A Note on the Lie Symmetry Analysis and Exact Solutions for the Extended mKdV Equation

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Abstract

We discuss the recent paper by Liu and Li [Liu H, Li J., Lie Symmetry Analysis and Exact Solutions for the Extended mKdV Equation, Acta Appl Math (2010) 109:1107 – 1119] and illustrate that so called the "extended mKdV equation" by authors can be reduced to the usual form of the modified Korteweg – de Vries equation. We also correct some other statements by authors with respect to exact solutions of solutions for the "extended mKdV equation".

1 Introduction

Nonlinear differential equations and their solutions play an important role in modern science and we can observe many publications in this area in the last years. However, in some cases supposedly successful approaches are inadequately applied in finding exact solutions of nonlinear ordinary differential equations and do not produce the expected useful results. Unfortunately, some authors claim that their solutions are "new", when the truth is that these solutions are merely "old" solutions in different guise. Recently, in enlightening papers [1, 2] we have warned researches and referees of the danger of not recognizing that apparently different solutions may simply be different forms of the same solution. We presented numerous examples to demonstrate this phenomenon.

In this short note we analyze the recent paper by Liu and Li [3] and show that authors studied in essence the modified Kprteweg – de Vries equation. Authors cannot find any new results for this equation.

2 Extended mKdV equation by Liu and Li.

In the recent paper [3] Liu and Li studied the nonlinear evolution equation in the form

$$u_t + a_1\, u_{xxx} + a_2\, u_x + a_3\, u\, u_x + a_4\, u^2\, u_x = 0 \tag{1}$$

Authors [3] called this equation "extended form of the mKdV equation". However it is well known that Eq.(1) was first derived in [4] and now is called as the Gardner equation [5]. Equation (1) was studied more than thirty years ago and we know all properties of this equation very well. No doubt this equation is integrable one by the inverse scattering transform, has infinitely many conservations laws and has many rational and solitary wave solutions.

All these properties can be determined taking the following transformations into account

$$t' = a_1\, t, \quad x' = x + \left(\frac{a_3^2}{4\, a_4 - a_2}\right)\, t, \quad u'(x', t') = \left(-\frac{a_4}{6\, a_1}\right)^{1/2} \left(u(x,t) + \frac{a_3}{2\, a_4}\right) \tag{2}$$

Using the transformations (2) in equation (1) we obtain the modified Korteweg – de Vries equation in the form (the primes are omitted)

$$u_t - 6\, u^2\, u_x + u_{xxx} = 0 \tag{3}$$

Equation (3) is the famous modified Korteweg – de Vries equation. We know the Lie symmetries, Lax pair, rational solutions, soliton solutions and other remarkable properties for this equation [6]. All properties of the Gardner equation (1) can be obtained from the well – known properties of the mKdV equation (3) taking transformation (2) into account.

It is well known that the similarity reduction of the mKdV equation (3) is the famous second Painlevé (P_2) equation. In fact, taking into account the self – similar solutions of equation (3) in the form

$$u(x,t) = \frac{1}{(3t)^{1/3}}\, f(\xi), \quad \xi = \frac{x}{(3t)^{1/3}} \tag{4}$$

we have the following nonlinear ordinary differential equation

$$f^{'''} = 6\, f^2\, f^{'} + \xi\, f^{'} + f \tag{5}$$

Integrating this equation with respect to ξ we obtain the P_2 equation

$$f^{''} = 2\, f^3 + \xi\, f + \alpha, \tag{6}$$

where α is a constant of integration.

Equation (6) was derived more then one hundred years ago by Painlevé and now is called as the second Painlevé equation (P_2). It is well known that solutions of equation (6) are Painlevé transcendents [7, 8].

Using the Lie symmetries for the Gardner equation (1) authors [3] obtained the nonlinear ordinary differential equation in the form

$$3\, a_1\, f^{'''} = -\frac{3}{4a_4}\, f^2\, f' + \xi\, f' + f. \tag{7}$$

However using transformations

$$\xi = (3\, a_1)^{1/3}\, \xi', \quad f = \sqrt{-8\, a_4\, (3\, a_1)^{1/3}}\, f' \tag{8}$$

we certainly have equation (6).

Authors of [3] claim: "In general, we can not obtain the exact and explicit solutions for the nonlinear ODEs such as (7) by using the elementary functions." However authors are wrong here because by definition the P_2 equation is integrated by the P_2 function [9]. There are also rational solutions of the second

Painlevé equation [10] in the case of integers $\alpha = n$. These solutions can be written in terms of special polynomials $Q_n(\xi)$ $(n > 1)$

$$f(\xi; n) = \frac{d}{dz}\left\{\ln\left[\frac{Q_{n-1}(\xi)}{Q_n(\xi)}\right]\right\}, \quad f(\xi; -n) = -f(z; n), \tag{9}$$

where $Q_n(\xi)$ are polynomials that were suggested by Yablonskii and Vorob'ev [11, 12] and now are called the Yablonskii–Vorob'ev polynomials [13].

At half–integers α the second Painlevé equation has solutions expressed via the Airy functions [10]. So, solutions of the Gardner equation (1) can be expressed via the Painlevé transcendents as well. In partial cases solutions of equation (1) can be expressed using rational function and the Airy functions.

Authors [3] presented solution of the P_2 equation in from of Taylor series in the neighborhood of $\xi = 0$. They show that this Taylor series is convergent as well. However this series is convergent under the Cauchy theorem because the point $\xi = 0$ is the regular point of the P_2 equation.

Using the travelling wave solutions $u(x,t) = \phi(\xi)$, where $\xi = x - c\,t$ from equation (3) we have after integrating the nonlinear ordinary differential equation in the form

$$\phi^{''} - 2\,\phi^3 - c\,\phi + c_1 = 0, \tag{10}$$

where c_1 is an arbitrary constant. Taking into account simple stretching and shift transformations one can see that equation (4) from [3] is equivalent to equation (10).

Multiplying equation (10) on $\phi^{'}$ and integrating again we obtain the equation

$$(\phi^{'})^2 - \phi^4 - c\,\phi^2 + 2\,c_1\,\phi + c_2 = 0, \tag{11}$$

where c_2 is an arbitrary constant.

The general solutions of equation (11) is expressed via the Jacobi elliptic functions [14]. Nevertheless authors [3] obtained several solutions ((8), (9), (12), (13), (16), (18), (20), (24), (26) in [3]) of equation (11). It is clear that all these solutions can be obtained from the general solution of equation (11). This fact was discussed in details in recent works [15]. So, all properties and all exact solutions of the Gardner equation are well known.

Also we should note that phase portraits of the differential equation system equivalent to equation (3) are very well studied at the present time. For example see classic books by Nayfeh [16].

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