A note on New kink-shaped solutions and periodic wave solutions for the (2+1)-dimensional Sine-Gordon equation

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Abstract

Exact solutions of the Nizhnik-Novikov-Veselov equation by Li [New kink-shaped solutions and periodic wave solutions for the (2+1)-dimensional Sine-Gordon equation, Appl. Math. Comput. 215 (2009). 3777-3781] are analyzed. We have observed that fourteen solutions by Li from thirty do not satisfy the equation. The other sixteen exact solutions by Li can be found from the general solutions of the well – known solution of the equation for the Weierstrass elliptic function.

In the paper [1], Li looked for the exact solutions of the nonlinear evolution equation

$$u_t + u_{xxx} + u_{yyy} + 3(u\partial_y^{-1}u_x)_x + 3(u\partial_x^{-1}u_y)_y = 0. \tag{1}$$

Author believes that Eq. (1) is the (2+1) dimensional Sine-Gordon equation. However, Eq. (1) was first derived in work [2] and now this equation is called as the Nizhnik-Novikov-Veselov equation [3, 4].

Author [1] have used the traveling wave transformation $u(x, y, t) = v(\xi)$ in Eq. (1), where $\xi = kx + ly + wt$. From Eq. (1) he obtained the nonlinear ordinary differential equations

$$kl(k^3 + l^3)v'' + 3(k^3 + l^3)v^2 + klwv = 0. \tag{2}$$

In fact, the general solution of Eq. (2) is well known. Let us present this solution. Multiplying Eq. (2) on v' and integrating with respect to ξ we obtain the equation in the form

$$(v')^2 = -\frac{2}{kl}v^3 - \frac{\omega}{k^3 + l^3}v^2 + c_1, \tag{3}$$

where c_1 is an arbitrary constant.

Substituting

$$v(\xi) = -2\,k\,l\,\wp(\xi) - \frac{k\,l\,w}{6\,(k^3 + l^3)}. \tag{4}$$

into Eq. (3) we have the equation for the Weierstrass elliptic function in the form

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{5}$$

where

$$g_2 = \frac{\omega^2}{12(k^3+l^3)}, \quad g_3 = \frac{\omega^3}{216(k^3+l^3)^3} - \frac{c_1}{4k^2l^2}. \tag{6}$$

So, the general solution of (2) is expressed via the Weierstrass elliptic function [5–7].

In the case $c_1 = 0$ and $\omega \neq 0$ from Eq. (3) we have the solitary wave solution of Eq. (2) [5, 6]

$$v(\xi) = -\frac{\omega kl}{2(k^3+l^3)} + \frac{\omega kl}{2(k^3+l^3)} \tanh^2\left(\sqrt{-\frac{\omega}{k^3+l^3}}(\xi-\xi_0)\right) \tag{7}$$

In the case $c_1 = 0$ and $\omega = 0$ from Eq. (3) we have the rational solution of Eq.(2) in the form

$$v(\xi) = -\frac{2kl}{(\xi+\xi_0)^2}. \tag{8}$$

Therefore, we do not need to look for the exact solutions of Eq. (2), because these solutions are well known.

Author [1] obtained thirty solutions of Eq. (2). However fourteen solutions $u_{3,4},\ u_{7,8},\ u_9,\ u_{11},\ u_{12,13},\ u_{21,22},\ u_{23,24},\ u_{25},\ u_{26}$ do not satisfy Eq. (2). All other his solutions can be obtained from solution (7). Let us illustrate that solutions $u_{1,2}, u_{5,6}, u_{10}, u_{14,15}, u_{16,17}, u_{18,19}, u_{20}, u_{27} - u_{29}$ can be obtained from solution (7).

Let us remind the following identities for the hyperbolic and the trigonometric functions

$$\tanh(i\,\xi) = i\tan(\xi), \tag{9}$$

$$\operatorname{sech}(\xi+\xi_0) = \frac{1}{\cosh(\xi+\xi_0)}, \tag{10}$$

$$1 - \tanh^2(\xi+\xi_0) = \frac{1}{\cosh^2(\xi+\xi_0)}, \tag{11}$$

$$\cosh(2(\xi+\xi_0)) = 2\cosh^2(\xi+\xi_0) - 1, \tag{12}$$

$$\cosh(i\xi) = \cos(\xi), \quad \sinh(i\xi) = i\sin(\xi), \tag{13}$$

$$\cos\left(\xi - \frac{\pi}{2}\right) = \sin(\xi),. \tag{14}$$

Using these identities we have that solution u_1 by Li is solution (7) at $\xi_0 = 0$. Solution u_2 coincides with solution u_1 because of $\tanh^2(\xi)$ is even function. Solution u_5 can be reduced to the solution (7) at $\xi_0 = 0$ taking into account (9). Solutions u_6 coincides with solutions u_5 because of $\tanh^2(\xi)$ is even function. Solutions u_{10} coincides with solution (7) at $\xi_0 = 0$ if we take into account relations (10),(11). Solutions $u_{14,15}$ can be obtained from (7) at $\xi_0 = 0$ taking into consideration (12). Solutions $u_{16,17}$ coincides with solution $u_{14,15}$ at $\xi_0 = 0$ if we use formulae (13). Solutions $u_{18,19}$ can be obtained from $u_{16,17}$ at $\xi_0 = 0$ taking into account (**??**). Solutions $u_{27},\ u_{28}$ coincides with solution (7) at $\xi_0 = \tanh^{-1}\frac{C_2}{C_1}$ if we take into account formulae (10),(11). Solutions u_{29} coincides with solution $u_{27},\ u_{28}$ taking into consideration (13). Solution u_{30} coincides with solution (8) at $\xi_0 = C$.

Thus we have that all thirty solutions by Li [1] of Eq. (2) are well known and author [1] has made a few common errors that were discussed in recent papers [6–16].

For finding the exact solutions of Eq. (2) Li in [1] has used three methods: the tanh–method, the $\frac{G'}{G}$–expansion method and the auxiliary function method. Likely the author believes that using different methods he can obtain many solutions of Eq. (2). However it is known that application of the $\frac{G'}{G}$ - method for finding exact solutions of nonlinear ordinary differential equations is equivalent to application of the tanh–method [10–17].

As to the application of the auxiliary function method we can consider this application for finding exact solutions of nonlinear differential equation by Li in [1] as confusion. Author [1] has used the equation in the form

$$(G')^2 = a\,G^2 + b\,G^4 + c\,G^6. \tag{15}$$

However , assuming in Eq.(15)

$$G = v^{1/2} \tag{16}$$

we obtain the equation

$$y_z^2 = 4\,a\,y^2 + 4\,b\,y^3 + 4\,c\,y^4 \tag{17}$$

The general solution of Eq.(17) is expressed via the Jacobi elliptic function. However author considered this equation at $c = 0$ and found twelve solutions of this equation. However we can see that Eq.(17) at $c = 0$ is exactly Eq.(3).

The general solution of Eq.(17) at $c = 0$ takes the form

$$y_z = \frac{a}{b}\coth^2\left(2\sqrt{a}(z+z_0)\right) - \frac{a}{b} \tag{18}$$

Assuming

$$a = -\frac{\omega}{4(k^3+l^3)}, \quad b = -\frac{1}{2kl} \tag{19}$$

we have solution (7) as well. So author [1] looked for the exact solutions of Eq.(2) taking into account exact solutions of Eq.(2) again.

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