Seven common errors in finding exact solutions of nonlinear differential equations

Nikolai A. Kudryashov *

Department of Applied Mathematics, Moscow Engineering and Physics Institute (State university), 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

---

Abstract

We analyze the common errors of the recent papers in which the solitary wave solutions of nonlinear differential equations are presented. Seven common errors are formulated and classified. These errors are illustrated by using multiple examples of the common errors from the recent publications. We show that many popular methods in finding of the exact solutions are equivalent each other. We demonstrate that some authors look for the solitary wave solutions of nonlinear ordinary differential equations and do not take into account the well - known general solutions of these equations. We illustrate several cases when authors present some functions for describing solutions but do not use arbitrary constants. As this fact takes place the redundant amount of the solutions of differential equations are found. A few examples of incorrect solutions by some authors are presented. Several other errors in finding the exact solutions of nonlinear differential equations are discussed.

Key words:
Nonlinear evolution equation, Exact solution, Common error, Truncated expansion method, Simplest equation method, Tanh - function method, Exp - function method
PACS: 02.30.Hq - Ordinary differential equations

---

1 Introduction

During the last forty years we have been observing many publications presenting the exact solutions of nonlinear evolution equations. The emergence of these publications results from the fact that there are a lot of applications

* Corresponding author.
Email address: `kudryashov@mephi.ru` (Nikolai A. Kudryashov).

of nonlinear differential equations describing different processes in many scientific areas.

The start of this science area was given in the famous work by Zabusky and Kruskal [1]. These authors showed that there are solitory waves with the property of the elastic particles in simple mathematical model. The result of the study of the Korteweg — de Vries equation was the discovery of the inverse scattering transform to solve the Cauchy problem for the integrable nonlinear differential equations [2]. Now the inverse scattering transform is used to find the solution of the Cauchy problem for many nonlinear evolution equations [3–5]. Later Hirota suggested the direct method [6] now called the Hirota method, that allows us to look for the solitary wave solutions and rational solutions for the exact solvable nonlinear differential equations [7, 8].

As to nonintegrable nonlinear evolution equations we cannot point out the best method to look for the exact solutions of nonlinear differential equations. However we prefer to use the truncated expansion method by Weiss, Tabor and Carnevalle [9] and the simplest equation method in finding the exact solutions. In section 1 we demonstrate that many popular methods to look for the exact solutions of nonlinear differential equations are based on the truncated expansion method. Many methods are obtained as consequence of the truncated expansion method and we are going to illustrate this fact in this paper.

Nowadays there are a lot of computer software programs like MATHEMATICA and MAPLE. Using these codes it is possible to have complicated analytical calculations to search for the different forms of the solutions for the nonlinear evolution equations and many authors use the computer codes to look for the exact solutions. However using the computer programs many investigators do not take into account some important properties of the differential equations. Therefore some authors obtain "new" cumbersome exact solutions of the nonlinear differential equations with some errors and mistakes.

The aim of this paper is to classify and to demonstrate some common errors that we have observed studying many publications in the last years. Using some examples of the nonlinear differential equations from the recent publications we illustrate these common errors.

The outline of this paper is as follows. In section 2 we present some popular methods to search for the exact solutions of nonintegrable differential equations and we show that in essence all these approaches are equivalent. In section 3 we analyze some reductions of partial differential equations to nonlinear ordinary differential equations. We demonstrate that many reductions have the well - known general solutions and there is no need to study the solitary wave solutions for these mathematical models. In section 4 we give some examples when the authors remove the constants of integration and lose some exact solutions. In section 5 we demonstrate that many publications contain the redundant expressions for the exact solutions and these expressions can be simplified by taking arbitrary constants into account. In section 6 we discuss that some solitary wave solutions can be simplified by the authors. In section 7 we point out that we have to check some exact solutions of nonlinear differential equations because in number of cases we have "solutions" which do not satisfy the equations studied. In section 8 we touch the solutions with redundant arbitrary constants. For many cases the redundant arbitrary constants do not lead to erroneous solutions but we believe that investigators have to take these facts into account.

2 First error: some authors use equivalent methods to find exact solutions

In this section we start to discuss common errors to search for the exact solutions of nonlinear differential equations. We observed these errors by studying many papers in the last years.

Many authors try to introduce "new methods" to look for "new solutions" of nonlinear differential equations and many investigators hope that using different approaches they can find new solutions. However analyzing numerous applications of many methods we discover that many of them are equivalent each other and in many cases it is impossible to obtain something new.

So the first error can be formulated as follows.

First error. *Some authors use the equivalent methods to find the exact solutions of nonlinear differential equations, but believe that they can find new exact solutions.*

Using the solitary wave solutions of the KdV - Burgers equation let us show that many methods to look for the exact solutions of nonlinear differential equations are equivalent.

Example 1a. *Application of the truncated expansion method.*

The truncated expansion method was introduced in [9, 10] and developed in many papers to obtain the Lax pairs, the Backlund transformations and the rational solutions for the integrable equations [11]. This approach was also used in the papers [12–21] to search for the exact solutions of nonintegrable differential equations.

Consider the application of the truncated expansion method in finding the exact solutions of the KdV - Burgers equation

$$u_t + u\,u_x + \beta\,u_{xxx} = \mu\,u_{xx}. \tag{2.1}$$

Substituting $u(x,t)$ in the form [9]

$$u(x,t) = \frac{u_0(x,t)}{F^p} + \frac{u_1(x,t)}{F^{p-1}} + ... + u_p \tag{2.2}$$

into the KdV - Burgers equation Eq.(2.1) and finding the order of the pole $p = 2$ for the solution $u(x,t)$ and functions $u_0(x,t)$ and $u_1(x,t)$ we obtain the transformation [12]

$$u(x,t) = 12\beta \frac{\partial^2 \ln F}{\partial x^2} - \frac{12\,\mu}{5}\,\frac{\partial \ln F}{\partial x} + u_2. \tag{2.3}$$

Assuming [12]

$$F = 1 + C_1\,\exp\left(k\,x - \omega\,t\right) \qquad u_2 = C_2, \tag{2.4}$$

(where C_1 and C_2 are arbitrary constants), we have the system of algebraic equations with respect to k and ω. Solving this algebraic system we have the values of k and ω as follows

$$k_{1,2} = \mp\frac{\mu}{5\,\beta}, \qquad \omega_{1,2} = \mp\frac{\mu\,C_2}{5\,\beta} - \frac{6\,\mu^3}{125\,\beta^2}. \tag{2.5}$$

For these values of k and ω the solitary wave solutions for the KdV - Burgers equation takes the form [12]

$$\begin{gathered} u(x,t) = C_2 - \frac{12\,\mu}{5}\,\frac{\partial}{\partial x}\ln\left(1 + C_1\,\exp\left(k\,x - \omega\,t\right)\right) + \\ +12\,\beta\,\frac{\partial^2}{\partial x^2}\ln\left(1 + C_1\,\exp\left(k\,x - \omega\,t\right)\right). \end{gathered} \tag{2.6}$$

We can see that solution (2.6) satisfies the nonlinear ordinary differential equation in the form

$$\beta\,k^3\,U_{\xi\xi\xi} - \mu\,k^2\,U_{\xi\xi} + k\,U\,U_\xi - \omega\,U_\xi = 0, \quad \xi = k\,x - \omega\,t. \tag{2.7}$$

We have found exact solution of the KdV — Burgers equation in essence using the travelling wave solutions although we tried to look for more general solutions. Many researches look for the exact solutions taking directly into account the travelling wave solutions.

Example 1b. *Application of the Riccati equation as the simplest equation (the first variant)* [22–26].

Solution (2.6) can be written as

$$u(\xi) = C_2 + \frac{C_1\,(60\beta\,k^2 - 12\,\mu\,k)\,\exp{(\xi)}}{5\,(1 + C_1\,\exp{(\xi)})} - \frac{12\beta\,C_1^2\,k^2\,\exp{(2\,\xi)}}{(1 + C_1\,\exp{(\xi)})^2}. \tag{2.8}$$

Consider the expression

$$H = \frac{C_1\,\exp{(\,\xi)}}{1 + C_1\,\exp{(\,\xi)}}. \tag{2.9}$$

Taking the derivative of the function H with respect to ξ we get

$$H_\xi = -\,H^2 +\,H. \tag{2.10}$$

We obtain that solution (2.8) can be written in the form

$$u(\xi) = C_2 + \left(12\beta\,k^2 - \frac{12\,\mu\,k}{5}\right)\,H - 12\,\beta\,k^2\,H^2. \tag{2.11}$$

Eq.(2.11) means that one can look for the solution of the KdV - Burgers equation using the expression

$$u(\xi) = A_0 +\,A_1\,H + A_2\,H^2, \tag{2.12}$$

where H is the solution of Eq.(2.10). It is seen, that the application of the simplest equation method with the Riccati equation gives the same results as the truncated expansion method.

Example 1c. *Application of the tanh - function method* [27–30].

Expression (2.9) can be transformed as follows

$$\begin{aligned} H &= \frac{C_1\,\exp{(\xi)}}{1 + C_1\,\exp{(\xi)}} = \frac{1}{2}\left(\frac{2\,\exp{(\xi - \xi_0)}}{1 + \exp{(\xi - \xi_0)}} - 1\right) + \frac{1}{2} = \\ &= \frac{1}{2}\left(1 + \tanh\left(\frac{\xi - \xi_0}{2}\right)\right), \quad C_1 = \exp{(-\xi_0)}. \end{aligned} \tag{2.13}$$

From (2.13) we have that solution (2.8) of the KdV - Burgers equation can be written in the form

$$\begin{aligned} u(\xi) = C_2 - \frac{6\,\mu\,k}{5} + 3\beta\,k^2 - \frac{6\,\mu\,k}{5}\tanh\left(\frac{\xi - \xi_0}{2}\right) - \\ -3\,\beta\,k^2\,\tanh^2\left(\frac{\xi - \xi_0}{2}\right). \end{aligned} \tag{2.14}$$

Substituting $k_{1,2}$ and $\omega_{1,2}$ into solution (2.14) we have the solution of the KdV - Burgers equation in the form [12]

$$u(\xi) = C_2 \pm \frac{6\mu^2}{25\,\beta}\left(1 + \tanh\left(\frac{\xi - \xi_0}{2}\right)\right) + \frac{3\mu^2}{25\beta}\left(1 + \tanh^2\left(\frac{\xi - \xi_0}{2}\right)\right),$$

$$\xi = \mp\frac{\mu}{5\,\beta}\,x - \left(\frac{6\,\mu^3}{125\,\beta^2} \pm \frac{\mu\,C_2}{5\,\beta}\right)\,t. \tag{2.15}$$

We obtain that the solution of the KdV - Burgers equation can be found as the sum of the hyperbolic tangents

$$u(\xi) = A_0 + A_1\,\tanh\left[m\,(\xi - \xi_0)\right] + A_2\,\tanh^2\left[m\,(\xi - \xi_0)\right], \tag{2.16}$$

where m is unknown parameter.

We have obtained that the tanh - function method is equivalent in essence to the truncated expansion method. As this fact takes place we can see that the maximum power of the hyperbolic tangent in (2.16) coincides with the order of the pole for the solution of the KdV - Burgers equation.

Example 1d. *Application of the Riccati equation as the simplest equation (the second variant).* [22–24].

Note that the function

$$Y(\xi) = m\,\tanh\left[m(\xi - \xi_0)\right] \tag{2.17}$$

is the general solution of the Riccati equation

$$Y_\xi = -Y^2 + m^2. \tag{2.18}$$

From (2.16) we obtain, that we can look for the solution of the KdV - Burgers equation taking into account the formula

$$u(\xi) = A_0 + A_1\,Y + A_2\,Y^2, \tag{2.19}$$

where Y satisfies Eq.(2.18). The maximum power of the function Y in (2.19) coincides with the pole of the solution for the KdV - Burgers equation.

Example 1e. *Application of the G_ξ/G method* [31–33].

Taking into account the transformation

$$Y = \frac{G_\xi}{G} \tag{2.20}$$

we reduce Eq.(2.18) to the linear equation

$$G_{\xi\xi} - \nu^2 G = 0. \tag{2.21}$$

As this fact takes place expression (2.19) can be written in the form

$$u(\xi) = A_0 + A_1 \frac{G_\xi}{G} + A_2 \left(\frac{G_\xi}{G}\right)^2. \tag{2.22}$$

This means, that we can search for the exact solutions of the KdV - Burgers equation in the form (2.22), where $G(\xi)$ satisfies Eq.(2.21). We obtain, that application of the G_ξ/G method is equivalent to application of the simplest equation method with the Riccati equation, to the tanh - function method and to the truncated expansion method.

Note that the G_ξ/G method follows directly from the truncated expansion method because solution (2.3) can be written as

$$u(x,t) = u_2 - \frac{12\,\mu}{5}\frac{F_x}{F} - 12\,\beta\,\frac{F_x^2}{F^2} + 12\,\beta\,\frac{F_{xx}}{F}. \tag{2.23}$$

Assuming that the function $F(x,t)$ satisfies linear equation of the second order

$$F_{xx} = m_0\,F_x + m_1\,F + m_2, \tag{2.24}$$

where m_0, m_1 and m_2 are unknown parameters we have the expression (2.22) for F_ξ/F. So, the G_ξ/G - method coincides with the truncated expansion method if we use the travelling wave solutions.

Example 1f. *Application of the tanh — coth method* [34–36].

Assuming in (2.16) $\xi_0 = \frac{i\pi}{2}$, we obtain the formula

$$u(\xi) = A_0 + A_1\,\coth(\nu\,\xi) + A_2\,\coth^2(\nu\,\xi). \tag{2.25}$$

This means that we can search for the solution of the KdV - Burgers equation in the form (2.25).

Using the identity for the hyperbolic functions in the form

$$2\,\coth(\nu\,\xi) = \tanh\left(\frac{\nu\,\xi}{2}\right) + \coth\left(\frac{\nu\,\xi}{2}\right), \tag{2.26}$$

and substituting (2.26) into (2.25) we have the tanh - coth method to look for the solitary wave solutions of the KdV - Burgers equation

$$\begin{gathered} u(\xi) = B_0 + B_1\,\tanh(\nu_1\,\xi) + B_2\coth(\nu_1\,\xi) + \\ + B_3\,\tanh^2(\nu_1\,\xi) + B_4\coth^2(\nu_1\,\xi). \end{gathered} \tag{2.27}$$

We have obtained, that the tanh - coth method is equivalent to the truncated expansion method too and we cannot find new exact solutions of the KdV - Burgers equation by the tanh - coth method.

Example 1g. *Application of the Exp - function method* [37–41].

Solution (2.8) of the KdV - Burgers equation can be written in the form

$$u(\xi) = \frac{5C_2 e^{-\xi} + C_1(60\beta k^2 - 12\mu k + 10C_2) + 5C_1^2(C_2 - 12\beta k^2)e^{\xi}}{5\,e^{-\xi} + 10\,C_1 + 5\,C_1^2\,e^{\,\xi}}. \quad (2.28)$$

This solution can be found if we look for a solution of the KdV — Burgers equation using the Exp - function method in the form

$$u(\xi) = \frac{a_0\,e^{-\xi} + a_1 + a_2\,e^{\xi}}{b_0\,e^{-\xi} + b_1 + b_2\,e^{\,\xi}}. \quad (2.29)$$

Using the Exp - function method to search for the exact solutions of the KdV - Burgers equation one can have again the solitary wave solutions (2.3) with the functions (2.4) and the parameters (2.5). Studying the application of the Exp - function method we obtain that this method provides no new exact solutions of nonlinear differential equations in comparison with other methods. Moreover the Exp - function method do not allow us to find the order of the pole for the solution of nonlinear differential equation [42]. Usually the authors use a few variants of fractions with the sum of exponential functions to look for the solitary wave solutions.

3 Second error: some authors do not use the known general solutions of ordinary differential equations

Many authors look for the solutions of nonlinear evolution equations using the travelling waves. As this fact takes place, these authors obtain nonlinear ordinary differential equations and search for the solutions of these equations. However nonlinear ordinary differential equations had been studied very well and the solutions of them were obtained many years ago. However some of the authors do not use the well - known general solutions of the ordinary differential equations. As a result these authors obtain the solutions that have already been found by other scientists.

The second error can be formulated as follows.

Second error. *Some authors search for solutions of nonlinear differential equations, but do not use the known general solutions of these equations.*

Example 2a. *Reduction of the (2+1) - dimensional Burgers equation by Li and Zhang* [43]

$$u_t = u\,u_y + \alpha\,v\,u_x + \beta\,u_{yy} + \alpha\,\beta\,u_{xx}, \qquad u_x = v_y. \tag{3.1}$$

The authors [43] considered the wave transformations in the form

$$u(x,y,t) = U(\xi), \quad v(x,y,t) = V(\xi), \quad \xi = k(x + ly + \lambda\,t), \tag{3.2}$$

and obtained the system of the ordinary differential equations

$$\beta\,k\,(\alpha + l^2)\,U_{\xi\xi} + \alpha\,V\,U_\xi + l\,U\,U_\xi - \lambda\,U_\xi = 0, \tag{3.3}$$

$$U_\xi - l\,V_\xi = 0. \tag{3.4}$$

Li and Zhang [43] proposed "a generalized multiple Riccati equation rational expansion method" to construct "a series of exact complex solutions" for the system of equations (3.3) and (3.4). The authors [43] found "new complex solutions" of the (2+1) - dimensional Burgers equation and "brought out rich complex solutions".

However, integrating Eq.(3.4) with respect to ξ we have

$$U = l\,V + C_1, \tag{3.5}$$

where C_1 is an arbitrary constant. Substituting U into Eq.(3.3) we obtain

$$\beta\,k\,(\alpha + l^2)\,V_{\xi\xi} + (\alpha + l^2)\,V\,V_\xi + (C_1\,l - \lambda)\,V_\xi = 0. \tag{3.6}$$

Integrating Eq.(3.6) with respect to ξ we get the Riccati equation in the form

$$V_\xi = -\frac{1}{2\beta\,k}\left(V^2 + \frac{2\,C_1\,l - 2\,\lambda}{\alpha + l^2}\,V + 2\,\beta\,k\,C_2\right). \tag{3.7}$$

Eq.(3.7) can be reduced to the form

$$\frac{dV}{d\xi} = -\frac{1}{2\beta\,k}\,(V - V_1)(V - V_2), \tag{3.8}$$

where V_1 and V_2 are the roots of the algebraic equation

$$V^2 + \frac{2\,C_1\,l - 2\,\lambda}{\alpha + l^2}\,V + 2\,\beta\,k\,C_2 = 0, \tag{3.9}$$

that take the form

$$V_{1,2} = \frac{\lambda - C_1\,l \pm \sqrt{(\lambda - C_1\,l)^2 - 2\,\beta\,k\,C_2\,(\alpha + l^2)^2}}{\alpha + l^2}. \tag{3.10}$$

Integrating Eq.(3.8) with respect to ξ, we find the general solution of Eq.(3.7) in the form

$$V(\xi)=\frac{V_1-V_2\,\exp\left(\frac{(V_2-V_1)(\xi-\xi_0)}{2\,\beta\,k}\right)}{1-\exp\left(\frac{(V_2-V_1)(\xi-\xi_0)}{2\,\beta\,k}\right)},\tag{3.11}$$

where ξ_0 is an arbitrary constant and $U(\xi)$ is determined by Eq.(3.5).

In the paper [43] the authors found 24 solitary wave solutions of the system (3.3) and (3.4), but we can see, that these solutions are useless for researches.

Example 2b. *Reduction of the (3+1) - dimensional Kadomtsev - Petviashvili equation by Zhang* [44]

$$u_{xt}+6\,u_x^2+6u\,u_{xx}-u_{xxxx}-u_{yy}-u_{zz}=0.\tag{3.12}$$

This equation was considered by Zhang [44], taking the travelling wave into account: $u=U(\eta)$, $\eta=k\,x+l\,y+\ s\,z+\omega\,t$. After reduction Zhang obtained the nonlinear ordinary differential equation in the form

$$k\,\omega\,U_{\eta\eta}+6\,k^2\,U_\eta^2+6\,k^2\,U\,U_{\eta\eta}-k^4\,U_{\eta\eta\eta\eta}-l^2\,U_{\eta\eta}-s^2\,U_{\eta\eta}=0.\tag{3.13}$$

The author [44] applied the Exp-function method and obtained the solitary wave solutions of Eq.(3.13).

However, denoting $p=\frac{k\,\omega-l^2-s^2}{k^4}$ from Eq. (3.13) we have the nonlinear ordinary differential equation in the form

$$U_{\eta\eta\eta\eta}-\frac{6}{k^2}\,U_\eta^2-\frac{6}{k^2}\,U\,U_{\eta\eta}-p\,U_{\eta\eta}=0.\tag{3.14}$$

Twice integrating Eq.(3.14) with respect to η we have

$$U_{\eta\eta}-\frac{3}{k^2}\,U^2-p\,U-C_1\,\eta+C_2=0,\tag{3.15}$$

where C_1 and C_2 are arbitrary constants. This equation is well known. Using the transformations for U and η by formulae

$$U=k^2\,\left(\frac{k^2}{C_1}\right)^{-\frac{2}{5}}\,w\;-\;\frac{p\,k^2}{6},\qquad \eta=\left(\frac{k^2}{C_1}\right)^{\frac{1}{5}}\,z\;+\;\frac{C_2}{C_1}+\frac{p^2\,k^2}{12\,C_1},\tag{3.16}$$

we have the first Painlevé equation [4, 7]

$$w_{zz}=3\,w^2+z.\tag{3.17}$$

The solutions of Eq. (3.17) are the Painlevé transcendents.

For the case $C_1 = 0$ from Eq.(3.15) we obtain the equation in the form

$$U_{\eta\eta} - \frac{3}{k^2}\, U^2 - p\, U + C_2 = 0. \tag{3.18}$$

Multiplying Eq.(3.18) on U_η we have

$$U_\eta^2 - \frac{2}{k^2}\, U^3 - p\, U^2 + 2\, C_2\, U + C_3 = 0, \tag{3.19}$$

where C_3 is an arbitrary constant.

The general solution of Eq. (3.19) is found via the Weierstrass elliptic function [45, 46]. We can see, that there is no need to look for the exact solutions of Eq. (3.13). This solution is expressed via the general solution of well known Eqs. (3.17) and (3.19).

Example 2c. *Reduction of the Ito equation by Khani* [47]

$$\phi_{xtt} + \phi_{xxxxt} + 6\, \phi_{xx}\, \phi_{xt} + 3\, \phi_x\, \phi_{xxt} + 3\, \phi_{xxx}\, \phi_t = 0. \tag{3.20}$$

Eq. (3.20) was studied by Khani [47]. The author looked for the solutions of Eq. (3.20) taking the travelling wave into account

$$\phi = \psi(\xi), \qquad \psi = k\, (x - \lambda\, t). \tag{3.21}$$

Substituting (3.21) into (3.20) when $\lambda \neq 0$ and $k \neq 0$ we obtain the equation in the form

$$\lambda\, \psi_{\xi\xi\xi} - k^2\, \psi_{\xi\xi\xi\xi\xi} - 6\, k\, \psi_{\xi\xi}\, \psi_{\xi\xi} - 6\, k\, \psi_\xi\, \psi_{\xi\xi\xi} = 0. \tag{3.22}$$

Twice integrating Eq. (3.22) with respect to ξ we have

$$\lambda\, \psi_\xi - k^2\, \psi_{\xi\xi\xi} - 3\, k\, (\psi_\xi)^2 + C_3\, \xi + C_4 = 0, \tag{3.23}$$

where C_3 and C_4 are arbitrary constants.

The author [47] looked for solution of Eq.(3.23) when $C_3 = C_4 = 0$ using the Exp - function method.

In fact, denoting $\psi_\xi = V(\xi)$ in Eq.(3.23) we get the following equation

$$k^2\, V_{\xi\xi} + 3\, k\, V^2 - \lambda\, V - C_3\, \xi - C_4 = 0. \tag{3.24}$$

Eq. (3.24) is equivalent to Eq. (3.15). The solutions of this equation are expressed for $C_3 \neq 0$ as the first Painlevé transcendents [4, 7] (see the previous example). For $C_3 = 0$ the solutions of Eq. (3.24) can be obtained using the Weierstrass elliptic function. So we need not to search for the solutions of Eq. (3.22) as well.

Example 2d. *Reduction of the (3+1) - dimensional Jimbo - Miva equation by Öziş and Aslan* [48]

$$u_{xxxy} + 3\,u_y\,u_{xx} + 3\,u_x\,u_{xy} + 2\,u_{yt} - 3\,u_{xz} = 0. \tag{3.25}$$

Using the travelling wave $u(x, y, z, t) = U(\xi)$, $\xi = k\,x + m\,y + r\,z + w\,t$ Eq. (3.25) can be written as the nonlinear ordinary differential equation

$$k^3\,m\,U_{\xi\xi\xi\xi} + 6\,k^2\,m\,U_\xi\,U_{\xi\xi} + (2\,m\,w - 3\,k\,r)\,U_{\xi\xi} = 0. \tag{3.26}$$

The authors [48] applied the Exp - function method to Eq.(3.26) to obtain "the exact and explicit generalized solitary solutions in more general forms".

But integrating Eq.(3.26) with respect to ξ we obtain

$$k^3\,m\,U_{\xi\xi\xi} + 3\,k^2\,m\,U_\xi^2 + (2\,m\,w - 3\,k\,r)\,U_\xi = C_5, \tag{3.27}$$

where C_5 is an arbitrary constant. Denoting $U_\xi = V$ we get

$$k^3\,m\,V_{\xi\xi} + 3\,k^2\,m\,V^2 + (2\,m\,w - 3\,k\,r)\,V = C_5. \tag{3.28}$$

Multiplying Eq.(3.28) on V_ξ we have the equation in the form

$$V_\xi^2 + \frac{2}{k}\,V^3 + \frac{2\,m\,w - 3\,k\,r}{m\,k^3}\,V^2 - \frac{2\,C_5}{mk^3}\,V - C_6 = 0, \tag{3.29}$$

where C_6 is an arbitrary constant. The general solution can be found using the Weierstrass elliptic function. The solution of Eq.(3.26) is found by the integral

$$U = \int V d\xi. \tag{3.30}$$

We can see that Eq.(3.27) has the general solution (3.30) and all partial cases can be found from the general solution of Eq.(3.30).

Example 2e. *Reduction of the Benjamin - Bona - Mahony equation* [49]

$$u_t - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0. \tag{3.31}$$

Eq. (3.31) was considered by Ganji and co - authors [49]. In terms of the travelling wave $u(x,t)=U(\eta)$, $\eta=k\,x-\omega\,t$ they obtained the equation

$$k^2\,\omega\;U_{\eta\eta\eta}+k\,U\,U_\eta+(k-\omega)\,U_\eta=0. \tag{3.32}$$

Using the Exp - function method the authors [49] looked for the solitary wave solutions of Eq.(3.32).

Integrating Eq. (3.32) with respect to η we have

$$k^2\,\omega\;U_{\eta\eta}+\frac{1}{2}\,k\,U^2+(k-\omega)\,U+C_1=0. \tag{3.33}$$

Multiplying Eq.(3.33) on U_η and integrating the result with respect to η again we obtain the equation

$$k^2\,\omega\;U_\eta^2+\frac{k}{3}\,U^3+(k-\omega)\,U^2+2\,C_1+C_2=0. \tag{3.34}$$

The solution of Eq.(3.34) can be given by the Weierstrass elliptic function [4, 7, 45, 46]. We can see that the authors [49] obtained the known solitary wave solutions of Eq.(3.31).

Example 2f. *Reduction of the Sharma - Tasso - Olver equation by Erbas and Yusufoglu* [50]

$$u_t+\alpha\,\left(u^3\right)_x+\frac{3}{2}\,\alpha\,\left(u^2\right)_{xx}+\alpha\,u_{xxx}=0. \tag{3.35}$$

Taking the travelling wave $u(x,t)=U(\xi)$, $\xi=\mu\,(x-c\,t)$ the authors [50] obtained the reduction of Eq.(3.35) in the form

$$\alpha\,\mu^2\,U_{\xi\xi}+3\,\alpha\,\mu\,U\,U_\xi+\alpha\,U^3-c\,U=0. \tag{3.36}$$

The authors of [50] used the Exp - function method to find "new solitonary solutions", but they left out of their account that using the transformation

$$U=\mu\,\frac{F_\xi}{F} \tag{3.37}$$

Eq.(3.36) can be transformed to the linear equation

$$F_{\xi\xi\xi}-\frac{c}{\alpha\,\mu^2}\,F_\xi=0. \tag{3.38}$$

The solution of Eq.(3.38) takes the form

$$F(\xi)=C_1+C_2\,\exp\left\{\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}+C_3\,\exp\left\{-\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}. \tag{3.39}$$

Substituting the solution (3.39) into the transformation (3.37) we obtain the solution of Eq.(3.36) in the form

$$u(\xi)=\frac{\sqrt{c}}{\sqrt{\alpha}}\,\frac{C_2\,\exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}-C_3\,\exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}{C_1+C_2\,\exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}+C_3\,\exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}. \tag{3.40}$$

Certainly all the solutions obtained by means of the Exp - function method can be found from solution (3.40).

Example 2g. *Reduction of the dispersive long wave equations by Abdou* [35]

$$\begin{gathered} v_t+v\,v_x+w_x=0,\\ w_t+(vw)_x+\frac{1}{3}\,v_{xxx}=0. \end{gathered} \tag{3.41}$$

Using the wave transformations $w(x,t)=\sigma(\xi)$, $v(x,t)=\phi(\xi)$, $\xi=k(x+\lambda\,t)$, the system of equations (3.41) can be written in the form

$$\lambda\,\phi_\xi+\phi\,\phi_\xi+\sigma_\xi=0, \tag{3.42}$$

$$\lambda\,\sigma_\xi+(\sigma\,\phi)_\xi+\frac{k^2}{2}\phi_{\xi\xi\xi}=0. \tag{3.43}$$

Abdou [35] looked for solutions of the system of equations (3.42) and (3.43) taking "the extended tanh method" into account.

Integrating Eqs. (3.42) and (3.43) with respect to ξ we obtain

$$\sigma=-C_1-\lambda\,\phi-\frac{1}{2}\,\phi^2, \tag{3.44}$$

$$\lambda\,\sigma+(\sigma\,\phi)+\frac{k^2}{2}\phi_{\xi\xi}+C_2=0. \tag{3.45}$$

Substituting the value of σ from (3.44) into Eq.(3.45) we have

$$\phi_{\xi\xi}-\frac{1}{k^3}\phi^3-\frac{3\,\lambda}{k^3}\phi^2-\frac{2(C_1+\lambda)}{k^3}\phi+\frac{2\,C_2}{k^3}=0. \tag{3.46}$$

Multiplying Eq.(3.46) on ϕ_ξ and integrating the equation with respect to ξ, we have the equation in the form

$$\phi_\xi^2-\frac{1}{2\,k^3}\phi^4-\frac{3\,\lambda}{k^3}\phi^3-\frac{(C_1+\lambda)}{k^3}\phi^2+\frac{2\,C_2}{k^3}\,\phi+C_3=0. \tag{3.47}$$

The general solution of Eq.(3.47) is determined via the Jacobi elliptic function [7]. The variable σ is found by the formula (3.44) and there is no need to look for the exact solutions of the Eqs.(3.42) and (3.43).

Unfortunately we have many similar examples. Some of them are also presented in our recent work [46].

4 Third error: some authors omit arbitrary constants after integration of equation

Reductions of nonlinear evolution equations to nonlinear ordinary differential equations can be often integrated. However, some authors assume, that the arbitrary constants of integration are equal to zero. This error potentially leads to the loss of the arbitrary constants in the final expression. So the solution obtained in such way is less general than it could be. The third common error can be formulated as follows.

Third error. *Some authors omit the arbitrary constants after integrating of the nonlinear ordinary differential equations.*

Example 3a. *Reduction of the Burgers equation by Soliman* [51].

Soliman [51] considered the Burgers equation

$$u_t + u\,u_x - \nu\,u_{xx} = 0 \tag{4.1}$$

to solve this equation by so called "the modified extended tanh - function method".

It is well known, that by using the Cole-Hopf transformation [52, 53]

$$u = -2\,\nu\,\frac{\partial}{\partial x}\ln F, \tag{4.2}$$

we can write the equality

$$u_t + u\,u_x - \nu\,u_{xx} = -2\,\nu\,\frac{\partial}{\partial x}\left(\frac{F_t - \nu\,F_{xx}}{F}\right). \tag{4.3}$$

From the last relation we can see, that each solution of the heat equation

$$F_t - \nu\,F_{xx} = 0, \tag{4.4}$$

gives the solution of the Burgers equation by formula (4.2).

However to find the solutions of the Burgers equation Soliman [51] used the travelling wave solutions $u(x,t) = U(\xi)$, $\xi = x - c\,t$ and from Eq.(4.1) after integration with respect to ξ the author obtained the equation in the form

$$\nu\, U_\xi - \frac{1}{2}\, U^2 + c\, U = 0. \tag{4.5}$$

The constant of integration he took to be equal to zero. The general solution of Eq.(4.5) takes the form

$$U(\xi) = \frac{2\,c\,C_2\,\exp\left\{-\frac{c\xi}{\nu}\right\}}{1 + C_2\,\exp\left\{-\frac{c\xi}{\nu}\right\}}, \tag{4.6}$$

where C_2 is an arbitrary constant.

The general solution of Eq.(4.5) has the only arbitrary constant. But if we take nonzero constant of integration in Eq.(4.5), we can have two arbitrary constants in the solution.

Example 3b. *Reduction of the (2+1) - dimensional Konopelchenko - Dubrovsky equation by Abdou* [54]

$$u_t - u_{xxx} - 6\,b\,u\,u_x + \frac{3\,a^2}{2}\,u^2\,u_x - 3\,v_y + 3\,a\,v\,u_x = 0, \qquad u_y = v_x. \tag{4.7}$$

Using the wave solutions

$$u(x,t) = U(\eta), \qquad \eta = k\,x + l\,y + \omega\,t \tag{4.8}$$

the author [54] looked for the exact solutions of Eq.(4.7). After integration with respect to η he obtained the second order differential equation

$$\left(\omega - \frac{3\,l^2}{k}\right)\,U - k^3\,U_{\eta\eta} + \left(\frac{3al}{2} - 3\,b\,k\right)\,U^2 + \frac{a^2}{2}\,k\,U^3 = 0, \tag{4.9}$$

but the zero constant of integration was taken. Abdou [54] used the Exp - function method to look for solitary wave solutions of Eq.(4.9).

However multiplying Eq.(4.9) on U_η and integrating this equation with respect to η again, we have the equation

$$\left(\omega - \frac{3\,l^2}{k}\right)\,U^2 - \,k^3\,U_\eta^2 + (al - 2\,b\,k)\,U^3 + \frac{a^2}{4}\,k\,U^4 = C_2, \tag{4.10}$$

where C_2 is a constant of integration. The general solution of Eq.(4.10) is expressed via the Jacobi elliptic function [7].

Example 3c. *Reduction of the Ito equation by Wazwaz* [55]

$$v_{xtt} + v_{xxxxt} + 6\, v_{xx}\, v_{xt} + 3\, v_x\, v_{xxt} + 3\, v_{xxx}\, v_t = 0. \tag{4.11}$$

The author [55] looked for the solutions of Eq. (4.11) taking into account the travelling wave

$$v = v(\xi), \qquad \xi = k\,(x - \lambda\, t). \tag{4.12}$$

Substituting (4.12) into (4.11) Wazwaz obtained when $\lambda \neq 0$ and $k \neq 0$ the equation in the form

$$\lambda\, v_{\xi\xi\xi} - k^2\, v_{\xi\xi\xi\xi\xi} - 6\, k\, v_{\xi\xi}\, v_{\xi\xi} - 6\, k\, v_{\xi}\, v_{\xi\xi\xi} = 0. \tag{4.13}$$

Integrating Eq.(4.13) twice with respect to ξ one can have the equation

$$\lambda\, v_{\xi} - k^2\, v_{\xi\xi\xi} - 3\, k\, (v_{\xi})^2 + C_8\, \xi + C_9 = 0, \tag{4.14}$$

where C_8 and C_9 are arbitrary constants. Denoting $v_{\xi} = V(\xi)$ in Eq. (4.14) we get the equation

$$k^2\, V_{\xi\xi} + 3\, k\, V^2 - \lambda\, V - C_8\, \xi - C_9 = 0. \tag{4.15}$$

The general solution of this equation was discussed above in example 2b.

However the author [55] looked for solutions of Eq. (4.14) for $C_8 = 0$ and $C_9 = 0$ taking into consideration the tanh - coth method and did not present the general solution of Eq. (4.13).

Example 3d. *Reduction of the Boussinesq equation by Bekir* [32]

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0. \tag{4.16}$$

Using the wave variable $\xi = x - c\, t$ the author [32] got the equation

$$u_{\xi\xi} - u^2 + (c^2 - 1)u = 0. \tag{4.17}$$

To look for the solitary wave solutions of Eq.(4.17) the author [32] used the G'/G - method, but he have been omitted two arbitrary constants after the integration.

In fact, from Eq.(4.16) we obtain the second order differential equation in the form

$$u_{\xi\xi} - u^2 + (c^2 - 1)u + C_1\, \xi + C_2 = 0. \tag{4.18}$$

The general solution of Eq.(4.18) is expressed at $C_1 \neq 0$ via the first Painlevé transcendents. In the case $C_1 = 0$ solutions of Eq.(4.18) is determined by the Weierstrass elliptic function. All possible solutions of Eq. (4.16) were obtained in work [56].

5 Fourth error: using some functions in finding exact solutions some authors lose arbitrary constants

Some authors do not include the arbitrary constants in finding the exact solutions of nonlinear differential equations. As a result these authors obtain many solutions, that can be determined as the only solution using some arbitrary constants. The arbitrary constants can be included, if we use the general solutions of the known differential equations. Sometimes this error can be corrected, using the property of the autonomous differential equation. So, the fourth error can be formulated as follows.

Fourth error. *Using some functions in finding the exact solutions of nonlinear differential equations some authors lose the arbitrary constants.*

Let us explain this error. Consider an ordinary differential equation in the general form

$$E(Y, Y_\xi, ...) = 0. \tag{5.1}$$

Let us assume that Eq. (5.1) is autonomous and this equation admits the shift of the independent variable $\xi \rightarrow \xi + C_2$ (where C_2 is an arbitrary constant). This means that the constant C_2 added to the variable ξ in Eq.(5.1) does not change the form of this equation. In this case Eq.(5.1) is invariant under the shift of the independent variable.

Taking this property into account, we obtain the advantage for the solution of Eq. (5.1). If we know a solution $Y = f(\xi)$ of Eq. (5.1), then for the autonomous equation we have a solution $Y = f(\xi - \xi_0)$ of this equation with additional arbitrary constant ξ_0.

The main feature of the autonomous equation is that the fact that the solution $Y = f(\xi - \xi_0)$ is more general, then $Y = f(\xi)$.

The error discussed often leads to a huge amount of different expressions for the solutions of nonlinear differential equations instead of choosing one solution with an arbitrary constant ξ_0.

The application of the tanh - function method [28–30] for finding the exact solutions allows us to have the special solutions of nonlinear differential equations as a sum of hyperbolic tangents $\tanh(\xi)$. However for the autonomous equation such types of the solutions can be taken as the more general solution in the form $\tanh(\xi - \xi_0)$.

Example 4a. *Solution of the Riccati equation*

$$Y_\xi = \beta(1 + Y^2). \tag{5.2}$$

Eq. (5.2) is of the first order, therefore the general solution of Eq. (5.2) depends on the only arbitrary constant. We can meet a lot of "different" solutions of Eq. (5.2). For example

$$Y_1(\xi) = \tan\{\beta\,\xi\}, \tag{5.3}$$

$$Y_2(\xi) = -\cot\{\beta\,\xi\}, \tag{5.4}$$

$$Y_3(\xi) = \tan\{2\beta\xi\} - \sec\{2\beta\xi\}, \tag{5.5}$$

$$Y_4(\xi) = \csc\{2\beta\xi\} - \cot\{2\beta\xi\}. \tag{5.6}$$

In fact, the general solution of Eq. (5.2) takes the form

$$Y(\xi) = \tan\{\beta(\xi - \xi_0)\}. \tag{5.7}$$

All solutions (5.3) - (5.6) can be obtained from the general solution (5.7) of the Riccati equation, because of the following equalities

$$\begin{gathered} -\cot\beta x = \tan(\beta x + \pi/2), \\ \tan 2\beta x - \sec 2\beta x = \tan(\beta x - \pi/4), \\ \csc 2\beta x - \cot 2\beta x = \tan\beta x. \end{gathered} \tag{5.8}$$

Example 4b. *Solution of the Cahn - Hilliard equation by Ugurlu and Kaya* [57]

$$u_t + u_{xxxx} = \left(u^3 - u\right)_{xx} + u_x. \tag{5.9}$$

Eq.(5.9) was considered in [57] by means of the modified extended tanh - function method by Ugurlu and Kaya. Using the travelling wave

$$u(x,t) = u(z), \qquad z = x + c\,t, \tag{5.10}$$

the authors obtained the exact solutions of the equation

$$c\,u_z + u_{zzzz} - 6\,u\,\left(u_z\right)^2 - 3\,u^2\,u_{zz} + u_{zz} - u_z = 0. \tag{5.11}$$

They found eight solutions of Eq.(5.11) at $c = 1$. Six solutions are the following

$$u_1 = \coth\left\{\frac{\sqrt{2}}{2}z\right\}, \quad u_2 = -\coth\left\{\frac{\sqrt{2}}{2}z\right\}, \quad u_3 = \tanh\left\{\frac{\sqrt{2}}{2}z\right\}, \tag{5.12}$$

$$u_4=\frac{1}{2}\left(\tanh\left\{\frac{z}{2\sqrt{2}}\right\}+\coth\left\{\frac{z}{2\sqrt{2}}\right\}\right),\quad u_5=-\tanh\left\{\frac{\sqrt{2}}{2}z\right\}, \tag{5.13}$$

$$u_6=-\frac{1}{2}\left(\tanh\left\{\frac{z}{2\sqrt{2}}\right\}+\coth\left\{\frac{z}{2\sqrt{2}}\right\}\right). \tag{5.14}$$

However all these solutions can be written as the only solution with an arbitrary constant z_0

$$u=\tanh\left\{\frac{\sqrt{2}}{2}(z-z_0)\right\}. \tag{5.15}$$

Note, that Eq.(5.11) at $c=1$ takes the form

$$u_{zzzz}-6\,u\,(u_z)^2-3\,u^2\,u_{zz}+u_{zz}=0. \tag{5.16}$$

Twice integrating Eq.(5.16) with respect to z, we have

$$u_{zz}-2\,u^3+u+C_1\,z+C_2=0, \tag{5.17}$$

where C_1 and C_2 are arbitrary constants. At $C_1=0$ the solution of Eq.(5.17) is expressed via the Jacobi elliptic function.

Example 4c. *Solution of the KdV - Burgers equation by Soliman* [51]

$$u_t+\varepsilon\,u\,u_x-\nu\,u_{xx}+\mu\,u_{xxx}=0. \tag{5.18}$$

Using the travelling wave and "the modified extended tanh - function method" the author [51] obtained four solutions of Eq.(5.18). Three of them take the form

$$u(x,t)=\frac{\nu^2}{25\varepsilon\mu}\left[9-6\coth\left(\frac{\nu\xi}{10\mu}\right)-3\coth^2\left(\frac{\nu\xi}{10\,\mu}\right)\right],\quad \xi=x-\frac{6\nu^2 t}{25\mu}; \tag{5.19}$$

$$u(x,t)=\frac{\nu^2}{25\varepsilon\mu}\left[9-6\tanh\left(\frac{\nu\xi}{10\mu}\right)-3\tanh^2\left(\frac{\nu\xi}{10\mu}\right)\right],\quad \xi=x-\frac{6\nu^2 t}{25\mu}; \tag{5.20}$$

$$\begin{gathered}u(x,t)=\frac{3\nu^2}{25\varepsilon\mu}\left[1-\frac{4}{10}\left(\tanh\left(\frac{\nu\xi}{20\mu}\right)+\coth\left(\frac{\nu\xi}{20\,\mu}\right)\right)-\right.\\ \left.-\frac{1}{10}\left(\tanh^2\left(\frac{\nu\xi}{20\mu}\right)+\coth^2\left(\frac{\nu\xi}{20\,\mu}\right)\right)\right],\quad \xi=x-\frac{6\nu^2 t}{25\mu}.\end{gathered} \tag{5.21}$$

All these solutions can be written in the form

$$u(x,t)=\frac{3\,\nu^2}{25\varepsilon\mu}\left[3-2\tanh\left(\frac{\nu\xi}{10\mu}-\xi_0\right)-\tanh^2\left(\frac{\nu\,\xi}{10\,\mu}-\xi_0\right)\right],$$
$$\xi=x-\frac{6\nu^2t}{25\mu}. \tag{5.22}$$

Assuming $\xi_0=0$ in (5.22), we obtain solution (5.20). In the case $\xi_0=\frac{i\pi}{2}$ in (5.22) we have solution (5.19). Taking into account the formulae

$$\tanh{(k\xi)}+\coth{(k\xi)}=2\,\coth{(2\,k\,\xi)}, \tag{5.23}$$

$$\tanh^2{(k\xi)}+\coth^2{(k\xi)}=4\,\coth^2{(2\,k\,\xi)}-2, \tag{5.24}$$

we can transform solution (5.22) into solution (5.21).

We can see, that these solutions do not differ, if we take the constant ξ_0 into account in one of these solutions.

The fourth solution by Soliman [51] can be simplified as well. All the solutions by the KdV — Burgers equation by Soliman coincides with (2.15).

Example 4d. *Solution of the combined KdV - mKdV equation by Bekir* [58]

$$u_t+p\,u\,u_x+q\,u^2\,u_x+r\,u_{xxx}=0. \tag{5.25}$$

Using the extended tanh method the author [58] have obtained the following solutions of Eq.(5.25)

$$u=\frac{12r}{p}[1+\tanh{(x-4rt)}], \tag{5.26}$$

$$u=\frac{12r}{p}[1+\coth{(x-4rt)}], \tag{5.27}$$

$$u=\frac{24r}{p}[(21+\tanh{(x-4rt)})+(1+\coth{(x-4rt)})]. \tag{5.28}$$

All these solutions can be written as the only solution with an arbitrary constant

$$u=\frac{12r}{p}[1+\tanh{(x-4rt+\varphi_0)}]. \tag{5.29}$$

Example 4e. *Solution of the coupled Hirota — Satsuma — KdV equation by Bekir* [59]

$$u_t=\frac{1}{4}\,u_{xxx}+3\,u\,u_x-6\,v\,u_x, \tag{5.30}$$

$$v_t = -\frac{1}{2}\, v_{xxx} - 3\, u\, v_x. \tag{5.31}$$

The system of Eqs.(5.30) and (5.31) was studied by means of the tanh - coth method by Bekir [59]. Using the travelling wave $\xi = (x - \beta\, t)$ the author obtained the system of equations

$$\begin{gathered}\frac{1}{4}\, U_{\xi\xi\xi} + 3\, U\, U_\xi - 6\, V\, V_\xi + \beta\, U_\xi = 0 \\ \frac{1}{2}\, V_{\xi\xi\xi} + 3\, U\, V_\xi - \beta\, V_\xi = 0,\end{gathered} \tag{5.32}$$

and gave ten solitary wave solutions. However all the solitary wave solutions of the system of equations (5.30) and (5.31) by Bekir [59] can be expressed by the formulae

$$\begin{gathered}U^{(1)} = \frac{\beta + 4\, k^2}{3} - 2\, k^2\, \tanh^2(k\, \xi - \xi_0), \\ V^{(1)} = \frac{2\, \beta + 2\, k^2}{3} - k^2\, \tanh^2(k\, \xi - \xi_0), \quad \xi = x - \beta\, t,\end{gathered} \tag{5.33}$$

(where k, ξ_0 and β are arbitrary constants)

$$\begin{gathered}U^{(2)} = k^2 - 2\, k^2\, \tanh^2(k\, \xi - \xi_0), \\ V^{(2)} = -k^2\, \tanh^2(k\, \xi - \xi_0), \quad \xi = x + k^2\, t, \quad \beta = -k^2,\end{gathered} \tag{5.34}$$

$$\begin{gathered}U^{(3,4)} = \frac{\beta + k^2}{3} - k^2 - k^2\, \tanh^2(k\, \xi - \xi_0), \\ V^{(3,4)} = \pm\sqrt{\frac{k^2 - 2\, \beta\, k}{3}}\, \tanh(k\, \xi - \xi_0), \quad \xi = x - \beta\, t.\end{gathered} \tag{5.35}$$

Example 4f. *"Twenty seven solution" of the "generalized Riccati equation" by Xie, Zhang and Lü* [60].

The authors [60] "firstly extend" the Riccati equation to the "general form"

$$\phi_\xi = r + p\, \phi + q\, \phi^2, \tag{5.36}$$

where r, p and q are the parameters. They "fortunately find twenty seven solutions" of Eq.(5.36).

It was very surprised that the authors are not aware that the solution of Eq.(5.36) was known more then one century ago. It is very strange but these 27 solutions was repeated by Zhang [61] as the important advantage.

Let us present the general solution of Eq.(5.36). Substituting

$$\phi = \frac{1}{q}\,V - \frac{p}{2\,q} \tag{5.37}$$

into Eq.(5.36) we have

$$V_\xi = V^2 + \frac{4rq - p^2}{4}. \tag{5.38}$$

Assuming

$$V = \frac{\psi_\xi}{\psi} \tag{5.39}$$

in Eq.(5.38) we obtain the linear equation of the second order

$$\psi_{\xi\xi} - \frac{4\,r\,q - p^2}{4}\,\psi = 0. \tag{5.40}$$

The general solution of Eq.(5.40) is well known

$$\psi(\xi) = C_1\,e^{\frac{\xi}{2}\sqrt{4rq-p^2}} + C_2\,e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}. \tag{5.41}$$

Using formula (5.39) and (5.37) we obtain the solution of Eq.(5.36) in the form

$$\phi(\xi) = \frac{\sqrt{4rq-p^2}}{2q}\,\frac{C_1\,e^{\frac{\xi}{2}\sqrt{4rq-p^2}} - C_2\,e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}}{C_1\,e^{\frac{\xi}{2}\sqrt{4rq-p^2}} + C_2\,e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}} - \frac{p}{2q}. \tag{5.42}$$

The general solution of Eq.(5.36) is found from (5.42). All 27 solutions by Xie, Zhang and Lü are found from solution (5.42) and we cannot obtain other solutions.

Example 4g. *"New solutions and kinks solutions" of the Sharma — Tasso — Olver equation by Wazwaz* [62].

Using the extended tanh method Wazwaz [62] have found 18 solitary wave and kink solutons of the Sharma — Tasso —- Olver equation

$$u_t + \alpha\,(u^3)_x + \frac{3}{2}\,\alpha\,(u^2)_{xx} + \,\alpha\,u_{xxx} = 0. \tag{5.43}$$

Taking the travelling wave solution $u(x,t) = u(\xi)$, $\xi = x - c\,t$ into account the author considered the nonlinear ordinary differential equation in the form

$$\alpha\,u_{\xi\xi} + 3\,\alpha\,u\,u_\xi + \alpha\,u^3 - c\,u = 0. \tag{5.44}$$

However Eq.(5.44) can be transformed to the second - order linear differential equation (see, example 2f)

$$F_{\xi\xi\xi} - \frac{c}{\alpha}\,F_\xi = 0 \tag{5.45}$$

by the transformation

$$u(\xi) = \frac{F_\xi}{F}. \qquad (5.46)$$

The solution of Eq.(5.44) is given by formula (3.40) at $\mu = 1$. Certainly all "new solutions" of the Sharma — Tasso — Olver equation by Wazwaz [62] are found from (3.40) at $\mu = 1$.

We can see, that there is no need to write a list of all possible expressions for the solutions at the given ξ_0. It is enough to present the solution of the equation with an arbitrary constant. Moreover, the solution with arbitrary constants looks better.

The simple and powerful tool to remove this error is to plot the graphs of the expressions obtained. The expressions having the same graphs usually are equivalent.

6 Fifth error: some authors do not simplify the solutions of differential equations

Using different approaches for nonlinear differential equations, the authors obtain different forms of the solutions and sometimes they find the solutions in cumbersome forms. Interpretation and understanding of such solutions are difficult and we believe, that these solutions can be simplified. The consequence of the cumbersome expressions is a risk of misprints. The solutions in cumbersome form give an illusion that the authors indeed find new solutions. We meet many such solutions of nonlinear differential equations especially found by means of the Exp — function method. Another consequence is the fact, that the authors obtain the same solutions expressed in different forms, because it is well known, that trigonometric and hyperbolic expressions admit different representation.

The fifth error can be formulated as follows.

Fifth error. *Expressions for solutions of nonlinear differential equations are often not simplified.*

Example 5a. *Solutions of the Jimbo - Miva equation* (3.26) *by Öziş and Aslan* [48].

Using the Exp - function method Öziş and Aslan [48] found some solitary wave solutions of Eq. (3.26), but a number of these solutions can be simplified. For example solution (30) in [48] is transformed by the following set of equalities

$$u(x,y,z,t)=\frac{a_2\,e^{2\,\xi}+a_2\,b_1\,e^{\xi}+b_0\,(a_2-4\,k)+b_0\,b_1\,(a_2-4\,k)\,e^{-\xi}}{e^{2\,\xi}+b_1\,e^{\xi}+b_0+b_0\,b_1\,e^{-\xi}}=$$
$$=\frac{\left(a_2\,e^{\xi}+b_0\,a_2\,e^{-\xi}-4\,b_0\,k\,e^{-\xi}\right)\left(e^{\xi}+b_1\right)}{\left(e^{\xi}+b_0\,e^{-\xi}\right)\,\left(e^{\xi}+b_1\right)}=a_2-\frac{4\,b_0\,k}{b_0+e^{2\,\xi}}. \tag{6.1}$$

It is seen, that the last expression is better for understanding, then the solution by Öziş and Aslan.

Example 5b. *Solutions of the Konopelchenko - Dubrovsky equation* (4.7) *by Abdou* [54].

By means of the Exp - function method Abdou [54] obtained some solitary wave solutions of Eq. (4.7), but some solutions can be simplified. For example solution (42) in [54] reduces to

$$u(x,t)=\frac{a_2\,e^{2\,\eta}+a_1\,e^{\eta}-\frac{a_1^2\,(a_2+2\,k^2)}{2\,k^4}+\frac{a_1^3}{4\,k^4}\,e^{-\eta}+\frac{a_2\,a_1^4}{16\,k^8}\,e^{-2\,\eta}}{\frac{a_1^4}{16\,k^8}\,e^{-2\,\eta}+\,e^{2\,\eta}-\frac{a_1^2}{2\,k^4}}=$$
$$=a_2+\frac{a_1\,\left(e^{\frac{\eta}{2}}-\frac{a_1}{2\,k^2}\,e^{-\frac{-\eta}{2}}\right)^2}{\left(e^{\eta}-\frac{a_1^2}{4\,k^4}\,e^{-\eta}\right)^2}=a_2+\frac{a_1}{\left(e^{\frac{\eta}{2}}+\frac{a_1}{2\,k}\,e^{-\frac{-\eta}{2}}\right)^2}. \tag{6.2}$$

Example 5c. *Solution of the modified Benjamin - Bona - Mahony equation by Yusufoğlu* [63]

$$u_t+u_x+\alpha\,u^2\,u_x+u_{xxx}=0. \tag{6.3}$$

Using the travelling wave the author [63] obtained the equation

$$\beta^2\,U_{\xi\xi}+\frac{1}{3}\,\alpha\,U^3+(1-\gamma)\,U=0. \tag{6.4}$$

Yusufoğlu [63] found the solution of Eq.(6.4) in the form

$$u(\xi)=\frac{\frac{24b_0b_1(c-1)}{a_0a}e^{\xi}+a_0}{\frac{24b_0b_1^2(c-1)}{a_0^2a}e^{2\xi}+b_1e^{\xi}+b_0+\frac{a_0^2a}{24b_1(c-1)}e^{-\xi}}. \tag{6.5}$$

Solution (6.5) has three arbitrary constants a_0, b_0 and b_1, but in fact this solution can be simplified using the set of equalities

$$u(\xi) = \frac{\frac{24 b_0 b_1 (c-1)}{a_0 a} e^{\xi} + a_0}{\frac{24 b_0 b_1^2 (c-1)}{a_0^2 a} e^{2\xi} + b_1 e^{\xi} + b_0 + \frac{a_0^2 a}{24 b_1 (c-1)} e^{-\xi}} =$$

$$= \frac{1}{\frac{a\,a_0}{24\,b_1 (c-1)} e^{-\xi} + \frac{b_1}{a_0}\,e^{\xi}} = \frac{24\,b\,(c-1)}{a\,e^{-\xi} + 24\,b^2\,(c-1)\,e^{\xi}}. \tag{6.6}$$

We can see, that solution (6.6) contains the only arbitrary constant $b = \frac{b_1}{a_0}$.

Note that Eq.(6.3) can be easy transformed to the modified KdV equation and this equation is not the BBM equation.

Example 5d. *Solutions of the fifth - order KdV equation by Chun* [64]

$$u_t + 30\,u^2\,u_x + 20\,u_x\,u_{xx} + 10\,u\,u_{xxx} + u_{xxxxx} = 0. \tag{6.7}$$

Using the Exp - function method, Chun obtained seventeen exact solutions of Eq. (6.7) [64]. His first solution (formula (25) in [64]) can be simplified to the trivial solution, because

$$u_1 = \frac{-k^4\,e^{\eta} + 2\,k^2\,a_0 - k^4\,b_{-1}\,e^{-\eta}}{2\,k^2\,e^{\eta} - 4\,a_0 + 2\,k^2\,b_{-1}\,e^{-\eta}} \equiv -\frac{k^2}{2}, \qquad \eta = k\,\left(x - \frac{7\,k^4}{4}\,t\right). \tag{6.8}$$

The sixth solution by Chun (formula (30) [64]) can be simplified to the trivial solution as well

$$u_6 = \frac{a_1\,e^{\eta} + \,a_1\,b_0 + a_1\,b_{-1}\,e^{-\eta}}{e^{\eta} + b_0 + \,b_{-1}\,e^{-\eta}} \equiv a_1, \qquad \eta = k\,\left(x + \omega\,t\right). \tag{6.9}$$

The thirteenth (as well as the fourteenth and the fifteenth) solution (formula (69) in paper [64]) is also constant as we can see from the equality

$$u_{13} = \frac{a_1\,e^{2\,\eta} + a_1\,b_1\,e^{\eta}}{b_1\,e^{2\,\eta} + b_1^2\,e^{-\,\eta}} \equiv \frac{a_1}{b_1}, \qquad \eta = (k\,x + \omega\,t)\,. \tag{6.10}$$

We recognized, that among seventeen solutions presented by Chun in his paper twelve solutions (25), (27), (28) (30), (49), (51) (53), (55), (57), (69), (70) and (71) satisfy the fifth-order KdV equation, but solutions (25), (30), (69), (70) and (71) are trivial ones (constants) and (51) is not the solitary wave solution. Six solutions (25), (27), (49), (53), (55) and (57) are solitary wave solutions, but these ones are known and can be found by means of other methods.

Example 5e. *Solutions of the improved Boussinesq equation by Abdou and co -authors* [65].

Using the Exp - function method Abdou and his co - authors looked for the solitary wave solutions of the improved Bouusinesq equation

$$u_{tt} - u_{xx} - u\,u_{xx} - (u_x)^2 - u_{xxtt} = 0. \tag{6.11}$$

In the travelling wave Eq.(6.13) takes the form

$$(w^2 - k^2 - k^2\,u)\,u_{\eta\eta} - k^2\,u_\eta^2 - k^2\,w^2\,u_{\eta\eta\eta\eta} = 0. \tag{6.12}$$

The authors found several solutions of Eq.(6.12). One of them takes the form

$$u(\eta) = \frac{\left(a_1\,e^{\eta} - \frac{b_0\left(5\,a_1\,k^2 + a_1 + 6\,k^2\right)}{k^2-1} + \frac{a_1\,b_0^2}{4}e^{-\eta}\right)}{\left(e^{\eta} + b_0 + \frac{b_0^2}{4}e^{-\eta}\right)}, \quad \eta = k\,x + t\sqrt{-\frac{a_1+1}{k^2-1}}. \tag{6.13}$$

However this solution can be presented as

$$u(\eta) = a_1 - \frac{6\,b_0\,k^2\,(a_1+1)}{(k^2-1)\left(e^{\frac{\eta}{2}} + \frac{b_0}{2}\,e^{-\frac{\eta}{2}}\right)^2}. \tag{6.14}$$

Solution (6.14) can be obtained by many methods.

We have often observed the fifth error studying the application of the Exp - function method.

Other examples of the fifth error are given in our recent papers [42, 46].

7 Sixth error: some authors do not check solutions of differential equations

Using the computer methods to look for the exact solutions of nonlinear differential equation we cannot remove our knowledge, our understanding and our attention to the theory of differential equations. Neglecting to do this we can obtain different mistakes in finding the exact solutions of nonlinear differential equations.

These mistakes lead to numerous misprints in the final expressions of the solutions and sometimes to the fatal errors. A need of verifying the obtained solutions is obvious and it does not take much time with the help of the modern computer algebra systems. We have to substitute the found solution into the equation and to check that the solution satisfies it.

The sixth error is formulated as follows.

Sixth error. *Some authors do not check the obtained solutions of nonlinear differential equations.*

Example 6a. *"Solutions" of the Burgers equation by Soliman* [51].

Soliman [51] found three "solutions" of the Burgers equation (4.1)

$$u_1(\xi) = \frac{c}{\alpha} + \frac{2\,\nu}{\alpha}\,\tanh\{x - c\,t\}, \tag{7.1}$$

$$u_2(\xi) = \frac{c}{\alpha} + \frac{c^2}{2\,\alpha\,\nu}\,\coth\{x - c\,t\}, \tag{7.2}$$

$$u_3(\xi) = \frac{c}{\alpha} + \frac{2\,\nu}{\alpha}\,\tanh\{x - c\,t\} + \frac{c^2}{8\,\alpha\,\nu}\,\coth\{x - c\,t\}. \tag{7.3}$$

However all these solutions are incorrect and do not satisfy Eq.(4.1). The solution of Eq.(4.1) can be found by many methods and can be presented in the form

$$u_4(\xi) = \frac{c}{\alpha} - \frac{2\,\nu}{\alpha}\,\tanh\{x - c\,t - x_0\}. \tag{7.4}$$

where x_0 is arbitrary constant.

Example 6b. *Solution of the foam drainage equation by Bekir and Cevikel* [66]

$$u_t + \frac{\partial}{\partial x}\left(u^2 - \frac{\sqrt{u}}{2}\frac{\partial u}{\partial x}\right) = 0. \tag{7.5}$$

Eq.(7.5) was studied in [66] by means of the tanh - coth method by Bekir and Cevikel [66]. The authors proposed the method to obtain "new travelling wave solutions".

Using the travelling wave

$$u(x,t) = u(\xi), \qquad \xi = k\,(x + c\,t), \tag{7.6}$$

they obtained the first - order equation in the form

$$c\,k\,u + k\,\left(u^2 - \frac{k}{2}\,\sqrt{u}\,u_\xi\right) = 0, \tag{7.7}$$

and obtained three solutions

$$\begin{aligned} u_1 &= k^2\,\tanh^2\Big(k(x - k^2\,t)\Big), \\ u_2 &= k^2\,\coth^2\Big(k(x - k^2\,t)\Big), \\ u_3 &= k^2\,\Big(\tanh\Big(k(x - 4\,k^2\,t)\Big) + \coth\Big(k(x - 4\,k^2\,t)\Big)\Big)^2. \end{aligned} \tag{7.8}$$

The solution of Eq.(7.7) can be found using the transformation $u = v^2$. For function $v(\xi)$ we obtain the Riccati equation in the form

$$c + v^2 = k\, v_\xi. \tag{7.9}$$

Solution of Eq.(7.9) can be written as

$$v(\xi) = -k\, \tanh(\xi - \xi_0), \qquad \xi = k\, x + k^3\, t. \tag{7.10}$$

We can see there are the misprints in solutions of Eq.(7.7)

Example 6c. *"Solutions" of the Fisher equation by Öziş and Köroğlu* [67]

$$u_t - u_{xx} - u\,(1-u) = 0. \tag{7.11}$$

Using the Exp-function method Öziş and Köroğlu [67] found four "solutions" of Eq. (7.11). These "solutions" were given in the form

$$u_{(1)} = \frac{(1 - 2\,k^2)\, b_{-1}}{b_0\, \exp{(\eta)} + b_{-1}}, \qquad \eta = k\, x + w\, t, \tag{7.12}$$

$$u_{(2)} = \frac{(1 - 8\,k^2)\, b_{-1}}{b_1\, \exp{(2\,\eta)} + b_{-1}}, \qquad \eta = k\, x + w\, t, \tag{7.13}$$

$$u_{(3)} = \frac{b_0 - 2\,k^2\, b_{-1}\, \exp{(-\eta)}}{b_0 + b_{-1} \exp{(-\eta)}}, \qquad \eta = k\, x + w\, t. \tag{7.14}$$

$$u_{(4)} = \frac{b_1\, \mathrm{e}^{\eta} - 8\,k^2 b_{-1}\, \mathrm{e}^{-\eta}}{b_1\, \mathrm{e}^{\eta} + b_{-1}\, \mathrm{e}^{-\eta}}, \qquad \eta = k\, x + w\, t. \tag{7.15}$$

However all these expressions do not satisfy equation (7.11). We can note this fact without substituting solutions (7.12) - (7.15) into Eq. (7.11), because a true solution of Eq. (7.11) has the pole of the second order, but all functions (7.12) - (7.15) are the first order poles and certainly by substituting expressions (7.12) - (7.15) into equation (7.11) we do not obtain zero.

Example 6d. *"Solutions" of the modified Benjamin - Bona - Mahony equation* (6.3) *found by Yusufoglu* [63].

The general solution of Eq.(6.4) can be found via the Jacobi elliptic function, but the author tried to obtain some solitary solutions by means of the Exp - function method using the travelling wave $U(\xi)$, $\xi = \beta\,(x - \gamma\, t)$. Some of his solutions are incorrect. For example the "solution" (formula (3.12) [63])

$$U(\xi)=\sqrt{\frac{3(c-1)}{\alpha}}\,\frac{\exp\{\frac{\beta(x-\gamma\,t)}{2}\}-\frac{b_0}{2\,b_1}\,\exp\{-\frac{\beta(x-\gamma\,t)}{2}\}}{\exp\{\frac{\beta(x-\gamma\,t)}{2}\}+\frac{b_0}{2\,b_1}\,\exp\{-\frac{\beta(x-\gamma\,t)}{2}\}} \tag{7.16}$$

do not satisfy Eq.(6.4).

The solution of Eq.(6.4) takes the form

$$U(\xi)=\pm\frac{6\,\beta}{\sqrt{-6\,\alpha}}\,\tanh{(\xi-\xi_0)}\,,\qquad \xi=\beta\,(x+\beta^2\,t-t),\quad \gamma=1-\beta^2 \tag{7.17}$$

This solution can be obtained by using different methods.

Example 6e. *"Solutions" of the Burgers - Huxley equation by Chun* [68].

Chun [68] applied the Exp - function method to obtain the generalized solitary wave solutions of the Burgers — Huxley equation

$$u_t+\alpha\,u\,u_x-u_{xx}=\beta\,u(1-u)(u-\gamma). \tag{7.18}$$

In the travelling wave the author looked for the solution of equation

$$\omega\,u_\eta+\alpha\,u\,u_\eta-u_{\eta\eta}=\beta\,u(1-u)(u-\gamma), \tag{7.19}$$

where $\eta=k\,x+\omega\,t$. Some of the solutions by Chun are incorrect. In particular, "solution" (25) in [68]

$$u(eta)=\frac{\gamma\,b_{-2}\,e^{-2\,\eta}}{e^{2\,\eta}+b_{-2}\,e^{-2\,\eta}},\qquad \eta=\frac{\gamma}{4}\,x+\frac{\gamma(1-\gamma)}{2}\,t \tag{7.20}$$

do not satisfy Eq.(7.19). The solutions of Eq.(7.18) were found in [69].

Example 6f. *"Solutions" of the Benjamin - Bona - Mahony - Burgers equation by El-Wakil, Abdou and Hendi* [70].

El-Wakil and co-authors [70] using the Exp - function method looked for the solitary wave solutions of the BBMB equation

$$u_t-u_{xxt}-\alpha\,u_{xx}+u\,u_x+\beta\,u_x=0. \tag{7.21}$$

Taking the travelling wave $\eta=k\,x+c\,t$ the authors searched for the solution of the equation

$$c\,u_\eta-c\,k^2\,u_{\eta\eta\eta}-\alpha\,k^2\,u_{\eta\eta}+\beta\,k\,u_\eta+k\,u\,u_\eta=0. \tag{7.22}$$

Some of solutions by El-Wakil and co-authors are incorrect. In particular, expression (18) in [70]

$$u(\eta)=\frac{\left(-\frac{\left(6\,b_0\,k^2+k^2a_0-a_0\right)}{b_0(1+5\,k^2)}e^{\eta}+a_0-\frac{1}{4}\,\frac{b_0\left(6\,b_0\,k^2+k^2a_0-a_0\right)}{1+5\,k^2}e^{-\eta}\right)}{\left(e^{\eta}+b_0+\frac{{b_0}^2}{4}\,e^{-\eta}\right)},$$

$$\eta=k\,x-\frac{k\,(b_0+a_0)}{b_0\,\left(1+5\,k^2\right)}\,t \tag{7.23}$$

do not satisfy Eq.(7.22).

The solutions of Eq.(7.22) can be found taking the different methods into account. Note that integrating Eq.(7.22) with respect to ξ we have

$$u_{\eta\eta}+\frac{\alpha}{c}\,u_{\eta}-\frac{1}{2\,c\,k}\,u^2-\frac{c+\beta\,k}{c\,k^2}\,u-C_2=0, \tag{7.24}$$

where C_2 is an arbitrary constant. Eq.(7.24) coincides with the KdV - Burgers equation in the travelling wave. The solitary wave solutions of this equation are given above in section 1. The general solution of Eq.(7.24) can be found by analogue with the general solution of Eq.(2.7) [46].

Example 6g. *"Solutions" of the Klein — Gordon equation with quadratic nonlinearity by Zhang* [71].

The Exp - function method was used by Zhang to obtain the generalized solitonary solutions of the Klein - Gordon equation with the quadratic nonlinearity

$$u_{tt}-u_{xx}+\beta\,u-\gamma\,u^2=0. \tag{7.25}$$

The author [71] applied the travelling wave $u=U(\eta)$ $\eta=k\,x+w\,t$ and searched for the solution of the equation

$$(w^2-k^2\,\alpha^2)\,U_{\eta\eta}+\beta\,U-\gamma\,U^2=0. \tag{7.26}$$

At least two solutions of Eq.(7.26) by Zhang [71] are incorrect and do not satisfy Eq.(7.26)

$$U_1=\frac{\beta}{\gamma}-\frac{3\,b_0\,\beta}{\gamma\left(b_1\,e^{k\,x+w\,t}+b_0+\frac{{b_0}^2}{4b_1}e^{-k\,x-w\,t}\right)},\quad w=k\,\sqrt{\alpha^2+\frac{\beta}{k^2}}, \tag{7.27}$$

$$U_2=\frac{\beta}{\gamma}+\frac{a_1}{\left(e^{kx+wt}-\frac{a_1\,\gamma^2}{3\,\beta}+\frac{{a_1}^2\gamma^2}{36\,\beta^2}\,e^{-kx-wt}\right)},\quad w=k\,\sqrt{\alpha^2+\frac{\beta}{k^2}}. \tag{7.28}$$

The solitary wave solutions of Eq.(7.26) was obtained in many papers, because this equation coincides with KdV equation in the travelling wave [46]. These solutions can be written as the following

$$U^{(1)} = \frac{3\,\beta}{2\,\gamma}\,\left(1 - \,\tanh^2(k\,\eta - k\,\eta_0)\right), \quad \eta = k\,x \pm \frac{1}{2}\sqrt{\frac{4\,\alpha^2\,k^4 - \beta}{k^2}}\,t, \tag{7.29}$$

$$U^{(2)} = -\frac{\beta}{2\,\gamma}\,\left(1 - 3\,\tanh^2(k\,\eta - k\,\eta_0)\right), \quad \eta = k\,x \pm \frac{1}{2}\sqrt{\frac{4\,\alpha^2\,k^4 + \beta}{k^2}}\,t, \tag{7.30}$$

where η_0 is an arbitrary constant.

Example 6h. *"Solutions" of the Kuramoto - Sivashinsky equation by Noor, Mohyud - Din and Waheed* [72].

Using the Exp - function method Noor, Mohyud-Din and Waheed [72] looked for the solutions of the Kuramoto—Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0. \tag{7.31}$$

These authors presented two expressions as the solutions of the Kuramoto - Sivashinsky equation. Their first "solution" is written as

$$u_1 = -\frac{\left(a_0 + a_{-1}\exp\left(-kx + \frac{\left(-a_0^2+2ka_{-1}\right)kt}{a_{-1}}\right)\right)(a_0^2 - ka_{-1} + a_{-1}k^3)}{a_{-1}^2\exp\left(-kx + \frac{\left(-a_0^2+2ka_{-1}\right)kt}{a_{-1}}\right)}, \tag{7.32}$$

where a_0, a_{-1}, k are arbitrary constants (formula (26) in [72]).

The second "solution" by the authors takes the form

$$u_2 = \frac{(a_2e^{2\eta} + a_0 + a_{-2}e^{-2\eta})\,a_{-2}}{b_2a_{-2}e^{2\eta} + b_{-2}a_0 + b_{-2}e^{-2\eta}}, \quad \eta = kx - \frac{k\,(a_{-2} + 8k^3b_{-2} + 2kb_{-2})}{b_{-2}}t, \tag{7.33}$$

where a_0, a_{-2}, a_2, b_0, b_2, b_{-2}, k are arbitrary constants (formula (34) in [72]).

Here we have the sixth common error in the fatal form. Substituting cited "solutions" (7.32) and (7.33) into (7.31), we obtain that these "solutions" do not satisfy the Kuramoto - Sivashinsky equation. Moreover, we cannot obtain zero with any nontrivial values of the parameters. We guess, that using the Exp - function method the authors [72] did not solve the system of the algebraic equations for the parameters.

The exact solutions of Eq.(7.31) were first found in [73]. These two solutions take the form

$$u^{(1)} = C_0 - \frac{45\sqrt{19}}{361}\,\tan\left(\frac{\sqrt{19}}{38}\,(x - C_0\,t - x_0)\right) - \\ -\frac{15\sqrt{19}}{361}\,\tan^3\left(\frac{\sqrt{19}}{38}\,(x - C_0\,t - x_0)\right)^3, \qquad (7.34)$$

$$u^{(2)} = C_0 - \frac{135\sqrt{209}}{361}\,\tanh\left(\frac{\sqrt{209}}{38}\,(x - C_0\,t - x_0)\right) + \\ +\frac{165\sqrt{209}}{361}\,\tanh^3\left(\frac{\sqrt{209}}{38}\,(x - C_0\,t - x_0)\right), \qquad (7.35)$$

where C_0 and x_0 are arbitrary constants.

Many authors tried to find new exact solutions of the Kuramoto - Sivashinsky equation (7.31). Some of authors [36, 74, 75] believe that they found new solutions but it is not this case. Nobody cannot find new exact solutions of Eq.(7.31).

8 Seventh error: some authors include additional arbitrary constants into solutions

In finding the exact solutions of nonlinear differential equation our goal is to find the general solution. But it is not possible in many cases. In this situation we can try to look for the exact solutions with the larger amount of arbitrary constants. However we need to remember that the general solution of the equation of the n - th order can have only n arbitrary constants. Nevertheless we meet the papers, in which the authors present the solution of the Riccati equation with two or even three arbitrary constants. Moreover these authors say about the advantage of their approach in comparison with other methods taking into consideration the amount of arbitrary constants. Unfortunately in these cases there are two possible variants for this kind of solutions. The first variant is the author have obtained the solution with extra arbitrary constants and the amount of these constants can be decreased by means of the transformations. In the second variant the large amount of arbitrary constants in the solution points out that the author have found wrong solution.

So, we can formulate the seventh error as follows.

Seventh error. *Some authors include additional arbitrary constants into solutions of nonlinear ordinary differential equations.*

Example 7a. *Solution of the Riccati equation*

$$Y_\xi = \beta(1+Y^2). \tag{8.1}$$

Using

$$Y(\xi) = -\frac{\psi_\xi}{\beta\,\psi}, \tag{8.2}$$

we obtain the linear equation of the second order

$$\psi_{\xi\xi} + \beta^2\,\psi = 0. \tag{8.3}$$

The solution of equation (8.1) takes the form

$$Y(\xi) = i\,\frac{C_3\,e^{-i\,\beta\,\xi} - C_4\,e^{i\,\beta\,\xi}}{C_3\,e^{-i\,\beta\,\xi} + C_4\,e^{i\,\beta\,\xi}.} \tag{8.4}$$

At first glance we obtain more general solution (8.4) of the Riccati equation then (5.7), but in fact these solutions are the same. We can see, that one of the constants can be removed by dividing the nominator and the denominator in solution (8.4) on C_3 (or on C_4). Denoting $C_5 = C_4/C_3$ (or $C_5 = C_3/C_4$) we obtain the solution with the only arbitrary constant.

Example 7b. *Solution of the Sharma - Tasso - Olver equation.*

In example 2f we have obtained the solution of Eq.(3.36) in the form

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{\alpha}}\,\frac{C_2\,\exp\left\{\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\} - C_3\,\exp\left\{-\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}}{C_1 + C_2\,\exp\left\{\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\} + C_3\,\exp\left\{-\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}}. \tag{8.5}$$

However Eq.(3.36) has the second order, but solution (8.5) contains tree arbitrary constants. Sometimes it is convenient to leave three constants in solution (8.5), but we have to remember that solution (8.5) is not the general solution and one of these constants is extra. We can remove one of the constants as in example 7a and can write the general solution of Eq.(3.36) in the form

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{\alpha}}\,\frac{C_4\,\exp\left\{\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\} - C_5\,\exp\left\{-\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}}{1 + C_4\,\exp\left\{\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\} + C_5\,\exp\left\{-\frac{\xi\,\sqrt{c}}{\mu\,\sqrt{\alpha}}\right\}}. \tag{8.6}$$

Solution (8.6) of Eq.(3.36) is not worse then solution (8.5), but this solution is the general solution by definition and all other solution can be found from this one.

Example 7c. *Solution* (6.5) *of the modified Benjamin - Bona - Mahony equa-*

tion (6.3) [63]

$$u(\xi) = \frac{\frac{24b_0b_1(c-1)}{a_0a}e^{\xi} + a_0}{\frac{24b_0b_1^2(c-1)}{a_0^2a}e^{2\xi} + b_1e^{\xi} + b_0 + \frac{a_0^2a}{24b_1(c-1)}e^{-\xi}}. \tag{8.7}$$

Solution (8.7) has three arbitrary constants a_0, b_0 and b_1, but in fact this solution can be simplified to the form with one arbitrary constant as it was demonstrated in example 5c.

Example 7d. *"New exact solution" of the Riccati equation by Dai and Wang* [76]

$$\varphi_{\xi} + l_0 + \varphi^2 = 0. \tag{8.8}$$

Dai and Wang [76] had been looking for the "new exact solutions" of the Riccati equation (8.8) and obtained five solutions. One of them (solution (17) [76]) takes the form

$$\varphi(\xi) = \frac{\sqrt{-l_0}\, b_1 \exp\left(\sqrt{-l_0}\xi + \xi_0\right) - \sqrt{-l_0} a_{-1} \exp\left(-\sqrt{-l_0}\xi - \xi_0\right)}{b_1\, \exp\left(\sqrt{-l_0}\xi + \xi_0\right) +\, a_{-1}\, \exp -\left(\sqrt{-l_0}\xi - \xi_0\right)}. \tag{8.9}$$

We can see, that solution (8.9) has three arbitrary constants, but two of them certainly can be removed.

Example 7e. *Solutions of the Burgers - Huxley equation by Chun* [68].

Using the Exp - function method Chun [68] obtained the generalized solitary wave solutions of the Burgers — Huxley equation (see, Eq.(8.10) in example 6e). One of his solution (solution (32) in [68]) takes the form

$$u(\eta) = \frac{\gamma(\gamma-1)(\gamma\, e^{2\eta} + a_1\, e^{\eta} + a_0 + a_{-1}\, \mathrm{e}^{-\eta})}{\gamma(\gamma-1)(e^{2\eta} + b_1e^{\eta} + a_{-1}e^{-\eta}) + \gamma(a_1b_1 + a_0 - b_1^2) - a_0 - a_1^2 + a_1b_1},$$

$$\eta = \frac{\gamma - 1}{2}\, x + \frac{\gamma^2 - 1}{4}\, t \tag{8.10}$$

The Burgers — Huxley equation is the second order one but solution (8.10) has three arbitrary constants. We suggested that this solution is incorrect. We checked this solution and have convinced that this solution does not satisfy Eq.(7.19).

Especially many solutions of equations with superfluous constants were obtained by means of the Exp - function method. Such examples can be found in many papers.

9 Conclusion

Let us shortly formulate the results of this paper. First of all we have tried to give some classification of the common errors and mistakes which occur in finding the solitary wave solutions of nonlinear differential equations. We have illustrated these errors using the examples from the recent publications in which the authors presented the exact solutions of nonlinear differential equations.

We have tried to classify the wide - spread errors in finding the exact solutions of nonlinear differential equations. Unfortunately we have no possibility to consider all errors and all problems of computer methods in finding the exact solutions. However we can note that the authors of many papers do not analyze the solutions of nonlinear differential equations that they obtain. Some errors are mentioned above due to this reason. It is important to discuss the solutions of nonlinear differential equations. We need to study how a solution depends on the parameters of nonlinear differential equation. It is useful to study the plots for the solutions of differential equation. We are interested in the stable exact solutions of nonlinear differential equations. These solutions are useful for the investigation of physical processes.

In our paper we have discussed the solitary wave solutions of nonlinear differential equations, but certainly we can find the same errors in looking for the other exact solutions.

We believe the results of this paper will be useful for young people who are going to study the exact solutions of nonlinear differential equations. We also hope that the material of this paper will be interesting for some referees.

10 Acknowledgements

This work was supported by the International Science and Technology Center under Project B 1213.

References

[1] Zabusky N.J., Kruskal M.D. Ineraction of "Solitons" in a collisioness plasma and the recurrence of initial states. Phys Rev Lett 1965; 15: 240-3

[2] Gardner C.S., Green J.M., Kruskal M.D., and Miura R. M. Method for solving the Korteweg-de Vries equations. Phys Rev Lett 1967; 19: 1095 - 97

[3] Lax P.D. Integrals of nonlinear equations of evolution and solitary waves. Communications on pure and applied mathematics 1968; 21: 467 - 90

[4] Ablowitz M.J., Kaup D. J., Newell A. C., Segur H. The Inverse Scattering Transform - Fourier Analysis for Nonlinear Problems Stud. Appl Math 1974; 53: 249 - 315.

[5] Ablowitz M.J. and Clarkson P.A. Solitons Nonlinear Evolution Equations and Inverse Scattering. Cambridge university press, 1991

[6] Hirota R. Exact solution of the Korteweg-de Vries for multiple collinsions of solutions. Phys Rev Lett 1971; 27: 1192 - 94

[7] Kudryashov N.A. Analitical theory of nonlinear differential equations. Moskow - Igevsk. Institute of computer investigations: 2004 [in Russian]

[8] Polyanin A.D., Zaitsev V.F., Zhyrov A.I. Methods of nonlinear equations of mathematical physics and mechanics. Moscow. Fizmatlit: 2005

[9] Weiss J., Tabor M., Carnevalle G. The Painleve property for partial differential equations. J Math Phys 1983; 24:522-26

[10] Weiss J. The Painleve property for partial differential equations. II: Backlund transformation, Lax pairs, and the Schwarzian derivative. J Math Phys 1983; 24:1405-13

[11] Conte R. The Painleve property, one century later, CRM series in mathematical physics, New York: Springer – Verlag; 1999. p. 77 - 180

[12] Kudryashov N.A. Exact soliton solutions of the generalized evolution equation of wave dynamics. Journal of Applied Mathematics and Mechanics 1988; 52(3):360-5

[13] Conte R., Musette M. Painlevé analysis and Backlund transformations in the Kuramoto - Sivashinsky equation. Journal of Physics A.: Math Gen 1989; 22:169-77

[14] Kudryashov N.A. Exact solutions of the generalized Kuramoto-Sivashinsky equation. Phys Lett A 1990; 147:287-91

[15] Kudryashov N.A. Exact soliton solutions of nonlinear wave equations arising in mechanics. Journal of Applied Mathematics and Mechanics 1990; 54(3):450-53

[16] Kudryashov N.A., On types of nonlinear nonintegrable equations with exact solutions Phys Lett A 1991; 155:269-75

[17] Kudryashov N.A. Partial differential equations with solutions having movable first - order singularities. Physics Letters A 1992; 169:237-42

[18] Kudryashov N.A., Truncated expansions and nonlinear integrable partial differential equations. Physics Letters A 1993; 178:99-104

[19] Kudryashov N.A. From singular manifold equations to integrable evolution equations. J Phys A Math Gen 1994; 27:2457-70

[20] Kudryashov N.A., Zargaryan E.D., Solitary waves in active-dissipative dispersive media. J Phys A Math Gen 1996;29:8067-77

[21] Peng Y.Z., Krishnan E.V. The singular manifold method and exact periodic wave solutions to a restricted BLP dispersive long wave system. Reports on mathematical physics 2005; 56:367 - 78

[22] Kudryashov N.A., Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos, Solitons and Fractals 2005; 24:1217-31

[23] Kudryashov N.A. Exact solitary waves of the Fisher equations. Physics Letters A 2005; 342:99-106

[24] Kudryashov N.A., Demina M.V. Polygons of differential equation for finding exact solutions. Chaos Solitons and Fractals 2007; 33:1480-96

[25] Kudryashov N.A. Solitary and Periodic Solutions of the Generalized Kuramoto -Sivashinsky Equation. Regular and Chaotic Dynamics 2008, 13:234-38

[26] Kudryashov N.A., Loguinova N.B. Extended simplest equation method for nonlinear differential equations. Applied Mathematics and Computation 2008; 205:396-402

[27] Lan H., Wang K. Exact solutions for some nonlinear equations. Physics Letters A 1989; 137: 369-73

[28] Lou S.Y.,Huang G.X., Ruan H.Y. Exact solitary waves in a convecting fluid. J Phys A Math Gen 1991; 24(11):L587-L590

[29] Malfliet W., Hereman W. The Tanh method: I Exact solutions of nonlinear evolution and wave equations. Phys Scripta 1996; 54:563-68

[30] Parkes E.J., Duffy B.R. An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations. Comput Phys Commun 1996; 98:288-300

[31] Wang M.L., Li X., Zhang J. The G'/G - expansion method and evolution erquations in mathematical physics. Phys Lett A 2008; 372(4):417

[32] Bekir A., Application of the G'/G - expansion method for nonlinear evolution equations. Phys Lett A 2008; 3400-06

[33] Zhang J., Wei X., Lu Y. A generalized G'/G - expansion method and its applicatioons, Phys. Lett. A 2008; 372:3653-58

[34] Wazwaz A. M. The tanh - coth method for solitons and kink solutions for nonlinear parabolic equations. Appl Math Comput 2007; 188:1467-75

[35] Abdou M. A. The extended tanh method and its applications for solving nonlinear physical models. Appl Math Comput 2007; 199:988-96

[36] Wazzan L. A modified tanh - coth method for solving the general Burgers - Fisher and the Kuramoto - Sivashinsky equations. Commun Nonlinear Sci Numer Simulat, (2008), doi: 10.1016/j.cnsns.2008.08.004.

[37] He J.H., Wu X.H. Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 2006; 30:700 - 08

[38] He J.H., Abdou M.A. New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos, Solitons and Fractals 2007; 34:1421 - 29

[39] Wu X.H., He J.H. Solitary solutions, peridic solutions and compaction - like solutions using the Exp-function method. Comput Math Appl 2007; 54:966 - 86

[40] Ebaid A. Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method. Phys Lett A 2007; 365:213 - 19

[41] El - wakil S.A., Madkour M.A., Abdou M.A. Application of the Exp-function method for nonlinear evolution equations with variable coefficients. Phys Lett A 2007; 369:62 - 9

[42] Kudryashov N.A., Loguinova N.B. Be carefull with the Exp - function method. Commun Nonlinear Sci Numer Simulat 2009; 14:1881 - 90

[43] Li W., Zhang H. Generalized multiple Riccati equations rational expansion method with symbolic computation to construct exact complexiton solutions of nonlinear partial differential equations. Appl Math Comput 2008; 197: 288 - 96

[44] Zhang S. Application of the Exp - function method to high - dimensional nonlinear evolution equation. Chaos, Solitons and Fractals 2008; 38:270 - 76

[45] Korteweg D.J., de Vries G. On the change of form of long waves advancing in a rectangular canal and on a new tipe of long stationary waves. Phil Mag 1895; 39:422-43

[46] Kudryashov N.A. , On "new travelave solutions" of the KdV and the KdV - Burgers equations. Commun Nonlinear Sci Numer Simulat 2009; 14:1891-1900

[47] Khani F. Analytic study on the higher order Ito equations: New solitary wave solutions using the Exp - function method. Chaos, Solitons and Fractals 2008; doi: 10.1016/j.chaos.2008.08.023

[48] Öziş T., Aslan I., Exact and explicit solutions to the (3+1) - dimensional Jimbo - Miva equation via the Exp - function method. Phys Lett A 2008; 372:7011 - 15

[49] Ganji Z.Z., Ganji D.D., Bararnia H. Approximate general and explicit solutions of nonlinear BBMB equations by Exp - function method. Applied Mathematical Modeling, 2008; doi: 10.1016/j.apm.2008.03.005

[50] Erbas B., Yusufoğlu E., Exp - function method for constructing exact solutions of Sharma - Tasso - Olver equation. Chaos, Solitons and Fractals 2008; doi: 10.1016/j.chaos.2008.09.003

[51] Soliman A.A., Exact solutions of the KdV - Burgers equation by Exp-function method. Chaos, Solitons and Fractals, 2008; doi: 10.1016/J.chaos. 2008.04.038

[52] Cole J.D. On a quasilinear parabolic equation occurring in aerodynamics. Quart Appl Math 1951 9:225 - 36

[53] Hopf E. The partial differential equation $u_t + uu_x = \mu\, u_{xx}$. Commun Pure AppL Math 1950; 13: 201 - 30

[54] Abdou M.A. Generalized solitonary and periodic solutions for nonlinear partial differential equations by the Exp - function method. Nonlinear Dynamics 2008: 52: 1-9, doi 10.1007/s1071-007-9250-1

[55] Wazwaz A.M. Multiple - soliton solutions for the generalized (1+1) - dimensional and the generalized (2+1) - dimensional Ito equations. Applied Mathematics and Computation 2008; 202: 840 - 49

[56] Clarkson P.A., Kruskal M.D. New similarity reductions of the Boussinesq equation. J Math Phys 1989; 30:2201 - 13

[57] Ugurlu Y., Kaya D. Solutions of the Cahn - Hillard equation. Computers and Mathematics with Applications 2008; doi:10.1016/j.camwa.2008.07.007

[58] Bekir A. On travelling wave solutions to combined KdV - mKdV and modified Burgers - KdV equation, Commun Nonlinear Sci Numer Simulat 2009 14:1038 - 42

[59] Bekir A., Applications of the extended tanh method for coupled nonlinear evolution equations. Commun Nonlinear Sci Numer Simulat 2008; 13:1748 - 57

[60] Xie F., Zhang Y., Lü Z. Symbolic computation in nonlinear evolution equation: application to (3+1) - dimensional Kadomttsev - Petviasvili equation. Chaos, Solitons and Fractals 2005; 24: 257 - 63

[61] Zhang S.Symbolic computation and new families of exact non-travelling wave solutions of (2+1) - dimensional Konopelchenko - Dubrovsky equations. Chaos, Solitons and fractals 2007; 31:951 - 59

[62] Wazwaz A.M. New solitons and kinks solutions to the Sharma - Tasso - Olver equation. Applied Mathematics and Computation 2007; 188:1205 - 13

[63] Yusufoğlu E. New solitary solutions for the MBBM equations using Exp-function method. Physics Letters A 2008; 372:442 - 46

[64] Chun C., Soliton and periodic solutions for the fifth-order KDV equation with the Exp - function method, Physics Letters A 2008; 372: 2760 - 66

[65] Abdou M.A., Soliman A.A., El-Basony S.T., New application of Exp - function method for improved Boussinesq equation. Physics Letters A 2007; 369:469 - 75

[66] Bekir A., Cevikel A.C. Solitary wave solutions of two nonlinear physical models by tanh - coth method. Commun Nonlinear Sci Numer Simulat 2009; 14: 1804 - 09

[67] *Öziş T., Köroğlu C.*, A novel approach for solving the Fisher equation using Exp-function method. Physics Letters A 2008; 372: 3836 - 3840

[68] Chun C. Application of Exp-function method to the generalized Burgers - Huxley equation. Journal of Physics: Conference Series 2008; 96:012217

[69] Efimova O.Yu., Kudryashov N.A. Exact solutions of the Burgers - Huxley equation. Journal of Applied Mathematics and Mechanics 2004; 68(3):413 - 20

[70] El - Wakil S.A., Abdou M.A., Hendi A. New periodic wave solutions via Exp-function method. Phys Lett A 2008; 372:830 - 40

[71] Zhang S. Exp-function method for Klein-Gordon equation with quadratic nonlinearity. J. Phys: Conference Series 2008; 96: 012002

[72] Noor M.A., Mohyud-Din S.T., Waheed A., Exp-function method for solving Kuramoto—Sivashinsky and Boussinesq equations. J Appl Math Comput 2008; doi: 10.1007/s12190-008-0083-y.

[73] Kuramoto Y., Tsuzuki T. Persistent Propagation of concentration waves in Dissipative media far from thermal equalibrium. Progress of theoretical Physics 1976; 55(2):356 - 69

[74] Wazwaz A.M. New solitary wave solutions to the Kuramoto—Sivashinsky and the Kawahara equations. Applied Mathematics and Computation 2006; 182:164250.

[75] Chen H., Zhang H. New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto - Sivashinsky equation. Chaos, Solitons and Fractals 2004; 19:71 6.

[76] Dai C.Q., Wang Y.Y. New exact solutions of the (3+1) - dimensional Burgers system. Physics Letters A 2008; doi:10.1016/j.physleta.2008.11.018