A note on the G'/G - expansion method

Nikolai A. Kudryashov

Department of Applied Mathematics, National Research Nuclear University "MEPHI" , Kashirskoe Shosse, 115409 Moscow, Russian Federation

Abstract

We demonstrate that the G'/G - expansion method which is often used in finding exact solutions of nonlinear differential equation is equivalent to the well – known tanh - method and application of these methods gives the same exact solutions of nonlinear differential equations.

1 Introduction

There are many different methods used for looking exact solutions of nonlinear partial differential equations. We know the inverse scattering transform [1–3] to solve the Cauchy problem for integrable partial differential equations. The Hirota method [4] is well known to obtain the solitary wave solutions.

Among non – integrable nonlinear differential equations there is a wide class of the equations that are referred to as the partially integrable. These equations have a limited number of exact solutions. To find some exact solutions of these equations the investigators usually use one of ansatz methods.

One of the typical methods in finding exact solutions of nonlinear differential equations is the tanh - method. This method was applied many times and presented widely in periodic literature. Recently a new approach for solving differential equations called the G'/G - expansion method was introduced. It has been widely used the last two years.

The aim of this letter is to demonstrate that the G'/G - expansion method and the famous tanh - method are equivalent and consequently these methods give the same solutions.

This letter is organized as follows. In the section 2 and 3 we present the descriptions of the tanh - method and the G'/G - expansion method. In section 4 we show that these methods are identical.

2 Description of the tanh - method

The tanh - method for finding exact solutions of nonlinear differential equations was introduced more than twenty years ago and now it is difficult to remember who was the first to use this effective approach. We should make some old publications of the application of the tanh - method to look for exact solutions of nonlinear differential equations [5–8]. Description of the tanh - method can be found in papers [9, 10]. The essence of this approach is as follows.

Assume we have nonlinear partial differential equation

$$P(u, u_x, u_t, \ldots, x, t) = 0, \tag{1}$$

where P is polynomial on $u(x,t)$ and its derivatives. One can say that the method contains five steps.

Step 1. Reduction of Eq.(1) *to nonlinear ordinary differential equation.*

At this step using the travelling wave ansatz $u(x,t) = U(z)$, $z = x - C_0\,t$ Eq.(1) is reduced to the nonlinear ordinary differential equation (ODE)

$$E(U, U_z, U_{zz} \ldots, z) = 0. \tag{2}$$

Step 2. Hypothesis that the solution for Eq.(2) *can be found in the form of the finite series on hyperbolic tangents.*

At this step we suppose that exact solutions of Eq.(2) can be found in the form

$$U = \sum_{k=0}^{N} b_k\, \tanh^k(m\,(z - z_0)), \tag{3}$$

where N is integer, coefficients b_k and parameter m are unknown values that can be found after substitution (3) into (2).

Step 3. Finding positive integer N in the finite series of the solution with the hyperbolic tangents.

Substituting

$$U = b_N\, \tanh^N(m\,z), \tag{4}$$

into Eq.(2) and equating the maximal power of the hyperbolic tangent to zero we find the values N and b_N. This step corresponds to the first step of analysis of the nonlinear differential equation on the Painlevé property. Some authors say that this step is the application of the homogenous method because integer N can be found taking the balance into account between the highest order derivatives and nonlinear terms in (2).

Step 4. Determination of coefficients b_k and parameter m.

Substituting expression (3) into Eq.(2) and equating expressions of different power of $\tanh^k\,(m\,(z - z_0))$ to zero we are looking for coefficients b_k and parameter m.

Step 5. Presentation of solutions for Eq.(2)*.*

Substituting b_k and m into formula (2) we obtain the exact solutions of nonlinear differential equation (2) as a result of the application of the tanh - method.

One can see that application of the tanh - method is a simple procedure for finding exact solutions of nonlinear differential equations. However many investigators want to have "new method" for finding exact solutions to find "new solutions" of nonlinear differential equations.

3 Description of the G'/G - expansion method

The G'/G - method for finding exact solutions of nonlinear differential equations was introduced in paper [11]. Currently this method is often used for searching exact solutions of nonlinear differential equations (see, for example, papers [12–23]).

The essence of this approach can be formulated as follows. The first step of this method is equivalent to the tanh - method.

Step 1. Reduction of Eq.(1) *to nonlinear ordinary differential equation* (2).

Taking the travelling wave ansatz into account again we have the nonlinear ordinary differential equation as a result.

Step 2. Hypothesis that the solution of Eq.(2) *can be searched for in the form of the finite series of the logarithmic derivative on the function* $G(z)$.

At this step we assume that the exact solutions of Eq.(2) can be found in the form of the finite series on the logarithmic derivatives

$$U=\sum_{k=0}^{N} a_k\,\left(\frac{G^{'}(z)}{G(z)}\right)^k, \quad G^{'}=\frac{dG}{dz} \tag{5}$$

where coefficients b_k can be found and $G(z)$ is a solution of the linear second - order differential equation

$$G^{''}-\lambda\,G^{'}-\mu\,G=0, \tag{6}$$

where λ and μ are constants determined as well.

Step 3. Finding positive integer N *in the finite series* (5) *on the logarithmic derivative of the function* $G(z)$.

Substituting $U(z)=a_N\,z^N$ into (2) and equating maximal power with respect to z we find values of N . Value N is determined by the value of the pole of the general solution of Eq.(2). Note that the value of integer N in the $(G^{'}/G)$ - expansion method corresponds to the value N in the tanh - method.

Step 4. Determination of coefficients a_k *and parameters* λ *and* μ.

Substituting expression (5) into Eq.(2) and equating expressions of different power of $G^{'}/G$ to zero, we obtain coefficients a_k in (5) and parameters λ and μ. Making the calculations we have to take Eq.(6) into account.

Step 5. Presentation of solutions of Eq.(2) *by means of formula* (5).

Substituting coefficients a_k into formula Eq.(5) we have the exact solution of Eq.(5). We can see that the $G^{'}/G$ - expansion method is similar to the tanh - method. In the next section we will demonstrate that these approaches are identical.

4 Equivalence of the $G^{'}/G$ - expansion method and the tanh - method

Assuming

$$Y(z)=\frac{G^{'}(z)}{G(z)}, \tag{7}$$

in (5) we obtain the expression for $U(z)$ in the form

$$U=\sum_{k=0}^{N} a_k\,Y(z)^k, \tag{8}$$

We also have formulae

$$G^{'}=Y(z)\,G(z), \quad G^{''}=\left(Y_z+Y^2\right)\,G. \tag{9}$$

Taking these formulae into account we obtain that Eq.(6) can be reduced to the Riccati equation [24]

$$Y_z = -Y^2 + \lambda\, Y + \mu. \tag{10}$$

Assuming

$$Y(z) = \tilde{Y}(z) + \frac{\lambda}{2}, \tag{11}$$

we can write Eq.(10) as

$$\tilde{Y}_z = -\tilde{Y}^2 + \alpha, \quad \alpha = \frac{\lambda^2}{4} + \mu. \tag{12}$$

The general solution of Eq.(12) takes the form

$$\tilde{Y}(z) = \sqrt{\alpha}\, \tanh\left(\sqrt{\alpha}(z - z_0)\right). \tag{13}$$

The general equation of Eq.(10) takes the form

$$Y(z) = \sqrt{\mu + \frac{\lambda^2}{4}}\, \tanh\left(\sqrt{\mu + \frac{\lambda^2}{4}}\, (z - z_0)\right) + \frac{\lambda}{2}. \tag{14}$$

Substituting solution (14) into expansion (8) we have

$$U = \sum_{k=0}^{N} a_k \left(\sqrt{\mu + \frac{\lambda^2}{4}}\, \tanh\left(\sqrt{\mu + \frac{\lambda^2}{4}}\, (z - z_0)\right) + \frac{\lambda}{2}\right)^k. \tag{15}$$

Using solution (15) we can write the following equalities

$$U = \sum_{k=0}^{N} a_k \left(\sqrt{\mu + \frac{\lambda^2}{4}}\, \tanh\left(\sqrt{\mu + \frac{\lambda^2}{4}}\, (z - z_0)\right) + \frac{\lambda}{2}\right)^k =$$
$$= \sum_{k=0}^{N} b_k\, \tanh^k\left(\sqrt{\mu + \frac{\lambda^2}{4}}\, (z - z_0)\right) = \sum_{k=0}^{N} b_k\, \tanh^k(m\,(z - z_0)). \tag{16}$$

From Eq.(16) we can find coefficients b_k and parameter m. Thus, we have obtained exact expression for formula that is used in finding exact solutions of nonlinear differential equations by means of the tanh - method.

Assuming $N = 1$ in Eq.(3) we have

$$b_0 = a_0 + \frac{a_1\, \lambda}{2}, \quad b_1 = a_1 \sqrt{\mu + \frac{\lambda^2}{4}}, \quad m = \sqrt{\mu + \frac{\lambda^2}{4}}. \tag{17}$$

In the case $N = 2$ we obtain

$$b_0 = a_0 + \frac{a_1\, \lambda}{2} + \frac{a_2\, \lambda^2}{4}, \quad b_1 = a_1 \sqrt{\mu + \frac{\lambda^2}{4}} + a_2 \lambda \sqrt{\mu + \frac{\lambda^2}{4}},$$
$$b_2 = a_2 \left(\mu + \frac{\lambda^2}{4}\right), \quad m = \sqrt{\mu + \frac{\lambda^2}{4}}. \tag{18}$$

We can find the values of coefficients b_k for other N as well. Considering Eqs.(17) and (18) we can see that coefficients b_k and parameter m in the tanh - method are determined unambiguously at given coefficients a_k and parameters λ and μ in the $G^{'}/G$ - expansion method and vice versa. Parameter λ in the $G^{'}/G$ - expansion method can be considered as zero without loss of the generality.

Before proceeding to an application of the $G^{'}/G$ - expansion method to look for exact solutions of nonlinear differential equations we have to remember it will be better to use the tanh - method because we cannot obtain any new solutions by means of the $G^{'}/G$ - method. In fact, there is only one case when the similar application has sense. This case was discussed in [25]. However using the $G^{'}/G$ - expansion method, when $G(z)$ satisfies the second order linear equation we make one of the errors that were discussed in recent critical papers [26–30].

References

[1] Gardner C.S., Green J.M., Kruskal M.D. , and Miura R. M., Method for solving the Korteweg-de Vries equations, Phys. Rev. Lett. 19 (1967) 1095 - 1097

[2] Lax P.D., Integrals of nonlinear equations of evolution and solitary waves, Communications on pure and applied mathematics, 21 (1968) 467 - 490

[3] Ablowitz M.J., Kaup D. J., Newell A. C., Segur H., The Inverse Scattering Transform - Fourier Analysis for Nonlinear Problems, Stud. Appl. Math. 53 (1974) 249 - 315

[4] Hirota R., Exact solution of the Korteweg-de Vries for multiple collinsions of solutions, Phys. Rev. Lett. 27 (1971) 1192 - 1194

[5] Kudryashov N.A. Exact soliton solutions of the generalized evolution equation of wave dynamics. Journal of Applied Mathematics and Mechanics, 52(3) (1988) 360 – 365

[6] Lan H., Wang K. Exact solutions for some nonlinear equations. Phys. Lett. A 137 (1989) 369 – 373

[7] Kudryashov, N.A., Exact solutions of the generalized Kuramoto— Sivashinsky equation, Phys. Lett. A 147 (1990) 287–291.

[8] Lou S.Y.,Huang G.X., Ruan H.Y. Exact solitary waves in a convecting fluid, J. Phys. A: Math. Gen. 24(11) (1991) L587– L590

[9] Malfliet W., Hereman W. The Tanh method: I Exact solutions of nonlinear evolution and wave equations. Phys Scripta 54 (1996) 563 – 568

[10] Parkes E.J., Duffy B.R. An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations. Comput Phys Commun 98 (1996) 288 – 300

[11] Wang M.L., Li X., Zhang J., The G'/G - expansion method and evolution erquations in mathematical physics. Phys. Lett. A, 372 (2008) 417 – 421

[12] Bekir A., Application of the G'/G - expansion method for nonlinear evolution equations. Phys. Lett. A 372 (2008) 3400 – 3406

[13] Zhang J., Wei X., Lu Y. A generalized G'/G - expansion method and its applicatioons, Phys. Lett. A 372 (2008) 3653 – 3658

[14] Li L.-X, Wang M.-L, The (G'/G) - expansion method and travelling wave solution for a higher - order nonlinear schrodinger equation, Appl Math Computation, 208 (2009) 440 – 445

[15] Aslan I., Osiz T, Analytical study on two nonlinear evolution equations by using $(G'/G\ )$ - expansion method, Appl Math Computation, 209 (2009) 425 – 429

[16] Zayed E.M.E., The G'/G - method and its application to some nonlinear evolution equations, J Appl Math Comput 30 (2009) 89 - 103

[17] Zayed E.M.E., New travelling wave solutions for higher dimensional nonlinear evolution equations using a generalized G'/G - expansion method, J Phys A; Math. Theor. 42 (2009) 0195202

[18] Berir A., Cevikel A.C., New exact travelling wave solutions of nonlinear physical models, Chaos, Solitons and Fractals, 41 (2009) 1733 – 1739

[19] Ganji D.D., Abdollahzadeh M., Exact travelling solutions of some nonlinear evolution equation by the (G'/G) - expansion method, Journal of Mathematical Physics, 50 (2009) 013519

[20] Gao H., Zhao R.X., New application of the $(G'/G\ )$ - expansion method to high-order nonlinear equations, Appl Math Computation, 215 (2009) 2781 – 2786

[21] Zhang H., New application of the $(G'/G\ )$ - expansion method, Commun Nonlinear Sci Numer Simulat, 14 (2009) 3220 – 3225

[22] Zhang S., Wang W., Tong J.-L., A generalized $(G'/G\ )$ - expansion method and it's application to the (2+1) - dimensional Broer-Kaup equations, Appl Math Computation, 209 (2009) 399 – 404

[23] Zhang S., Dong L., Ba J.-M., Siu Y.-N., The $(G'/G\ )$ - expansion method for nonlinear differential - difference equations, Physics Letters A, 373 (2009) 905 – 910

[24] Kudryashov N.A., Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos, Solitons and Fractals, 24 (2005) 1217 – 1231

[25] Kudryashov N.A., Loguinova N.B., Extended simplest equation method for nonlinear differential equations, Appl Math Comput 205 (2008) 396 – 402

[26] Kudryashov N.A., Loguinova N.B., Be careful with Exp – function method, Commun Nonlinear Sci Numer Simulat 14 (2009) 1881 – 1890

[27] Kudryashov N.A., On "new traveling wave solutions" of the KdV and the KdV—Burgers equations, Nonlinear Sci. Numer. Simulat. 14 (2009) 1891 – 1900

[28] Kudryashov N.A., Seven common errors in finding exact solutions of nonlinear differential equations, Commun Nonlinear Sci Numer Simulat. 14 (2009) 3507 – 3529

[29] Kudryashov N.A., Meromorphic solutions of nonlinear differential equations, Commun Nonlinear Sci Numer Simulat, (2009) doi: 10:10.1016/j.cnsns.2009.11.013

[30] Parkes E.J., A note on travelling - wave solutions to Lax's seventh - order KdV equation, Appl Math Comput, 215 (2009), 864 - 865