m\,\xi}, \qquad v_{\xi\xi} = (m^2\, W - 2\,m\,W_\xi + W_{\xi\xi})\, e^{-m\,\xi}. \tag{5.11}$$

Substituting (5.10) and (5.11) into Eq. (5.9) we obtain the equation

$$W_{\xi\xi} - \left(2m + \frac{\alpha}{\beta}\right) W_\xi + \left(m^2 + \frac{m\alpha}{\beta} + \frac{b-\ \omega}{\beta}\right) W - \frac{1}{2\beta} e^{-m\,\xi}\, W^2 = 0. \tag{5.12}$$

Suppose

$$W(\xi) = w(z), \qquad z = \varphi(\xi), \tag{5.13}$$

we have

$$W_\xi = w_z\, \frac{dz}{d\xi}, \qquad W_{\xi\xi} = w_{zz}\, \left(\frac{dz}{d\xi}\right)^2 + w_z\, \frac{d^2 z}{d\xi^2}. \tag{5.14}$$

Substituting (5.13) and (5.14) into Eq. (5.12) we obtain the equation

$$w_{zz}\left(\frac{dz}{d\xi}\right)^2-\frac{1}{2\,\beta}\,e^{-m\,\xi}\,w^2+w_z\left(\frac{d^2z}{d\xi^2}-(2\,m+\frac{\alpha}{\beta})\frac{dz}{d\xi}\right)+$$
$$+\left(m^2+\frac{\alpha}{\beta}+\frac{b-\omega}{\beta}\right)\,w=0. \tag{5.15}$$

Assuming in Eq.(5.15)

$$\left(\frac{dz}{d\xi}\right)^2=\frac{1}{12\,\beta}\,e^{-m\,\xi}, \tag{5.16}$$

$$\frac{d^2z}{d\xi^2}=\left(2\,m+\frac{\alpha}{\beta}\right)\,\frac{dz}{d\xi}, \tag{5.17}$$

$$m^2+m\,\frac{\alpha}{\beta}+\frac{b-\omega}{\beta}=0, \tag{5.18}$$

we have equation

$$w_{zz}=6\,w^2. \tag{5.19}$$

Multiplying Eq.(5.19) by $w_z$ and integrating with respect to $z$, we have

$$w_z^2=4\,w^2-C_7, \tag{5.20}$$

where $C_7$ is an arbitrary constant. The solution of Eq.(5.20) is found by means of integral

$$\int\frac{dw}{\sqrt{4\,w^3-C_7}}=z \tag{5.21}$$

and is expressed via the Weierstrass function

$$w(z)=\wp(z+C_8,\,0,\,C_7) \tag{5.22}$$

with invariants $g_2=0$ and $g_3=C_7$. ($C_8$ is an arbitrary constant).

From Eq. (5.16) we find $z(\xi)$ in the form

$$z(\xi)=C_9-\frac{1}{m\,\sqrt{3\,\beta}}\,e^{-m\,\xi/2}, \tag{5.23}$$

where $C_9$ is arbitrary constant. Using $z(\xi)$ we obtain from Eqs. (5.17) and (5.18) values $m$ and $b$

$$m = -\frac{2\alpha}{5\,\beta}, \qquad b = \omega + \frac{6\,\alpha^2}{25\,\beta}. \tag{5.24}$$

Using this value of $b$ we obtain $C_6$ from Eq.(5.8) in the form

$$C_6 = \frac{\omega^2}{2} - \frac{18\,\alpha^4}{625\,\beta^2}. \tag{5.25}$$

Taking (5.6), (5.10), (5.19), (5.23) and (5.25) into account we have the general solution of Eq. (5.2) in the form

$$y(\xi) = \omega_k + \frac{6\alpha^2}{25\beta} - \exp\left\{\frac{2\alpha\xi}{5\,\beta}\right\}\wp\left(C_8 - \frac{5\beta}{\alpha\sqrt{12\beta}}\,\exp\left\{\frac{\alpha\,\xi}{5\,\beta}\right\},\,0,\,C_7\right),$$

$$\xi = x - \omega_k\,t, \qquad (k=1,2), \qquad \omega_{1,2} = \sqrt{2\,C_6 + \frac{36\,\alpha^4}{625\,\beta^2}}. \tag{5.26}$$

We have not seen the exact solution (5.26) of the KdV-Burgers equation in the literature but we cannot tell that this solution is new. We applied the well known approach [28] and we are sure that some people could find this solution many years ago.

The solitary travelling wave solutions of the KdV-Burgers equation can be obtained from solution (5.26) in the case $C_7 = 0$. As this takes place the solution of Eq. (5.20) takes the form

$$w(z) = \frac{1}{(C_8 \pm z)^2} \tag{5.27}$$

Using the solutions (5.26) and (5.27) we obtain the solitary travelling wave solutions in the form

$$y(\xi) = \omega + \frac{6\,\alpha^2}{25\,\beta} - \frac{\exp\left\{\frac{2\alpha\xi}{5\beta}\right\}}{\left(C_8 \pm \frac{5\,\beta}{\alpha\,\sqrt{12\,\beta}}\exp\left\{\frac{\alpha\,\xi}{5\,\beta}\right\}\right)^2}. \tag{5.28}$$

The solution (5.28) can be transformed to the usual form

$$y(\xi) = \omega + \frac{6\,\alpha^2}{25\,\beta} - \frac{12\,\alpha^2}{25\,\beta\,\left(1 \pm C_8\,\exp\left\{-\frac{\alpha\,\xi}{5\,\beta}\right\}\right)^2}. \tag{5.29}$$

At $C_6 = 0$ we have for the solutions (5.26) and (5.29) $\omega$ and $\xi$ in the form

$$\omega_{1,2} = \pm \frac{6\,\alpha^2}{25\,\beta}, \qquad \xi = x \mp \frac{6\,\alpha^2}{25\,\beta}\,t. \tag{5.30}$$

The solitary wave solutions (5.29) were first found twenty years ago in [29] by means of the singular manifold method [30–37]. The solution (5.29) can also be obtained by the simplest equation method [19–22], by the tanh-function method [38–41] and so on. The solitary wave solutions (5.29) of the KdV-Burgers equation (5.1) were obtained many times [42–49]. Certainly new solutions of this equation were not found. Any method cannot give new exact solutions to the KdV-Burgers equation.

# 6 Conclusion

Let us shortly formulate the results of our paper. We have considered two famous nonlinear evolution equations: the KdV and the KdV-Burgers equations. We have demonstrated that using the travelling wave one can find the general solutions of these equations. However Wazzan using a modified tanh-coth method obtained [1] "new solitary wave solutions" of the KdV and the KdV-Burgers equations. We have illustrated that the exact solutions by Wazzan can be transformed to more simple forms and his solutions coincide with the known solutions of the KdV and the KdV-Burgers equations. We have confirmed that using the travelling wave nobody can find new exact solutions of the KdV and the KdV-Burgers equations by any method.

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