

# Seven common errors in finding exact solutions of nonlinear differential equations

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## Abstract

We analyze the common errors of the recent papers in which the solitary wave solutions of nonlinear differential equations are presented. Seven common errors are formulated and classified. These errors are illustrated by using multiple examples of the common errors from the recent publications. We show that many popular methods in finding of the exact solutions are equivalent each other. We demonstrate that some authors look for the solitary wave solutions of nonlinear ordinary differential equations and do not take into account the well - known general solutions of these equations. We illustrate several cases when authors present some functions for describing solutions but do not use arbitrary constants. As this fact takes place the redundant amount of the solutions of differential equations are found. A few examples of incorrect solutions by some authors are presented. Several other errors in finding the exact solutions of nonlinear differential equations are discussed.

### *Key words:*

Nonlinear evolution equation, Exact solution, Common error, Truncated expansion method, Simplest equation method, Tanh - function method, Exp - function method

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## 1 Introduction

During the last forty years we have been observing many publications presenting the exact solutions of nonlinear evolution equations. The emergence of these publications results from the fact that there are a lot of applications

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of nonlinear differential equations describing different processes in many scientific areas.

The start of this science area was given in the famous work by Zabusky and Kruskal [1]. These authors showed that there are solitary waves with the property of the elastic particles in simple mathematical model. The result of the study of the Korteweg — de Vries equation was the discovery of the inverse scattering transform to solve the Cauchy problem for the integrable nonlinear differential equations [2]. Now the inverse scattering transform is used to find the solution of the Cauchy problem for many nonlinear evolution equations [3–5]. Later Hirota suggested the direct method [6] now called the Hirota method, that allows us to look for the solitary wave solutions and rational solutions for the exact solvable nonlinear differential equations [7, 8].

As to nonintegrable nonlinear evolution equations we cannot point out the best method to look for the exact solutions of nonlinear differential equations. However we prefer to use the truncated expansion method by Weiss, Tabor and Carnevale [9] and the simplest equation method in finding the exact solutions. In section 1 we demonstrate that many popular methods to look for the exact solutions of nonlinear differential equations are based on the truncated expansion method. Many methods are obtained as consequence of the truncated expansion method and we are going to illustrate this fact in this paper.

Nowadays there are a lot of computer software programs like MATHEMATICA and MAPLE. Using these codes it is possible to have complicated analytical calculations to search for the different forms of the solutions for the nonlinear evolution equations and many authors use the computer codes to look for the exact solutions. However using the computer programs many investigators do not take into account some important properties of the differential equations. Therefore some authors obtain "new" cumbersome exact solutions of the nonlinear differential equations with some errors and mistakes.

The aim of this paper is to classify and to demonstrate some common errors that we have observed studying many publications in the last years. Using some examples of the nonlinear differential equations from the recent publications we illustrate these common errors.

The outline of this paper is as follows. In section 2 we present some popular methods to search for the exact solutions of nonintegrable differential equations and we show that in essence all these approaches are equivalent. In section 3 we analyze some reductions of partial differential equations to nonlinear ordinary differential equations. We demonstrate that many reductions have the well - known general solutions and there is no need to study the solitary wave solutions for these mathematical models. In section 4 we give some

examples when the authors remove the constants of integration and lose some exact solutions. In section 5 we demonstrate that many publications contain the redundant expressions for the exact solutions and these expressions can be simplified by taking arbitrary constants into account. In section 6 we discuss that some solitary wave solutions can be simplified by the authors. In section 7 we point out that we have to check some exact solutions of nonlinear differential equations because in number of cases we have "solutions" which do not satisfy the equations studied. In section 8 we touch the solutions with redundant arbitrary constants. For many cases the redundant arbitrary constants do not lead to erroneous solutions but we believe that investigators have to take these facts into account.

## 2 First error: some authors use equivalent methods to find exact solutions

In this section we start to discuss common errors to search for the exact solutions of nonlinear differential equations. We observed these errors by studying many papers in the last years.

Many authors try to introduce "new methods" to look for "new solutions" of nonlinear differential equations and many investigators hope that using different approaches they can find new solutions. However analyzing numerous applications of many methods we discover that many of them are equivalent each other and in many cases it is impossible to obtain something new.

So the first error can be formulated as follows.

***First error.** Some authors use the equivalent methods to find the exact solutions of nonlinear differential equations, but believe that they can find new exact solutions.*

Using the solitary wave solutions of the KdV - Burgers equation let us show that many methods to look for the exact solutions of nonlinear differential equations are equivalent.

***Example 1a.** Application of the truncated expansion method.*

The truncated expansion method was introduced in [9, 10] and developed in many papers to obtain the Lax pairs, the Backlund transformations and the rational solutions for the integrable equations [11]. This approach was also used in the papers [12–21] to search for the exact solutions of nonintegrable differential equations.

Consider the application of the truncated expansion method in finding the

exact solutions of the KdV - Burgers equation

$$u_t + u u_x + \beta u_{xxx} = \mu u_{xx}. \quad (2.1)$$

Substituting  $u(x, t)$  in the form [9]

$$u(x, t) = \frac{u_0(x, t)}{F^p} + \frac{u_1(x, t)}{F^{p-1}} + \dots + u_p \quad (2.2)$$

into the KdV - Burgers equation Eq.(2.1) and finding the order of the pole  $p = 2$  for the solution  $u(x, t)$  and functions  $u_0(x, t)$  and  $u_1(x, t)$  we obtain the transformation [12]

$$u(x, t) = 12\beta \frac{\partial^2 \ln F}{\partial x^2} - \frac{12\mu}{5} \frac{\partial \ln F}{\partial x} + u_2. \quad (2.3)$$

Assuming [12]

$$F = 1 + C_1 \exp(kx - \omega t) \quad u_2 = C_2, \quad (2.4)$$

(where  $C_1$  and  $C_2$  are arbitrary constants), we have the system of algebraic equations with respect to  $k$  and  $\omega$ . Solving this algebraic system we have the values of  $k$  and  $\omega$  as follows

$$k_{1,2} = \mp \frac{\mu}{5\beta}, \quad \omega_{1,2} = \mp \frac{\mu C_2}{5\beta} - \frac{6\mu^3}{125\beta^2}. \quad (2.5)$$

For these values of  $k$  and  $\omega$  the solitary wave solutions for the KdV - Burgers equation takes the form [12]

$$\begin{aligned} u(x, t) = C_2 - \frac{12\mu}{5} \frac{\partial}{\partial x} \ln(1 + C_1 \exp(kx - \omega t)) + \\ + 12\beta \frac{\partial^2}{\partial x^2} \ln(1 + C_1 \exp(kx - \omega t)). \end{aligned} \quad (2.6)$$

We can see that solution (2.6) satisfies the nonlinear ordinary differential equation in the form

$$\beta k^3 U_{\xi\xi\xi} - \mu k^2 U_{\xi\xi} + k U U_{\xi} - \omega U_{\xi} = 0, \quad \xi = kx - \omega t. \quad (2.7)$$

We have found exact solution of the KdV — Burgers equation in essence using the travelling wave solutions although we tried to look for more general solutions. Many researches look for the exact solutions taking directly into account the travelling wave solutions.

**Example 1b.** Application of the Riccati equation as the simplest equation (the first variant) [22–26].

Solution (2.6) can be written as

$$u(\xi) = C_2 + \frac{C_1 (60\beta k^2 - 12\mu k) \exp(\xi)}{5(1 + C_1 \exp(\xi))} - \frac{12\beta C_1^2 k^2 \exp(2\xi)}{(1 + C_1 \exp(\xi))^2}. \quad (2.8)$$

Consider the expression

$$H = \frac{C_1 \exp(\xi)}{1 + C_1 \exp(\xi)}. \quad (2.9)$$

Taking the derivative of the function  $H$  with respect to  $\xi$  we get

$$H_\xi = -H^2 + H. \quad (2.10)$$

We obtain that solution (2.8) can be written in the form

$$u(\xi) = C_2 + \left(12\beta k^2 - \frac{12\mu k}{5}\right) H - 12\beta k^2 H^2. \quad (2.11)$$

Eq.(2.11) means that one can look for the solution of the KdV - Burgers equation using the expression

$$u(\xi) = A_0 + A_1 H + A_2 H^2, \quad (2.12)$$

where  $H$  is the solution of Eq.(2.10). It is seen, that the application of the simplest equation method with the Riccati equation gives the same results as the truncated expansion method.

**Example 1c.** Application of the tanh - function method [27–30].

Expression (2.9) can be transformed as follows

$$\begin{aligned} H &= \frac{C_1 \exp(\xi)}{1 + C_1 \exp(\xi)} = \frac{1}{2} \left( \frac{2 \exp(\xi - \xi_0)}{1 + \exp(\xi - \xi_0)} - 1 \right) + \frac{1}{2} = \\ &= \frac{1}{2} \left( 1 + \tanh \left( \frac{\xi - \xi_0}{2} \right) \right), \quad C_1 = \exp(-\xi_0). \end{aligned} \quad (2.13)$$

From (2.13) we have that solution (2.8) of the KdV - Burgers equation can be written in the form

$$\begin{aligned} u(\xi) &= C_2 - \frac{6\mu k}{5} + 3\beta k^2 - \frac{6\mu k}{5} \tanh \left( \frac{\xi - \xi_0}{2} \right) - \\ &\quad - 3\beta k^2 \tanh^2 \left( \frac{\xi - \xi_0}{2} \right). \end{aligned} \quad (2.14)$$

Substituting  $k_{1,2}$  and  $\omega_{1,2}$  into solution (2.14) we have the solution of the KdV - Burgers equation in the form [12]

$$u(\xi) = C_2 \pm \frac{6\mu^2}{25\beta} \left( 1 + \tanh \left( \frac{\xi - \xi_0}{2} \right) \right) + \frac{3\mu^2}{25\beta} \left( 1 + \tanh^2 \left( \frac{\xi - \xi_0}{2} \right) \right),$$

$$\xi = \mp \frac{\mu}{5\beta} x - \left( \frac{6\mu^3}{125\beta^2} \pm \frac{\mu C_2}{5\beta} \right) t. \quad (2.15)$$

We obtain that the solution of the KdV - Burgers equation can be found as the sum of the hyperbolic tangents

$$u(\xi) = A_0 + A_1 \tanh [m(\xi - \xi_0)] + A_2 \tanh^2 [m(\xi - \xi_0)], \quad (2.16)$$

where  $m$  is unknown parameter.

We have obtained that the tanh - function method is equivalent in essence to the truncated expansion method. As this fact takes place we can see that the maximum power of the hyperbolic tangent in (2.16) coincides with the order of the pole for the solution of the KdV - Burgers equation.

**Example 1d.** Application of the Riccati equation as the simplest equation (the second variant). [22–24].

Note that the function

$$Y(\xi) = m \tanh [m(\xi - \xi_0)] \quad (2.17)$$

is the general solution of the Riccati equation

$$Y_\xi = -Y^2 + m^2. \quad (2.18)$$

From (2.16) we obtain, that we can look for the solution of the KdV - Burgers equation taking into account the formula

$$u(\xi) = A_0 + A_1 Y + A_2 Y^2, \quad (2.19)$$

where  $Y$  satisfies Eq.(2.18). The maximum power of the function  $Y$  in (2.19) coincides with the pole of the solution for the KdV - Burgers equation.

**Example 1e.** Application of the  $G_\xi/G$  method [31–33].

Taking into account the transformation

$$Y = \frac{G_\xi}{G} \quad (2.20)$$

we reduce Eq.(2.18) to the linear equation

$$G_{\xi\xi} - \nu^2 G = 0. \quad (2.21)$$

As this fact takes place expression (2.19) can be written in the form

$$u(\xi) = A_0 + A_1 \frac{G_\xi}{G} + A_2 \left( \frac{G_\xi}{G} \right)^2. \quad (2.22)$$

This means, that we can search for the exact solutions of the KdV - Burgers equation in the form (2.22), where  $G(\xi)$  satisfies Eq.(2.21). We obtain, that application of the  $G_\xi/G$  method is equivalent to application of the simplest equation method with the Riccati equation, to the tanh - function method and to the truncated expansion method.

Note that the  $G_\xi/G$  method follows directly from the truncated expansion method because solution (2.3) can be written as

$$u(x, t) = u_2 - \frac{12\mu}{5} \frac{F_x}{F} - 12\beta \frac{F_x^2}{F^2} + 12\beta \frac{F_{xx}}{F}. \quad (2.23)$$

Assuming that the function  $F(x, t)$  satisfies linear equation of the second order

$$F_{xx} = m_0 F_x + m_1 F + m_2, \quad (2.24)$$

where  $m_0$ ,  $m_1$  and  $m_2$  are unknown parameters we have the expression (2.22) for  $F_\xi/F$ . So, the  $G_\xi/G$  - method coincides with the truncated expansion method if we use the travelling wave solutions.

**Example 1f.** *Application of the tanh — coth method [34–36].*

Assuming in (2.16)  $\xi_0 = \frac{i\pi}{2}$ , we obtain the formula

$$u(\xi) = A_0 + A_1 \coth(\nu \xi) + A_2 \coth^2(\nu \xi). \quad (2.25)$$

This means that we can search for the solution of the KdV - Burgers equation in the form (2.25).

Using the identity for the hyperbolic functions in the form

$$2 \coth(\nu \xi) = \tanh\left(\frac{\nu \xi}{2}\right) + \coth\left(\frac{\nu \xi}{2}\right), \quad (2.26)$$

and substituting (2.26) into (2.25) we have the tanh - coth method to look for the solitary wave solutions of the KdV - Burgers equation

$$\begin{aligned} u(\xi) = & B_0 + B_1 \tanh(\nu_1 \xi) + B_2 \coth(\nu_1 \xi) + \\ & + B_3 \tanh^2(\nu_1 \xi) + B_4 \coth^2(\nu_1 \xi). \end{aligned} \quad (2.27)$$

We have obtained, that the tanh - coth method is equivalent to the truncated expansion method too and we cannot find new exact solutions of the KdV - Burgers equation by the tanh - coth method.

**Example 1g.** *Application of the Exp - function method [37–41].*

Solution (2.8) of the KdV - Burgers equation can be written in the form

$$u(\xi) = \frac{5C_2 e^{-\xi} + C_1(60\beta k^2 - 12\mu k + 10C_2) + 5C_1^2(C_2 - 12\beta k^2)e^{\xi}}{5e^{-\xi} + 10C_1 + 5C_1^2 e^{\xi}}. \quad (2.28)$$

This solution can be found if we look for a solution of the KdV — Burgers equation using the Exp - function method in the form

$$u(\xi) = \frac{a_0 e^{-\xi} + a_1 + a_2 e^{\xi}}{b_0 e^{-\xi} + b_1 + b_2 e^{\xi}}. \quad (2.29)$$

Using the Exp - function method to search for the exact solutions of the KdV - Burgers equation one can have again the solitary wave solutions (2.3) with the functions (2.4) and the parameters (2.5). Studying the application of the Exp - function method we obtain that this method provides no new exact solutions of nonlinear differential equations in comparison with other methods. Moreover the Exp - function method do not allow us to find the order of the pole for the solution of nonlinear differential equation [42]. Usually the authors use a few variants of fractions with the sum of exponential functions to look for the solitary wave solutions.

### 3 Second error: some authors do not use the known general solutions of ordinary differential equations

Many authors look for the solutions of nonlinear evolution equations using the travelling waves. As this fact takes place, these authors obtain nonlinear ordinary differential equations and search for the solutions of these equations. However nonlinear ordinary differential equations had been studied very well and the solutions of them were obtained many years ago. However some of the authors do not use the well - known general solutions of the ordinary differential equations. As a result these authors obtain the solutions that have already been found by other scientists.

The second error can be formulated as follows.

**Second error.** *Some authors search for solutions of nonlinear differential equations, but do not use the known general solutions of these equations.*



**Example 2a.** Reduction of the (2+1) - dimensional Burgers equation by Li and Zhang [43]

$$u_t = u u_y + \alpha v u_x + \beta u_{yy} + \alpha \beta u_{xx}, \quad u_x = v_y. \quad (3.1)$$

The authors [43] considered the wave transformations in the form

$$u(x, y, t) = U(\xi), \quad v(x, y, t) = V(\xi), \quad \xi = k(x + ly + \lambda t), \quad (3.2)$$

and obtained the system of the ordinary differential equations

$$\beta k (\alpha + l^2) U_{\xi\xi} + \alpha V U_{\xi} + l U U_{\xi} - \lambda U_{\xi} = 0, \quad (3.3)$$

$$U_{\xi} - l V_{\xi} = 0. \quad (3.4)$$

Li and Zhang [43] proposed "a generalized multiple Riccati equation rational expansion method" to construct "a series of exact complex solutions" for the system of equations (3.3) and (3.4). The authors [43] found "new complex solutions" of the (2+1) - dimensional Burgers equation and "brought out rich complex solutions".

However, integrating Eq.(3.4) with respect to  $\xi$  we have

$$U = l V + C_1, \quad (3.5)$$

where  $C_1$  is an arbitrary constant. Substituting  $U$  into Eq.(3.3) we obtain

$$\beta k (\alpha + l^2) V_{\xi\xi} + (\alpha + l^2) V V_{\xi} + (C_1 l - \lambda) V_{\xi} = 0. \quad (3.6)$$

Integrating Eq.(3.6) with respect to  $\xi$  we get the Riccati equation in the form

$$V_{\xi} = -\frac{1}{2\beta k} \left( V^2 + \frac{2C_1 l - 2\lambda}{\alpha + l^2} V + 2\beta k C_2 \right). \quad (3.7)$$

Eq.(3.7) can be reduced to the form

$$\frac{dV}{d\xi} = -\frac{1}{2\beta k} (V - V_1)(V - V_2), \quad (3.8)$$

where  $V_1$  and  $V_2$  are the roots of the algebraic equation

$$V^2 + \frac{2C_1 l - 2\lambda}{\alpha + l^2} V + 2\beta k C_2 = 0, \quad (3.9)$$

that take the form

$$V_{1,2} = \frac{\lambda - C_1 l \pm \sqrt{(\lambda - C_1 l)^2 - 2\beta k C_2 (\alpha + l^2)^2}}{\alpha + l^2}. \quad (3.10)$$

Integrating Eq.(3.8) with respect to  $\xi$ , we find the general solution of Eq.(3.7) in the form

$$V(\xi) = \frac{V_1 - V_2 \exp\left(\frac{(V_2-V_1)(\xi-\xi_0)}{2\beta k}\right)}{1 - \exp\left(\frac{(V_2-V_1)(\xi-\xi_0)}{2\beta k}\right)}, \quad (3.11)$$

where  $\xi_0$  is an arbitrary constant and  $U(\xi)$  is determined by Eq.(3.5).

In the paper [43] the authors found 24 solitary wave solutions of the system (3.3) and (3.4), but we can see, that these solutions are useless for researches.

**Example 2b.** *Reduction of the (3+1) - dimensional Kadomtsev - Petviashvili equation by Zhang [44]*

$$u_{xt} + 6u_x^2 + 6u u_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \quad (3.12)$$

This equation was considered by Zhang [44], taking the travelling wave into account:  $u = U(\eta)$ ,  $\eta = kx + ly + sz + \omega t$ . After reduction Zhang obtained the nonlinear ordinary differential equation in the form

$$k\omega U_{\eta\eta} + 6k^2 U_{\eta}^2 + 6k^2 U U_{\eta\eta} - k^4 U_{\eta\eta\eta\eta} - l^2 U_{\eta\eta} - s^2 U_{\eta\eta} = 0. \quad (3.13)$$

The author [44] applied the Exp-function method and obtained the solitary wave solutions of Eq.(3.13).

However, denoting  $p = \frac{k\omega - l^2 - s^2}{k^4}$  from Eq. (3.13) we have the nonlinear ordinary differential equation in the form

$$U_{\eta\eta\eta\eta} - \frac{6}{k^2} U_{\eta}^2 - \frac{6}{k^2} U U_{\eta\eta} - p U_{\eta\eta} = 0. \quad (3.14)$$

Twice integrating Eq.(3.14) with respect to  $\eta$  we have

$$U_{\eta\eta} - \frac{3}{k^2} U^2 - pU - C_1 \eta + C_2 = 0, \quad (3.15)$$

where  $C_1$  and  $C_2$  are arbitrary constants. This equation is well known. Using the transformations for  $U$  and  $\eta$  by formulae

$$U = k^2 \left(\frac{k^2}{C_1}\right)^{-\frac{2}{5}} w - \frac{pk^2}{6}, \quad (3.16)$$

$$\eta = \left(\frac{k^2}{C_1}\right)^{\frac{1}{5}} z + \frac{C_2}{C_1} + \frac{p^2 k^2}{12 C_1},$$

we have the first Painlevé equation [4, 7]

$$w_{zz} = 3w^2 + z. \quad (3.17)$$

The solutions of Eq. (3.17) are the Painlevé transcendents.

For the case  $C_1 = 0$  from Eq.(3.15) we obtain the equation in the form

$$U_{\eta\eta} - \frac{3}{k^2} U^2 - pU + C_2 = 0. \quad (3.18)$$

Multiplying Eq.(3.18) on  $U_\eta$  we have

$$U_\eta^2 - \frac{2}{k^2} U^3 - pU^2 + 2C_2 U + C_3 = 0, \quad (3.19)$$

where  $C_3$  is an arbitrary constant.

The general solution of Eq. (3.19) is found via the Weierstrass elliptic function [45, 46]. We can see, that there is no need to look for the exact solutions of Eq. (3.13). This solution is expressed via the general solution of well known Eqs. (3.17) and (3.19).

**Example 2c.** *Reduction of the Ito equation by Khani [47]*

$$\phi_{xtt} + \phi_{xxxxt} + 6\phi_{xx}\phi_{xt} + 3\phi_x\phi_{xxt} + 3\phi_{xxx}\phi_t = 0. \quad (3.20)$$

Eq. (3.20) was studied by Khani [47]. The author looked for the solutions of Eq. (3.20) taking the travelling wave into account

$$\phi = \psi(\xi), \quad \psi = k(x - \lambda t). \quad (3.21)$$

Substituting (3.21) into (3.20) when  $\lambda \neq 0$  and  $k \neq 0$  we obtain the equation in the form

$$\lambda\psi_{\xi\xi\xi} - k^2\psi_{\xi\xi\xi\xi} - 6k\psi_{\xi\xi}\psi_{\xi\xi} - 6k\psi_\xi\psi_{\xi\xi\xi} = 0. \quad (3.22)$$

Twice integrating Eq. (3.22) with respect to  $\xi$  we have

$$\lambda\psi_\xi - k^2\psi_{\xi\xi\xi} - 3k(\psi_\xi)^2 + C_3\xi + C_4 = 0, \quad (3.23)$$

where  $C_3$  and  $C_4$  are arbitrary constants.

The author [47] looked for solution of Eq.(3.23) when  $C_3 = C_4 = 0$  using the Exp - function method.

In fact, denoting  $\psi_\xi = V(\xi)$  in Eq.(3.23) we get the following equation

$$k^2 V_{\xi\xi} + 3kV^2 - \lambda V - C_3\xi - C_4 = 0. \quad (3.24)$$

Eq. (3.24) is equivalent to Eq. (3.15). The solutions of this equation are expressed for  $C_3 \neq 0$  as the first Painlevé transcendents [4, 7] (see the previous example). For  $C_3 = 0$  the solutions of Eq. (3.24) can be obtained using the Weierstrass elliptic function. So we need not to search for the solutions of Eq. (3.22) as well.

**Example 2d.** *Reduction of the (3+1) - dimensional Jimbo - Miva equation by Öziş and Aslan [48]*

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \quad (3.25)$$

Using the travelling wave  $u(x, y, z, t) = U(\xi)$ ,  $\xi = kx + my + rz + wt$  Eq. (3.25) can be written as the nonlinear ordinary differential equation

$$k^3 m U_{\xi\xi\xi\xi} + 6k^2 m U_{\xi} U_{\xi\xi} + (2mw - 3kr) U_{\xi\xi} = 0. \quad (3.26)$$

The authors [48] applied the Exp - function method to Eq.(3.26) to obtain "the exact and explicit generalized solitary solutions in more general forms".

But integrating Eq.(3.26) with respect to  $\xi$  we obtain

$$k^3 m U_{\xi\xi\xi} + 3k^2 m U_{\xi}^2 + (2mw - 3kr) U_{\xi} = C_5, \quad (3.27)$$

where  $C_5$  is an arbitrary constant. Denoting  $U_{\xi} = V$  we get

$$k^3 m V_{\xi\xi} + 3k^2 m V^2 + (2mw - 3kr) V = C_5. \quad (3.28)$$

Multiplying Eq.(3.28) on  $V_{\xi}$  we have the equation in the form

$$V_{\xi}^2 + \frac{2}{k} V^3 + \frac{2mw - 3kr}{mk^3} V^2 - \frac{2C_5}{mk^3} V - C_6 = 0, \quad (3.29)$$

where  $C_6$  is an arbitrary constant. The general solution can be found using the Weierstrass elliptic function. The solution of Eq.(3.26) is found by the integral

$$U = \int V d\xi. \quad (3.30)$$

We can see that Eq.(3.27) has the general solution (3.30) and all partial cases can be found from the general solution of Eq.(3.30).

**Example 2e.** *Reduction of the Benjamin - Bona - Mahony equation [49]*

$$u_t - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0. \quad (3.31)$$

Eq. (3.31) was considered by Ganji and co - authors [49]. In terms of the

travelling wave  $u(x, t) = U(\eta)$ ,  $\eta = kx - \omega t$  they obtained the equation

$$k^2 \omega U_{\eta\eta\eta} + kU U_{\eta} + (k - \omega) U_{\eta} = 0. \quad (3.32)$$

Using the Exp - function method the authors [49] looked for the solitary wave solutions of Eq.(3.32).

Integrating Eq. (3.32) with respect to  $\eta$  we have

$$k^2 \omega U_{\eta\eta} + \frac{1}{2} k U^2 + (k - \omega) U + C_1 = 0. \quad (3.33)$$

Multiplying Eq.(3.33) on  $U_{\eta}$  and integrating the result with respect to  $\eta$  again we obtain the equation

$$k^2 \omega U_{\eta}^2 + \frac{k}{3} U^3 + (k - \omega) U^2 + 2C_1 + C_2 = 0. \quad (3.34)$$

The solution of Eq.(3.34) can be given by the Weierstrass elliptic function [4, 7, 45, 46]. We can see that the authors [49] obtained the known solitary wave solutions of Eq.(3.31).

**Example 2f.** *Reduction of the Sharma - Tasso - Olver equation by Erbas and Yusufoglu [50]*

$$u_t + \alpha (u^3)_x + \frac{3}{2} \alpha (u^2)_{xx} + \alpha u_{xxx} = 0. \quad (3.35)$$

Taking the travelling wave  $u(x, t) = U(\xi)$ ,  $\xi = \mu(x - ct)$  the authors [50] obtained the reduction of Eq.(3.35) in the form

$$\alpha \mu^2 U_{\xi\xi} + 3\alpha \mu U U_{\xi} + \alpha U^3 - cU = 0. \quad (3.36)$$

The authors of [50] used the Exp - function method to find "new solitary solutions", but they left out of their account that using the transformation

$$U = \mu \frac{F_{\xi}}{F} \quad (3.37)$$

Eq.(3.36) can be transformed to the linear equation

$$F_{\xi\xi\xi} - \frac{c}{\alpha \mu^2} F_{\xi} = 0. \quad (3.38)$$

The solution of Eq.(3.38) takes the form

$$F(\xi) = C_1 + C_2 \exp \left\{ \frac{\xi \sqrt{c}}{\mu \sqrt{\alpha}} \right\} + C_3 \exp \left\{ -\frac{\xi \sqrt{c}}{\mu \sqrt{\alpha}} \right\}. \quad (3.39)$$

Substituting the solution (3.39) into the transformation (3.37) we obtain the solution of Eq.(3.36) in the form

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{\alpha}} \frac{C_2 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} - C_3 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}{C_1 + C_2 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} + C_3 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}. \quad (3.40)$$

Certainly all the solutions obtained by means of the Exp - function method can be found from solution (3.40).

**Example 2g.** Reduction of the dispersive long wave equations by Abdou [35]

$$\begin{aligned} v_t + v v_x + w_x &= 0, \\ w_t + (vw)_x + \frac{1}{3} v_{xxx} &= 0. \end{aligned} \quad (3.41)$$

Using the wave transformations  $w(x, t) = \sigma(\xi)$ ,  $v(x, t) = \phi(\xi)$ ,  $\xi = k(x + \lambda t)$ , the system of equations (3.41) can be written in the form

$$\lambda \phi_\xi + \phi \phi_\xi + \sigma_\xi = 0, \quad (3.42)$$

$$\lambda \sigma_\xi + (\sigma \phi)_\xi + \frac{k^2}{2} \phi_{\xi\xi\xi} = 0. \quad (3.43)$$

Abdou [35] looked for solutions of the system of equations (3.42) and (3.43) taking "the extended tanh method" into account.

Integrating Eqs. (3.42) and (3.43) with respect to  $\xi$  we obtain

$$\sigma = -C_1 - \lambda \phi - \frac{1}{2} \phi^2, \quad (3.44)$$

$$\lambda \sigma + (\sigma \phi) + \frac{k^2}{2} \phi_{\xi\xi} + C_2 = 0. \quad (3.45)$$

Substituting the value of  $\sigma$  from (3.44) into Eq.(3.45) we have

$$\phi_{\xi\xi} - \frac{1}{k^3} \phi^3 - \frac{3\lambda}{k^3} \phi^2 - \frac{2(C_1 + \lambda)}{k^3} \phi + \frac{2C_2}{k^3} = 0. \quad (3.46)$$

Multiplying Eq.(3.46) on  $\phi_\xi$  and integrating the equation with respect to  $\xi$ , we have the equation in the form

$$\phi_\xi^2 - \frac{1}{2k^3} \phi^4 - \frac{3\lambda}{k^3} \phi^3 - \frac{(C_1 + \lambda)}{k^3} \phi^2 + \frac{2C_2}{k^3} \phi + C_3 = 0. \quad (3.47)$$

The general solution of Eq.(3.47) is determined via the Jacobi elliptic function [7]. The variable  $\sigma$  is found by the formula (3.44) and there is no need to look for the exact solutions of the Eqs.(3.42) and (3.43).

Unfortunately we have many similar examples. Some of them are also presented in our recent work [46].

#### 4 Third error: some authors omit arbitrary constants after integration of equation

Reductions of nonlinear evolution equations to nonlinear ordinary differential equations can be often integrated. However, some authors assume, that the arbitrary constants of integration are equal to zero. This error potentially leads to the loss of the arbitrary constants in the final expression. So the solution obtained in such way is less general than it could be. The third common error can be formulated as follows.

**Third error.** *Some authors omit the arbitrary constants after integrating of the nonlinear ordinary differential equations.*

**Example 3a.** *Reduction of the Burgers equation by Soliman [51].*

Soliman [51] considered the Burgers equation

$$u_t + u u_x - \nu u_{xx} = 0 \quad (4.1)$$

to solve this equation by so called "the modified extended tanh - function method".

It is well known, that by using the Cole-Hopf transformation [52, 53]

$$u = -2\nu \frac{\partial}{\partial x} \ln F, \quad (4.2)$$

we can write the equality

$$u_t + u u_x - \nu u_{xx} = -2\nu \frac{\partial}{\partial x} \left( \frac{F_t - \nu F_{xx}}{F} \right). \quad (4.3)$$

From the last relation we can see, that each solution of the heat equation

$$F_t - \nu F_{xx} = 0, \quad (4.4)$$

gives the solution of the Burgers equation by formula (4.2).

However to find the solutions of the Burgers equation Soliman [51] used the travelling wave solutions  $u(x, t) = U(\xi)$ ,  $\xi = x - ct$  and from Eq.(4.1) after integration with respect to  $\xi$  the author obtained the equation in the form

$$\nu U_\xi - \frac{1}{2} U^2 + cU = 0. \quad (4.5)$$

The constant of integration he took to be equal to zero. The general solution of Eq.(4.5) takes the form

$$U(\xi) = \frac{2cC_2 \exp\left\{-\frac{c\xi}{\nu}\right\}}{1 + C_2 \exp\left\{-\frac{c\xi}{\nu}\right\}}, \quad (4.6)$$

where  $C_2$  is an arbitrary constant.

The general solution of Eq.(4.5) has the only arbitrary constant. But if we take nonzero constant of integration in Eq.(4.5), we can have two arbitrary constants in the solution.

**Example 3b.** Reduction of the (2+1) - dimensional Konopelchenko - Dubrovsky equation by Abdou [54]

$$u_t - u_{xxx} - 6buu_x + \frac{3a^2}{2}u^2u_x - 3v_y + 3av u_x = 0, \quad u_y = v_x. \quad (4.7)$$

Using the wave solutions

$$u(x, t) = U(\eta), \quad \eta = kx + ly + \omega t \quad (4.8)$$

the author [54] looked for the exact solutions of Eq.(4.7). After integration with respect to  $\eta$  he obtained the second order differential equation

$$\left(\omega - \frac{3l^2}{k}\right)U - k^3U_{\eta\eta} + \left(\frac{3al}{2} - 3bk\right)U^2 + \frac{a^2}{2}kU^3 = 0, \quad (4.9)$$

but the zero constant of integration was taken. Abdou [54] used the Exp - function method to look for solitary wave solutions of Eq.(4.9).

However multiplying Eq.(4.9) on  $U_\eta$  and integrating this equation with respect to  $\eta$  again, we have the equation

$$\left(\omega - \frac{3l^2}{k}\right)U^2 - k^3U_\eta^2 + (al - 2bk)U^3 + \frac{a^2}{4}kU^4 = C_2, \quad (4.10)$$

where  $C_2$  is a constant of integration. The general solution of Eq.(4.10) is expressed via the Jacobi elliptic function [7].



**Example 3c.** Reduction of the Ito equation by Wazwaz [55]

$$v_{xtt} + v_{xxxxt} + 6 v_{xx} v_{xt} + 3 v_x v_{xxt} + 3 v_{xxx} v_t = 0. \quad (4.11)$$

The author [55] looked for the solutions of Eq. (4.11) taking into account the travelling wave

$$v = v(\xi), \quad \xi = k(x - \lambda t). \quad (4.12)$$

Substituting (4.12) into (4.11) Wazwaz obtained when  $\lambda \neq 0$  and  $k \neq 0$  the equation in the form

$$\lambda v_{\xi\xi\xi} - k^2 v_{\xi\xi\xi\xi\xi} - 6k v_{\xi\xi} v_{\xi\xi} - 6k v_{\xi} v_{\xi\xi\xi} = 0. \quad (4.13)$$

Integrating Eq.(4.13) twice with respect to  $\xi$  one can have the equation

$$\lambda v_{\xi} - k^2 v_{\xi\xi\xi} - 3k (v_{\xi})^2 + C_8 \xi + C_9 = 0, \quad (4.14)$$

where  $C_8$  and  $C_9$  are arbitrary constants. Denoting  $v_{\xi} = V(\xi)$  in Eq. (4.14) we get the equation

$$k^2 V_{\xi\xi} + 3k V^2 - \lambda V - C_8 \xi - C_9 = 0. \quad (4.15)$$

The general solution of this equation was discussed above in example 2b.

However the author [55] looked for solutions of Eq. (4.14) for  $C_8 = 0$  and  $C_9 = 0$  taking into consideration the tanh - coth method and did not present the general solution of Eq. (4.13).

**Example 3d.** Reduction of the Boussinesq equation by Bekir [32]

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0. \quad (4.16)$$

Using the wave variable  $\xi = x - ct$  the author [32] got the equation

$$u_{\xi\xi} - u^2 + (c^2 - 1)u = 0. \quad (4.17)$$

To look for the solitary wave solutions of Eq.(4.17) the author [32] used the  $G'/G$  - method, but he have been omitted two arbitrary constants after the integration.

In fact, from Eq.(4.16) we obtain the second order differential equation in the form

$$u_{\xi\xi} - u^2 + (c^2 - 1)u + C_1 \xi + C_2 = 0. \quad (4.18)$$

The general solution of Eq.(4.18) is expressed at  $C_1 \neq 0$  via the first Painlevé transcendents. In the case  $C_1 = 0$  solutions of Eq.(4.18) is determined by the Weierstrass elliptic function. All possible solutions of Eq. (4.16) were obtained in work [56].

## 5 Fourth error: using some functions in finding exact solutions some authors lose arbitrary constants

Some authors do not include the arbitrary constants in finding the exact solutions of nonlinear differential equations. As a result these authors obtain many solutions, that can be determined as the only solution using some arbitrary constants. The arbitrary constants can be included, if we use the general solutions of the known differential equations. Sometimes this error can be corrected, using the property of the autonomous differential equation. So, the fourth error can be formulated as follows.

**Fourth error.** *Using some functions in finding the exact solutions of nonlinear differential equations some authors lose the arbitrary constants.*

Let us explain this error. Consider an ordinary differential equation in the general form

$$E(Y, Y_\xi, \dots) = 0. \quad (5.1)$$

Let us assume that Eq. (5.1) is autonomous and this equation admits the shift of the independent variable  $\xi \rightarrow \xi + C_2$  (where  $C_2$  is an arbitrary constant). This means that the constant  $C_2$  added to the variable  $\xi$  in Eq.(5.1) does not change the form of this equation. In this case Eq.(5.1) is invariant under the shift of the independent variable.

Taking this property into account, we obtain the advantage for the solution of Eq. (5.1). If we know a solution  $Y = f(\xi)$  of Eq. (5.1), then for the autonomous equation we have a solution  $Y = f(\xi - \xi_0)$  of this equation with additional arbitrary constant  $\xi_0$ .

The main feature of the autonomous equation is that the fact that the solution  $Y = f(\xi - \xi_0)$  is more general, then  $Y = f(\xi)$ .

The error discussed often leads to a huge amount of different expressions for the solutions of nonlinear differential equations instead of choosing one solution with an arbitrary constant  $\xi_0$ .

The application of the tanh - function method [28–30] for finding the exact solutions allows us to have the special solutions of nonlinear differential equations as a sum of hyperbolic tangents  $\tanh(\xi)$ . However for the autonomous

equation such types of the solutions can be taken as the more general solution in the form  $\tanh(\xi - \xi_0)$ .

**Example 4a.** *Solution of the Riccati equation*

$$Y_\xi = \beta(1 + Y^2). \quad (5.2)$$

Eq. (5.2) is of the first order, therefore the general solution of Eq. (5.2) depends on the only arbitrary constant. We can meet a lot of "different" solutions of Eq. (5.2). For example

$$Y_1(\xi) = \tan\{\beta\xi\}, \quad (5.3)$$

$$Y_2(\xi) = -\cot\{\beta\xi\}, \quad (5.4)$$

$$Y_3(\xi) = \tan\{2\beta\xi\} - \sec\{2\beta\xi\}, \quad (5.5)$$

$$Y_4(\xi) = \csc\{2\beta\xi\} - \cot\{2\beta\xi\}. \quad (5.6)$$

In fact, the general solution of Eq. (5.2) takes the form

$$Y(\xi) = \tan\{\beta(\xi - \xi_0)\}. \quad (5.7)$$

All solutions (5.3) - (5.6) can be obtained from the general solution (5.7) of the Riccati equation, because of the following equalities

$$\begin{aligned} -\cot\beta x &= \tan(\beta x + \pi/2), \\ \tan 2\beta x - \sec 2\beta x &= \tan(\beta x - \pi/4), \\ \csc 2\beta x - \cot 2\beta x &= \tan\beta x. \end{aligned} \quad (5.8)$$

**Example 4b.** *Solution of the Cahn - Hilliard equation by Ugurlu and Kaya [57]*

$$u_t + u_{xxxx} = (u^3 - u)_{xx} + u_x. \quad (5.9)$$

Eq.(5.9) was considered in [57] by means of the modified extended tanh - function method by Ugurlu and Kaya. Using the travelling wave

$$u(x, t) = u(z), \quad z = x + ct, \quad (5.10)$$

the authors obtained the exact solutions of the equation

$$cu_z + u_{zzzz} - 6u(u_z)^2 - 3u^2u_{zz} + u_{zz} - u_z = 0. \quad (5.11)$$

They found eight solutions of Eq.(5.11) at  $c = 1$ . Six solutions are the following

$$u_1 = \coth\left\{\frac{\sqrt{2}}{2}z\right\}, \quad u_2 = -\coth\left\{\frac{\sqrt{2}}{2}z\right\}, \quad u_3 = \tanh\left\{\frac{\sqrt{2}}{2}z\right\}, \quad (5.12)$$

$$u_4 = \frac{1}{2} \left( \tanh \left\{ \frac{z}{2\sqrt{2}} \right\} + \coth \left\{ \frac{z}{2\sqrt{2}} \right\} \right), \quad u_5 = -\tanh \left\{ \frac{\sqrt{2}}{2} z \right\}, \quad (5.13)$$

$$u_6 = -\frac{1}{2} \left( \tanh \left\{ \frac{z}{2\sqrt{2}} \right\} + \coth \left\{ \frac{z}{2\sqrt{2}} \right\} \right). \quad (5.14)$$

However all these solutions can be written as the only solution with an arbitrary constant  $z_0$

$$u = \tanh \left\{ \frac{\sqrt{2}}{2} (z - z_0) \right\}. \quad (5.15)$$

Note, that Eq.(5.11) at  $c = 1$  takes the form

$$u_{zzzz} - 6u(u_z)^2 - 3u^2 u_{zz} + u_{zz} = 0. \quad (5.16)$$

Twice integrating Eq.(5.16) with respect to  $z$ , we have

$$u_{zz} - 2u^3 + u + C_1 z + C_2 = 0, \quad (5.17)$$

where  $C_1$  and  $C_2$  are arbitrary constants. At  $C_1 = 0$  the solution of Eq.(5.17) is expressed via the Jacobi elliptic function.

**Example 4c.** *Solution of the KdV - Burgers equation by Soliman [51]*

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0. \quad (5.18)$$

Using the travelling wave and "the modified extended tanh - function method" the author [51] obtained four solutions of Eq.(5.18). Three of them take the form

$$u(x, t) = \frac{\nu^2}{25\varepsilon\mu} \left[ 9 - 6 \coth \left( \frac{\nu\xi}{10\mu} \right) - 3 \coth^2 \left( \frac{\nu\xi}{10\mu} \right) \right], \quad \xi = x - \frac{6\nu^2 t}{25\mu}; \quad (5.19)$$

$$u(x, t) = \frac{\nu^2}{25\varepsilon\mu} \left[ 9 - 6 \tanh \left( \frac{\nu\xi}{10\mu} \right) - 3 \tanh^2 \left( \frac{\nu\xi}{10\mu} \right) \right], \quad \xi = x - \frac{6\nu^2 t}{25\mu}; \quad (5.20)$$

$$u(x, t) = \frac{3\nu^2}{25\varepsilon\mu} \left[ 1 - \frac{4}{10} \left( \tanh \left( \frac{\nu\xi}{20\mu} \right) + \coth \left( \frac{\nu\xi}{20\mu} \right) \right) - \frac{1}{10} \left( \tanh^2 \left( \frac{\nu\xi}{20\mu} \right) + \coth^2 \left( \frac{\nu\xi}{20\mu} \right) \right) \right], \quad \xi = x - \frac{6\nu^2 t}{25\mu}. \quad (5.21)$$

All these solutions can be written in the form

$$u(x, t) = \frac{3\nu^2}{25\varepsilon\mu} \left[ 3 - 2 \tanh \left( \frac{\nu\xi}{10\mu} - \xi_0 \right) - \tanh^2 \left( \frac{\nu\xi}{10\mu} - \xi_0 \right) \right], \quad (5.22)$$

$$\xi = x - \frac{6\nu^2 t}{25\mu}.$$

Assuming  $\xi_0 = 0$  in (5.22), we obtain solution (5.20). In the case  $\xi_0 = \frac{i\pi}{2}$  in (5.22) we have solution (5.19). Taking into account the formulae

$$\tanh(k\xi) + \coth(k\xi) = 2 \coth(2k\xi), \quad (5.23)$$

$$\tanh^2(k\xi) + \coth^2(k\xi) = 4 \coth^2(2k\xi) - 2, \quad (5.24)$$

we can transform solution (5.22) into solution (5.21).

We can see, that these solutions do not differ, if we take the constant  $\xi_0$  into account in one of these solutions.

The fourth solution by Soliman [51] can be simplified as well. All the solutions by the KdV — Burgers equation by Soliman coincides with (2.15).

**Example 4d.** *Solution of the combined KdV - mKdV equation by Bekir [58]*

$$u_t + pu u_x + qu^2 u_x + ru_{xxx} = 0. \quad (5.25)$$

Using the extended tanh method the author [58] have obtained the following solutions of Eq.(5.25)

$$u = \frac{12r}{p} [1 + \tanh(x - 4rt)], \quad (5.26)$$

$$u = \frac{12r}{p} [1 + \coth(x - 4rt)], \quad (5.27)$$

$$u = \frac{24r}{p} [(21 + \tanh(x - 4rt)) + (1 + \coth(x - 4rt))]. \quad (5.28)$$

All these solutions can be written as the only solution with an arbitrary constant

$$u = \frac{12r}{p} [1 + \tanh(x - 4rt + \varphi_0)]. \quad (5.29)$$

**Example 4e.** *Solution of the coupled Hirota — Satsuma — KdV equation by Bekir [59]*

$$u_t = \frac{1}{4} u_{xxx} + 3u u_x - 6v u_x, \quad (5.30)$$

$$v_t = -\frac{1}{2} v_{xxx} - 3u v_x. \quad (5.31)$$

The system of Eqs.(5.30) and (5.31) was studied by means of the tanh - coth method by Bekir [59]. Using the travelling wave  $\xi = (x - \beta t)$  the author obtained the system of equations

$$\begin{aligned} \frac{1}{4} U_{\xi\xi\xi} + 3U U_{\xi} - 6V V_{\xi} + \beta U_{\xi} &= 0 \\ \frac{1}{2} V_{\xi\xi\xi} + 3U V_{\xi} - \beta V_{\xi} &= 0, \end{aligned} \quad (5.32)$$

and gave ten solitary wave solutions. However all the solitary wave solutions of the system of equations (5.30) and (5.31) by Bekir [59] can be expressed by the formulae

$$U^{(1)} = \frac{\beta + 4k^2}{3} - 2k^2 \tanh^2(k\xi - \xi_0), \quad (5.33)$$

$$V^{(1)} = \frac{2\beta + 2k^2}{3} - k^2 \tanh^2(k\xi - \xi_0), \quad \xi = x - \beta t,$$

(where  $k$ ,  $\xi_0$  and  $\beta$  are arbitrary constants)

$$U^{(2)} = k^2 - 2k^2 \tanh^2(k\xi - \xi_0), \quad (5.34)$$

$$V^{(2)} = -k^2 \tanh^2(k\xi - \xi_0), \quad \xi = x + k^2 t, \quad \beta = -k^2,$$

$$U^{(3,4)} = \frac{\beta + k^2}{3} - k^2 - k^2 \tanh^2(k\xi - \xi_0), \quad (5.35)$$

$$V^{(3,4)} = \pm \sqrt{\frac{k^2 - 2\beta k}{3}} \tanh(k\xi - \xi_0), \quad \xi = x - \beta t.$$

**Example 4f.** "Twenty seven solution" of the "generalized Riccati equation" by Xie, Zhang and Lü [60].

The authors [60] "firstly extend" the Riccati equation to the "general form"

$$\phi_{\xi} = r + p\phi + q\phi^2, \quad (5.36)$$

where  $r$ ,  $p$  and  $q$  are the parameters. They "fortunately find twenty seven solutions" of Eq.(5.36).

It was very surprised that the authors are not aware that the solution of Eq.(5.36) was known more then one century ago. It is very strange but these 27 solutions was repeated by Zhang [61] as the important advantage.

Let us present the general solution of Eq.(5.36). Substituting

$$\phi = \frac{1}{q} V - \frac{p}{2q} \quad (5.37)$$

into Eq.(5.36) we have

$$V_\xi = V^2 + \frac{4rq - p^2}{4}. \quad (5.38)$$

Assuming

$$V = \frac{\psi_\xi}{\psi} \quad (5.39)$$

in Eq.(5.38) we obtain the linear equation of the second order

$$\psi_{\xi\xi} - \frac{4rq - p^2}{4} \psi = 0. \quad (5.40)$$

The general solution of Eq.(5.40) is well known

$$\psi(\xi) = C_1 e^{\frac{\xi}{2}\sqrt{4rq-p^2}} + C_2 e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}. \quad (5.41)$$

Using formula (5.39) and (5.37) we obtain the solution of Eq.(5.36) in the form

$$\phi(\xi) = \frac{\sqrt{4rq - p^2}}{2q} \frac{C_1 e^{\frac{\xi}{2}\sqrt{4rq-p^2}} - C_2 e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}}{C_1 e^{\frac{\xi}{2}\sqrt{4rq-p^2}} + C_2 e^{-\frac{\xi}{2}\sqrt{4rq-p^2}}} - \frac{p}{2q}. \quad (5.42)$$

The general solution of Eq.(5.36) is found from (5.42). All 27 solutions by Xie, Zhang and Lü are found from solution (5.42) and we cannot obtain other solutions.

**Example 4g.** "New solutions and kinks solutions" of the Sharma — Tasso — Olver equation by Wazwaz [62].

Using the extended tanh method Wazwaz [62] have found 18 solitary wave and kink solutions of the Sharma — Tasso — Olver equation

$$u_t + \alpha (u^3)_x + \frac{3}{2} \alpha (u^2)_{xx} + \alpha u_{xxx} = 0. \quad (5.43)$$

Taking the travelling wave solution  $u(x, t) = u(\xi)$ ,  $\xi = x - ct$  into account the author considered the nonlinear ordinary differential equation in the form

$$\alpha u_{\xi\xi} + 3\alpha u u_\xi + \alpha u^3 - c u = 0. \quad (5.44)$$

However Eq.(5.44) can be transformed to the second - order linear differential equation (see, example 2f)

$$F_{\xi\xi\xi} - \frac{c}{\alpha} F_\xi = 0 \quad (5.45)$$

by the transformation

$$u(\xi) = \frac{F_\xi}{F}. \quad (5.46)$$

The solution of Eq.(5.44) is given by formula (3.40) at  $\mu = 1$ . Certainly all "new solutions" of the Sharma — Tasso — Olver equation by Wazwaz [62] are found from (3.40) at  $\mu = 1$ .

We can see, that there is no need to write a list of all possible expressions for the solutions at the given  $\xi_0$ . It is enough to present the solution of the equation with an arbitrary constant. Moreover, the solution with arbitrary constants looks better.

The simple and powerful tool to remove this error is to plot the graphs of the expressions obtained. The expressions having the same graphs usually are equivalent.

## 6 Fifth error: some authors do not simplify the solutions of differential equations

Using different approaches for nonlinear differential equations, the authors obtain different forms of the solutions and sometimes they find the solutions in cumbersome forms. Interpretation and understanding of such solutions are difficult and we believe, that these solutions can be simplified. The consequence of the cumbersome expressions is a risk of misprints. The solutions in cumbersome form give an illusion that the authors indeed find new solutions. We meet many such solutions of nonlinear differential equations especially found by means of the Exp — function method. Another consequence is the fact, that the authors obtain the same solutions expressed in different forms, because it is well known, that trigonometric and hyperbolic expressions admit different representation.

The fifth error can be formulated as follows.

**Fifth error.** *Expressions for solutions of nonlinear differential equations are often not simplified.*

**Example 5a.** *Solutions of the Jimbo - Miva equation (3.26) by Öziş and Aslan [48].*

Using the Exp - function method Öziş and Aslan [48] found some solitary wave solutions of Eq. (3.26), but a number of these solutions can be simplified. For



example solution (30) in [48] is transformed by the following set of equalities

$$\begin{aligned}
u(x, y, z, t) &= \frac{a_2 e^{2\xi} + a_2 b_1 e^\xi + b_0 (a_2 - 4k) + b_0 b_1 (a_2 - 4k) e^{-\xi}}{e^{2\xi} + b_1 e^\xi + b_0 + b_0 b_1 e^{-\xi}} = \\
&= \frac{(a_2 e^\xi + b_0 a_2 e^{-\xi} - 4 b_0 k e^{-\xi}) (e^\xi + b_1)}{(e^\xi + b_0 e^{-\xi}) (e^\xi + b_1)} = a_2 - \frac{4 b_0 k}{b_0 + e^{2\xi}}.
\end{aligned} \tag{6.1}$$

It is seen, that the last expression is better for understanding, then the solution by Öziş and Aslan.

**Example 5b.** *Solutions of the Konopelchenko - Dubrovsky equation (4.7) by Abdou [54].*

By means of the Exp - function method Abdou [54] obtained some solitary wave solutions of Eq. (4.7), but some solutions can be simplified. For example solution (42) in [54] reduces to

$$\begin{aligned}
u(x, t) &= \frac{a_2 e^{2\eta} + a_1 e^\eta - \frac{a_1^2 (a_2 + 2k^2)}{2k^4} + \frac{a_1^3}{4k^4} e^{-\eta} + \frac{a_2 a_1^4}{16k^8} e^{-2\eta}}{\frac{a_1^4}{16k^8} e^{-2\eta} + e^{2\eta} - \frac{a_1^2}{2k^4}} = \\
&= a_2 + \frac{a_1 \left( e^{\frac{\eta}{2}} - \frac{a_1}{2k^2} e^{-\frac{\eta}{2}} \right)^2}{\left( e^\eta - \frac{a_1^2}{4k^4} e^{-\eta} \right)^2} = a_2 + \frac{a_1}{\left( e^{\frac{\eta}{2}} + \frac{a_1}{2k} e^{-\frac{\eta}{2}} \right)^2}.
\end{aligned} \tag{6.2}$$

**Example 5c.** *Solution of the modified Benjamin - Bona - Mahony equation by Yusufoglu [63]*

$$u_t + u_x + \alpha u^2 u_x + u_{xxx} = 0. \tag{6.3}$$

Using the travelling wave the author [63] obtained the equation

$$\beta^2 U_{\xi\xi} + \frac{1}{3} \alpha U^3 + (1 - \gamma) U = 0. \tag{6.4}$$

Yusufoglu [63] found the solution of Eq.(6.4) in the form

$$u(\xi) = \frac{\frac{24b_0 b_1 (c-1)}{a_0 a} e^\xi + a_0}{\frac{24b_0 b_1^2 (c-1)}{a_0^2 a} e^{2\xi} + b_1 e^\xi + b_0 + \frac{a_0^2 a}{24b_1 (c-1)} e^{-\xi}}. \tag{6.5}$$

Solution (6.5) has three arbitrary constants  $a_0$ ,  $b_0$  and  $b_1$ , but in fact this

solution can be simplified using the set of equalities

$$\begin{aligned}
u(\xi) &= \frac{\frac{24b_0b_1(c-1)}{a_0a}e^\xi + a_0}{\frac{24b_0b_1^2(c-1)}{a_0^2a}e^{2\xi} + b_1e^\xi + b_0 + \frac{a_0^2a}{24b_1(c-1)}e^{-\xi}} = \\
&= \frac{1}{\frac{a_0a}{24b_1(c-1)}e^{-\xi} + \frac{b_1}{a_0}e^\xi} = \frac{24b(c-1)}{ae^{-\xi} + 24b^2(c-1)e^\xi}.
\end{aligned} \tag{6.6}$$

We can see, that solution (6.6) contains the only arbitrary constant  $b = \frac{b_1}{a_0}$ .

Note that Eq.(6.3) can be easy transformed to the modified KdV equation and this equation is not the BBM equation.

**Example 5d.** *Solutions of the fifth - order KdV equation by Chun [64]*

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0. \tag{6.7}$$

Using the Exp - function method, Chun obtained seventeen exact solutions of Eq. (6.7) [64]. His first solution (formula (25) in [64]) can be simplified to the trivial solution, because

$$u_1 = \frac{-k^4e^\eta + 2k^2a_0 - k^4b_{-1}e^{-\eta}}{2k^2e^\eta - 4a_0 + 2k^2b_{-1}e^{-\eta}} \equiv -\frac{k^2}{2}, \quad \eta = k\left(x - \frac{7k^4}{4}t\right). \tag{6.8}$$

The sixth solution by Chun (formula (30) [64]) can be simplified to the trivial solution as well

$$u_6 = \frac{a_1e^\eta + a_1b_0 + a_1b_{-1}e^{-\eta}}{e^\eta + b_0 + b_{-1}e^{-\eta}} \equiv a_1, \quad \eta = k(x + \omega t). \tag{6.9}$$

The thirteenth (as well as the fourteenth and the fifteenth) solution (formula (69) in paper [64]) is also constant as we can see from the equality

$$u_{13} = \frac{a_1e^{2\eta} + a_1b_1e^\eta}{b_1e^{2\eta} + b_1^2e^{-\eta}} \equiv \frac{a_1}{b_1}, \quad \eta = (kx + \omega t). \tag{6.10}$$

We recognized, that among seventeen solutions presented by Chun in his paper twelve solutions (25), (27), (28) (30), (49), (51) (53), (55), (57), (69), (70) and (71) satisfy the fifth-order KdV equation, but solutions (25), (30), (69), (70) and (71) are trivial ones (constants) and (51) is not the solitary wave solution. Six solutions (25), (27), (49), (53), (55) and (57) are solitary wave solutions, but these ones are known and can be found by means of other methods.

**Example 5e.** Solutions of the improved Boussinesq equation by Abdou and co -authors [65].

Using the Exp - function method Abdou and his co - authors looked for the solitary wave solutions of the improved Boussinesq equation

$$u_{tt} - u_{xx} - u u_{xx} - (u_x)^2 - u_{xxt} = 0. \quad (6.11)$$

In the travelling wave Eq.(6.11) takes the form

$$(w^2 - k^2 - k^2 u) u_{\eta\eta} - k^2 u_{\eta}^2 - k^2 w^2 u_{\eta\eta\eta\eta} = 0. \quad (6.12)$$

The authors found several solutions of Eq.(6.12). One of them takes the form

$$u(\eta) = \frac{\left( a_1 e^{\eta} - \frac{b_0 (5 a_1 k^2 + a_1 + 6 k^2)}{k^2 - 1} + \frac{a_1 b_0^2}{4} e^{-\eta} \right)}{\left( e^{\eta} + b_0 + \frac{b_0^2}{4} e^{-\eta} \right)}, \quad \eta = k x + t \sqrt{-\frac{a_1 + 1}{k^2 - 1}}. \quad (6.13)$$

However this solution can be presented as

$$u(\eta) = a_1 - \frac{6 b_0 k^2 (a_1 + 1)}{(k^2 - 1) \left( e^{\frac{\eta}{2}} + \frac{b_0}{2} e^{-\frac{\eta}{2}} \right)^2}. \quad (6.14)$$

Solution (6.14) can be obtained by many methods.

We have often observed the fifth error studying the application of the Exp - function method.

Other examples of the fifth error are given in our recent papers [42, 46].

## 7 Sixth error: some authors do not check solutions of differential equations

Using the computer methods to look for the exact solutions of nonlinear differential equation we cannot remove our knowledge, our understanding and our attention to the theory of differential equations. Neglecting to do this we can obtain different mistakes in finding the exact solutions of nonlinear differential equations.

These mistakes lead to numerous misprints in the final expressions of the solutions and sometimes to the fatal errors. A need of verifying the obtained solutions is obvious and it does not take much time with the help of the modern computer algebra systems. We have to substitute the found solution into the equation and to check that the solution satisfies it.

The sixth error is formulated as follows.

**Sixth error.** *Some authors do not check the obtained solutions of nonlinear differential equations.*

**Example 6a.** *"Solutions" of the Burgers equation by Soliman [51].*

Soliman [51] found three "solutions" of the Burgers equation (4.1)

$$u_1(\xi) = \frac{c}{\alpha} + \frac{2\nu}{\alpha} \tanh \{x - ct\}, \quad (7.1)$$

$$u_2(\xi) = \frac{c}{\alpha} + \frac{c^2}{2\alpha\nu} \coth \{x - ct\}, \quad (7.2)$$

$$u_3(\xi) = \frac{c}{\alpha} + \frac{2\nu}{\alpha} \tanh \{x - ct\} + \frac{c^2}{8\alpha\nu} \coth \{x - ct\}. \quad (7.3)$$

However all these solutions are incorrect and do not satisfy Eq.(4.1). The solution of Eq.(4.1) can be found by many methods and can be presented in the form

$$u_4(\xi) = \frac{c}{\alpha} - \frac{2\nu}{\alpha} \tanh \{x - ct - x_0\}. \quad (7.4)$$

where  $x_0$  is arbitrary constant.

**Example 6b.** *Solution of the foam drainage equation by Bekir and Cevikel [66]*

$$u_t + \frac{\partial}{\partial x} \left( u^2 - \frac{\sqrt{u}}{2} \frac{\partial u}{\partial x} \right) = 0. \quad (7.5)$$

Eq.(7.5) was studied in [66] by means of the tanh - coth method by Bekir and Cevikel [66]. The authors proposed the method to obtain "new travelling wave solutions".

Using the travelling wave

$$u(x, t) = u(\xi), \quad \xi = k(x + ct), \quad (7.6)$$

they obtained the first - order equation in the form

$$cku + k \left( u^2 - \frac{k}{2} \sqrt{u} u_\xi \right) = 0, \quad (7.7)$$

and obtained three solutions

$$\begin{aligned} u_1 &= k^2 \tanh^2 (k(x - k^2 t)), \\ u_2 &= k^2 \coth^2 (k(x - k^2 t)), \\ u_3 &= k^2 \left( \tanh (k(x - 4k^2 t)) + \coth (k(x - 4k^2 t)) \right)^2. \end{aligned} \quad (7.8)$$

The solution of Eq.(7.7) can be found using the transformation  $u = v^2$ . For function  $v(\xi)$  we obtain the Riccati equation in the form

$$c + v^2 = k v_\xi. \quad (7.9)$$

Solution of Eq.(7.9) can be written as

$$v(\xi) = -k \tanh(\xi - \xi_0), \quad \xi = kx + k^3 t. \quad (7.10)$$

We can see there are the misprints in solutions of Eq.(7.7)

**Example 6c.** "Solutions" of the Fisher equation by Öziş and Köroğlu [67]

$$u_t - u_{xx} - u(1 - u) = 0. \quad (7.11)$$

Using the Exp-function method Öziş and Köroğlu [67] found four "solutions" of Eq. (7.11). These "solutions" were given in the form

$$u_{(1)} = \frac{(1 - 2k^2)b_{-1}}{b_0 \exp(\eta) + b_{-1}}, \quad \eta = kx + wt, \quad (7.12)$$

$$u_{(2)} = \frac{(1 - 8k^2)b_{-1}}{b_1 \exp(2\eta) + b_{-1}}, \quad \eta = kx + wt, \quad (7.13)$$

$$u_{(3)} = \frac{b_0 - 2k^2 b_{-1} \exp(-\eta)}{b_0 + b_{-1} \exp(-\eta)}, \quad \eta = kx + wt. \quad (7.14)$$

$$u_{(4)} = \frac{b_1 e^\eta - 8k^2 b_{-1} e^{-\eta}}{b_1 e^\eta + b_{-1} e^{-\eta}}, \quad \eta = kx + wt. \quad (7.15)$$

However all these expressions do not satisfy equation (7.11). We can note this fact without substituting solutions (7.12) - (7.15) into Eq. (7.11), because a true solution of Eq. (7.11) has the pole of the second order, but all functions (7.12) - (7.15) are the first order poles and certainly by substituting expressions (7.12) - (7.15) into equation (7.11) we do not obtain zero.

**Example 6d.** "Solutions" of the modified Benjamin - Bona - Mahony equation (6.3) found by Yusufoglu [63].

The general solution of Eq.(6.4) can be found via the Jacobi elliptic function, but the author tried to obtain some solitary solutions by means of the Exp - function method using the travelling wave  $U(\xi)$ ,  $\xi = \beta(x - \gamma t)$ . Some of his

solutions are incorrect. For example the "solution" (formula (3.12) [63])

$$U(\xi) = \sqrt{\frac{3(c-1)}{\alpha}} \frac{\exp\left\{\frac{\beta(x-\gamma t)}{2}\right\} - \frac{b_0}{2b_1} \exp\left\{-\frac{\beta(x-\gamma t)}{2}\right\}}{\exp\left\{\frac{\beta(x-\gamma t)}{2}\right\} + \frac{b_0}{2b_1} \exp\left\{-\frac{\beta(x-\gamma t)}{2}\right\}} \quad (7.16)$$

do not satisfy Eq.(6.4).

The solution of Eq.(6.4) takes the form

$$U(\xi) = \pm \frac{6\beta}{\sqrt{-6\alpha}} \tanh(\xi - \xi_0), \quad \xi = \beta(x + \beta^2 t - t), \quad \gamma = 1 - \beta^2 \quad (7.17)$$

This solution can be obtained by using different methods.

**Example 6e.** "Solutions" of the Burgers - Huxley equation by Chun [68].

Chun [68] applied the Exp - function method to obtain the generalized solitary wave solutions of the Burgers — Huxley equation

$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u)(u-\gamma). \quad (7.18)$$

In the travelling wave the author looked for the solution of equation

$$\omega u_\eta + \alpha u u_\eta - u_{\eta\eta} = \beta u(1-u)(u-\gamma), \quad (7.19)$$

where  $\eta = kx + \omega t$ . Some of the solutions by Chun are incorrect. In particular, "solution" (25) in [68]

$$u(\eta) = \frac{\gamma b_{-2} e^{-2\eta}}{e^{2\eta} + b_{-2} e^{-2\eta}}, \quad \eta = \frac{\gamma}{4} x + \frac{\gamma(1-\gamma)}{2} t \quad (7.20)$$

do not satisfy Eq.(7.19). The solutions of Eq.(7.18) were found in [69].

**Example 6f.** "Solutions" of the Benjamin - Bona - Mahony - Burgers equation by El-Wakil, Abdou and Hendi [70].

El-Wakil and co-authors [70] using the Exp - function method looked for the solitary wave solutions of the BBMB equation

$$u_t - u_{xxt} - \alpha u_{xx} + u u_x + \beta u_x = 0. \quad (7.21)$$

Taking the travelling wave  $\eta = kx + ct$  the authors searched for the solution of the equation

$$c u_\eta - c k^2 u_{\eta\eta\eta} - \alpha k^2 u_{\eta\eta} + \beta k u_\eta + k u u_\eta = 0. \quad (7.22)$$

Some of solutions by El-Wakil and co-authors are incorrect. In particular,

expression (18) in [70]

$$u(\eta) = \frac{\left(-\frac{(6b_0k^2+k^2a_0-a_0)}{b_0(1+5k^2)}e^\eta + a_0 - \frac{1}{4}\frac{b_0(6b_0k^2+k^2a_0-a_0)}{1+5k^2}e^{-\eta}\right)}{\left(e^\eta + b_0 + \frac{b_0^2}{4}e^{-\eta}\right)}, \quad (7.23)$$

$$\eta = kx - \frac{k(b_0 + a_0)}{b_0(1 + 5k^2)}t$$

do not satisfy Eq.(7.22).

The solutions of Eq.(7.22) can be found taking the different methods into account. Note that integrating Eq.(7.22) with respect to  $\xi$  we have

$$u_{\eta\eta} + \frac{\alpha}{c}u_\eta - \frac{1}{2ck}u^2 - \frac{c + \beta k}{ck^2}u - C_2 = 0, \quad (7.24)$$

where  $C_2$  is an arbitrary constant. Eq.(7.24) coincides with the KdV - Burgers equation in the travelling wave. The solitary wave solutions of this equation are given above in section 1. The general solution of Eq.(7.24) can be found by analogue with the general solution of Eq.(2.7) [46].

**Example 6g.** "Solutions" of the Klein — Gordon equation with quadratic nonlinearity by Zhang [71].

The Exp - function method was used by Zhang to obtain the generalized solitary solutions of the Klein - Gordon equation with the quadratic nonlinearity

$$u_{tt} - u_{xx} + \beta u - \gamma u^2 = 0. \quad (7.25)$$

The author [71] applied the travelling wave  $u = U(\eta)$   $\eta = kx + wt$  and searched for the solution of the equation

$$(w^2 - k^2\alpha^2)U_{\eta\eta} + \beta U - \gamma U^2 = 0. \quad (7.26)$$

At least two solutions of Eq.(7.26) by Zhang [71] are incorrect and do not satisfy Eq.(7.26)

$$U_1 = \frac{\beta}{\gamma} - \frac{3b_0\beta}{\gamma\left(b_1e^{kx+wt} + b_0 + \frac{b_0^2}{4b_1}e^{-kx-wt}\right)}, \quad w = k\sqrt{\alpha^2 + \frac{\beta}{k^2}}, \quad (7.27)$$

$$U_2 = \frac{\beta}{\gamma} + \frac{a_1}{\left(e^{kx+wt} - \frac{a_1\gamma^2}{3\beta} + \frac{a_1^2\gamma^2}{36\beta^2}e^{-kx-wt}\right)}, \quad w = k\sqrt{\alpha^2 + \frac{\beta}{k^2}}. \quad (7.28)$$

The solitary wave solutions of Eq.(7.26) was obtained in many papers, because this equation coincides with KdV equation in the travelling wave [46]. These

solutions can be written as the following

$$U^{(1)} = \frac{3\beta}{2\gamma} \left(1 - \tanh^2(k\eta - k\eta_0)\right), \quad \eta = kx \pm \frac{1}{2}\sqrt{\frac{4\alpha^2 k^4 - \beta}{k^2}} t, \quad (7.29)$$

$$U^{(2)} = -\frac{\beta}{2\gamma} \left(1 - 3 \tanh^2(k\eta - k\eta_0)\right), \quad \eta = kx \pm \frac{1}{2}\sqrt{\frac{4\alpha^2 k^4 + \beta}{k^2}} t, \quad (7.30)$$

where  $\eta_0$  is an arbitrary constant.

**Example 6h.** "Solutions" of the Kuramoto - Sivashinsky equation by Noor, Mohyud - Din and Waheed [72].

Using the Exp - function method Noor, Mohyud-Din and Waheed [72] looked for the solutions of the Kuramoto—Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0. \quad (7.31)$$

These authors presented two expressions as the solutions of the Kuramoto - Sivashinsky equation. Their first "solution" is written as

$$u_1 = -\frac{\left(a_0 + a_{-1} \exp\left(-kx + \frac{(-a_0^2 + 2ka_{-1})kt}{a_{-1}}\right)\right) (a_0^2 - ka_{-1} + a_{-1}k^3)}{a_{-1}^2 \exp\left(-kx + \frac{(-a_0^2 + 2ka_{-1})kt}{a_{-1}}\right)}, \quad (7.32)$$

where  $a_0, a_{-1}, k$  are arbitrary constants (formula (26) in [72]).

The second "solution" by the authors takes the form

$$u_2 = \frac{(a_2 e^{2\eta} + a_0 + a_{-2} e^{-2\eta}) a_{-2}}{b_2 a_{-2} e^{2\eta} + b_{-2} a_0 + b_{-2} e^{-2\eta}}, \quad \eta = kx - \frac{k(a_{-2} + 8k^3 b_{-2} + 2kb_{-2})}{b_{-2}} t, \quad (7.33)$$

where  $a_0, a_{-2}, a_2, b_0, b_2, b_{-2}, k$  are arbitrary constants (formula (34) in [72]).

Here we have the sixth common error in the fatal form. Substituting cited "solutions" (7.32) and (7.33) into (7.31), we obtain that these "solutions" do not satisfy the Kuramoto - Sivashinsky equation. Moreover, we cannot obtain zero with any nontrivial values of the parameters. We guess, that using the Exp - function method the authors [72] did not solve the system of the algebraic equations for the parameters.

The exact solutions of Eq.(7.31) were first found in [73]. These two solutions



take the form

$$u^{(1)} = C_0 - \frac{45\sqrt{19}}{361} \tan\left(\frac{\sqrt{19}}{38}(x - C_0 t - x_0)\right) - \frac{15\sqrt{19}}{361} \tan^3\left(\frac{\sqrt{19}}{38}(x - C_0 t - x_0)\right), \quad (7.34)$$

$$u^{(2)} = C_0 - \frac{135\sqrt{209}}{361} \tanh\left(\frac{\sqrt{209}}{38}(x - C_0 t - x_0)\right) + \frac{165\sqrt{209}}{361} \tanh^3\left(\frac{\sqrt{209}}{38}(x - C_0 t - x_0)\right), \quad (7.35)$$

where  $C_0$  and  $x_0$  are arbitrary constants.

Many authors tried to find new exact solutions of the Kuramoto - Sivashinsky equation (7.31). Some of authors [36, 74, 75] believe that they found new solutions but it is not this case. Nobody cannot find new exact solutions of Eq.(7.31).

## 8 Seventh error: some authors include additional arbitrary constants into solutions

In finding the exact solutions of nonlinear differential equation our goal is to find the general solution. But it is not possible in many cases. In this situation we can try to look for the exact solutions with the larger amount of arbitrary constants. However we need to remember that the general solution of the equation of the  $n$  - th order can have only  $n$  arbitrary constants. Nevertheless we meet the papers, in which the authors present the solution of the Riccati equation with two or even three arbitrary constants. Moreover these authors say about the advantage of their approach in comparison with other methods taking into consideration the amount of arbitrary constants. Unfortunately in these cases there are two possible variants for this kind of solutions. The first variant is the author have obtained the solution with extra arbitrary constants and the amount of these constants can be decreased by means of the transformations. In the second variant the large amount of arbitrary constants in the solution points out that the author have found wrong solution.

So, we can formulate the seventh error as follows.

***Seventh error.*** *Some authors include additional arbitrary constants into solutions of nonlinear ordinary differential equations.*

**Example 7a.** *Solution of the Riccati equation*

$$Y_\xi = \beta(1 + Y^2). \quad (8.1)$$

Using

$$Y(\xi) = -\frac{\psi_\xi}{\beta\psi}, \quad (8.2)$$

we obtain the linear equation of the second order

$$\psi_{\xi\xi} + \beta^2\psi = 0. \quad (8.3)$$

The solution of equation (8.1) takes the form

$$Y(\xi) = i \frac{C_3 e^{-i\beta\xi} - C_4 e^{i\beta\xi}}{C_3 e^{-i\beta\xi} + C_4 e^{i\beta\xi}}. \quad (8.4)$$

At first glance we obtain more general solution (8.4) of the Riccati equation then (5.7), but in fact these solutions are the same. We can see, that one of the constants can be removed by dividing the nominator and the denominator in solution (8.4) on  $C_3$  (or on  $C_4$ ). Denoting  $C_5 = C_4/C_3$  (or  $C_5 = C_3/C_4$ ) we obtain the solution with the only arbitrary constant.

**Example 7b.** *Solution of the Sharma - Tasso - Olver equation.*

In example 2f we have obtained the solution of Eq.(3.36) in the form

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{\alpha}} \frac{C_2 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} - C_3 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}{C_1 + C_2 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} + C_3 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}. \quad (8.5)$$

However Eq.(3.36) has the second order, but solution (8.5) contains tree arbitrary constants. Sometimes it is convenient to leave three constants in solution (8.5), but we have to remember that solution (8.5) is not the general solution and one of these constants is extra. We can remove one of the constants as in example 7a and can write the general solution of Eq.(3.36) in the form

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{\alpha}} \frac{C_4 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} - C_5 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}{1 + C_4 \exp\left\{\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\} + C_5 \exp\left\{-\frac{\xi\sqrt{c}}{\mu\sqrt{\alpha}}\right\}}. \quad (8.6)$$

Solution (8.6) of Eq.(3.36) is not worse then solution (8.5), but this solution is the general solution by definition and all other solution can be found from this one.

**Example 7c.** *Solution (6.5) of the modified Benjamin - Bona - Mahony equa-*

tion (6.3) [63]

$$u(\xi) = \frac{\frac{24b_0b_1(c-1)}{a_0a}e^\xi + a_0}{\frac{24b_0b_1^2(c-1)}{a_0^2a}e^{2\xi} + b_1e^\xi + b_0 + \frac{a_0^2a}{24b_1(c-1)}e^{-\xi}}. \quad (8.7)$$

Solution (8.7) has three arbitrary constants  $a_0$ ,  $b_0$  and  $b_1$ , but in fact this solution can be simplified to the form with one arbitrary constant as it was demonstrated in example 5c.

**Example 7d.** "New exact solution" of the Riccati equation by Dai and Wang [76]

$$\varphi_\xi + l_0 + \varphi^2 = 0. \quad (8.8)$$

Dai and Wang [76] had been looking for the "new exact solutions" of the Riccati equation (8.8) and obtained five solutions. One of them (solution (17) [76]) takes the form

$$\varphi(\xi) = \frac{\sqrt{-l_0} b_1 \exp(\sqrt{-l_0}\xi + \xi_0) - \sqrt{-l_0} a_{-1} \exp(-\sqrt{-l_0}\xi - \xi_0)}{b_1 \exp(\sqrt{-l_0}\xi + \xi_0) + a_{-1} \exp(-\sqrt{-l_0}\xi - \xi_0)}. \quad (8.9)$$

We can see, that solution (8.9) has three arbitrary constants, but two of them certainly can be removed.

**Example 7e.** Solutions of the Burgers - Huxley equation by Chun [68].

Using the Exp - function method Chun [68] obtained the generalized solitary wave solutions of the Burgers — Huxley equation (see, Eq.(8.10) in example 6e). One of his solution (solution (32) in [68]) takes the form

$$u(\eta) = \frac{\gamma(\gamma - 1)(\gamma e^{2\eta} + a_1 e^\eta + a_0 + a_{-1} e^{-\eta})}{\gamma(\gamma - 1)(e^{2\eta} + b_1 e^\eta + a_{-1} e^{-\eta}) + \gamma(a_1 b_1 + a_0 - b_1^2) - a_0 - a_1^2 + a_1 b_1},$$

$$\eta = \frac{\gamma - 1}{2} x + \frac{\gamma^2 - 1}{4} t \quad (8.10)$$

The Burgers — Huxley equation is the second order one but solution (8.10) has three arbitrary constants. We suggested that this solution is incorrect. We checked this solution and have convinced that this solution does not satisfy Eq.(7.19).

Especially many solutions of equations with superfluous constants were obtained by means of the Exp - function method. Such examples can be found in many papers.

## 9 Conclusion

Let us shortly formulate the results of this paper. First of all we have tried to give some classification of the common errors and mistakes which occur in finding the solitary wave solutions of nonlinear differential equations. We have illustrated these errors using the examples from the recent publications in which the authors presented the exact solutions of nonlinear differential equations.

We have tried to classify the wide - spread errors in finding the exact solutions of nonlinear differential equations. Unfortunately we have no possibility to consider all errors and all problems of computer methods in finding the exact solutions. However we can note that the authors of many papers do not analyze the solutions of nonlinear differential equations that they obtain. Some errors are mentioned above due to this reason. It is important to discuss the solutions of nonlinear differential equations. We need to study how a solution depends on the parameters of nonlinear differential equation. It is useful to study the plots for the solutions of differential equation. We are interested in the stable exact solutions of nonlinear differential equations. These solutions are useful for the investigation of physical processes.

In our paper we have discussed the solitary wave solutions of nonlinear differential equations, but certainly we can find the same errors in looking for the other exact solutions.

We believe the results of this paper will be useful for young people who are going to study the exact solutions of nonlinear differential equations. We also hope that the material of this paper will be interesting for some referees.

## 10 Acknowledgements

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