Redundant exact solutions of nonlinear differential equations

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Abstract

We analyze the paper by Wazwaz and Mehanna [Wazwaz A.M., Mehanna M.S., A variety of exact travelling wave solutions for the (2+1) – dimensional Boiti – Leon – Pempinelli equation, Appl. Math. Comp. 217 (2010) 1484 – 1490]. The authors claim that they have found exact solutions of the (2+1) – dimensional Boiti – Leon – Pempinelli equation using the tanh – coth method and the Exp – function method. We demonstrate that two of their solutions are incorrect. All the others can be simplified and they are the cases of the well-known solution. Wazwaz and Mehanna made a number of typical mistakes in finding exact solutions of nonlinear differential equations. Taking the results of this paper we introduce the definition of redundant exact solutions for the nonlinear ordinary differential equations.

1 Introduction

Construction of exact solutions for nonlinear differential equations is an important part of nonlinear science and we can see significant progress in this area in the last years [1–6]. Many of these achievements were reached using symbolic calculations by means of such software as MAPLE and MATHEMATICA. However there are some shortcomings of this approach. Of course, computers can help investigators to do calculating routine but they cannot completely replace the investigators since computers do not know mathematics. Total reliance on computers without knowledge of mathematics can lead to various errors in finding exact solutions of nonlinear differential equations.

We have seen many papers in different journals with such examples but for this note we selected one of them. Our aim is to demonstrate the mistakes of [7] which were made in finding exact solutions for the system of nonlinear differential equations.

In the paper [7] Wazwaz and Mehanna considered the system of equations

$$u_{ty} = (u^2 - u_x)_{xy} + 2\,v_{xxx}, \tag{1}$$

$$v_t = v_{xx} + 2\,u\,v_x. \tag{2}$$

To look for exact solutions of this system the authors used the traveling wave solutions $u(x,y,t) = u(\xi)$, $v(x,y,t) = v(\xi)$, $\xi = \mu(x + y - c\,t)$. Wazwaz and

Mehanna have looked for solutions of the following system of nonlinear ordinary differential equations

$$-c\,u^{''} = (u^2)^{''} - \mu\,u^{'''} + 2\mu\,v^{'''}, \tag{3}$$

$$-c\,v^{'} = \mu\,v^{''} + 2\,u\,v^{'}, \tag{4}$$

where

$$u^{'} = \frac{du}{d\xi}, \quad v^{'} = \frac{d\,v}{d\xi}$$

and so on.

The authors wrote in [7]: "integrating the first equation twice with respect to ξ gives"

$$v^{'} = \frac{1}{2}\,u^{'} - \frac{u^2 + c\,u}{2\,\mu}. \tag{5}$$

In fact, after integration Eq.(3) twice with respect to ξ we obtain

$$v^{'} = \frac{1}{2}\,u^{'} - \frac{u^2 + c\,u}{2\,\mu} + A_1\,\xi + A_2, \tag{6}$$

where A_1 and A_2 are arbitrary constants. Wazwaz and Mehanna omitted these constants of integration and reduced a class of exact solutions for Eqs.(3)–(4). Thus we can see that the authors made the third error from the list of errors by Kudryashov [8] while finding exact solutions of nonlinear differential equations.

Substituting (5) into (4) Wazwaz and Mehanna obtained the equation in the form

$$\mu^2\,u^{''} - 2\,u^3 - 3\,c\,u^2 - c^2\,u = 0. \tag{7}$$

They investigated this second-order ordinary differential equation using the tanh-coth function and the Exp-function methods for finding exact solutions.

The outline of this note is the following: in Section 2 we give the general solution of (7). In Section 3 we analyze the application of the tanh - coth method for finding exact solutions of Eq.(7) and show that all the solutions presented by Wazwaz and Mehanna can be reduced to a single one. In Section 4 we consider the application of the Exp - function method to Eq.(7) and illustrate that two of the found solutions are incorrect and all the others can be simplified. In Section 5 we introduce a definition of the redundant exact solutions for a nonlinear ordinary differential equation and discuss some examples.

2 General solution of Eq.(7)

Let us show that the general solution of Eq. (7) can be expressed via the Jacobi elliptic function and consequently all the other exact solutions can be found from this general solution.

Multiplying Eq.(7) on $u^{'}$ and integrating Eq. (7) once with respect to ξ we have

$$\mu^2\,(u^{'})^2 = u^4 + 2\,c\,u^3 + c^2\,u^2 - \alpha, \tag{8}$$

where α is an integration constant.

Eq.(8) has the following general solution [5]

$$u = \frac{\sqrt{c^2 - 4\sqrt{\alpha}}}{2}\,\mathrm{sn}\left\{\frac{1}{2\,\mu}\sqrt{c^2 + 4\sqrt{\alpha}}(\xi - \xi_0), \sqrt{\frac{c^2 - 4\sqrt{\alpha}}{c^2 + 4\sqrt{\alpha}}}\right\} - \frac{c}{2}, \qquad (9)$$

where ξ_0 is an arbitrary constant.

In the case $\alpha = 0$ from solution (9) we have

$$u = -\frac{c}{1 + C_1\,\mathrm{e}^{\pm\frac{c}{\mu}\eta}}, \qquad (10)$$

where C_1 is an arbitrary constant.

Note that solution (10) can be presented in the form

$$u = \frac{c}{2}\left(\pm\tanh\left\{\frac{c}{2\,\mu}(\xi - \xi_0)\right\} - 1\right). \qquad (11)$$

In the case $\alpha = \frac{c^4}{16}$ solution of Eq. (7) has the following form

$$u = -\frac{2\,c^2\,C_2\,e^{\mp\frac{\sqrt{-c^2}}{\sqrt{2}\mu}\eta}}{C_2^2\,e^{\mp\frac{2\sqrt{-c^2}}{\sqrt{2}\mu}\eta} + 2\,c^2} - \frac{c}{2}. \qquad (12)$$

All the exact solutions of Eq.(7) can be obtained from solution (9). However Wazwaz and Mehanna have looked for the exact solutions of Eq.(7) using the tanh – coth and the Exp – function methods.

3 Application of the tanh-coth method by Wazwaz and Mehanna to Eq.(7)

Let us show explicitly that all solutions of Eq. (7) found by Wazwaz and Mehanna by means of the tanh – coth method can be reduced to solution (11) and the authors of Ref. [7] have made the second error from the Kudryashov list of typical errors [8].

At the beginning Wazwaz and Mehanna applied the tanh – coth method to obtain solutions of Eq.(7). They have found the following three solutions

$$u_1(\xi) = -\frac{c}{2} \pm \frac{c}{2}\,\tanh\xi, \qquad (13)$$

$$u_2(\xi) = -\frac{c}{2} \pm \frac{c}{2}\,\coth\xi, \qquad (14)$$

$$u_3(\xi) = -\frac{c}{2} \pm \frac{c}{4}\,\tanh\xi \pm \frac{c}{4}\,\coth\xi. \qquad (15)$$

Let us obtain solutions (13)-(15) from solution (11). In the case of $\mu = \pm\frac{c}{2}$ and $\xi_0 = 0$ from (11) we have solution (13).

Let us note that following relations take place

$$\tanh\left(\xi - \frac{i\,\pi}{2}\right) = \coth\xi, \qquad (16)$$

$$\tanh\left(\xi - \frac{i\,\pi}{2}\right) = \frac{1}{2}\left(\tanh\frac{\xi}{2} + \coth\frac{\xi}{2}\right). \qquad (17)$$

Solutions (14) and (15) can be easily found using equalities (16) and (17) from solution (11) with $\xi_0 = \frac{i\pi}{2}$ and $\mu = \pm\frac{c}{2}$ and $\mu = \pm\frac{c}{4}$ respectively.

Thus solutions (13)-(15) are partial cases of solution (11) of Eq.(7). Therefore the authors of Ref. [7] also made the fourth mistake from the Kudryashov list of errors in finding exact solutions of nonlinear differential equations.

4 Application of the Exp – function method by Wazwaz and Mehanna to Eq.(7)

Using the Exp – function method Wazwaz and Mehanna also found 16 exact solutions of Eq.(7). These exact solutions will be given later after the first sign of equality. Afterwards we will present our transformations of these solutions to (10) and (12) to simple forms.

Exact solutions u_1 and u_3 do not satisfy Eq.(7). Actually, they are wrong. Let us demonstrate that all the other solutions coincide with solutions (10) and (12). It is easy to see that the solutions u_2, u_4, u_5, u_8, u_{11} and u_{13} have 2 arbitrary constants, but these solutions are not reduced to the general solution. The solutions u_1, u_3, u_6, u_9, u_{12}, u_{14} and u_{15} contain 3 arbitrary constants. This situation is not possible for a second-order ordinary differential equation. The solutions u_7, u_{10} and u_{16} contain 4 arbitrary constants. The authors of Ref. [7] should ask each other: how can it be possible?

We can easily see that Eq.(7) is the equation of the second order. As a consequence we can obtain only two arbitrary constants for the general solution. For a special case we can have less than two arbitrary constants. It is amusing that the authors of Ref. [7] do not know this fact. So they made the seventh error from the list of errors [8] as well.

Now let us illustrate that the solutions by Wazwaz and Mehanna can be simplified to two solutions (10) and (12).

The solution u_1 is wrong but if we take $\mathrm{e}^{-\eta}$ in place of e^{η} in the last expression of denominator we can transform the solution u_1

$$u_1(\eta) = \frac{a_0 - cb_{-1}\,\mathrm{e}^{-\eta}}{-\frac{a_0\,(a_0+b_0\,c)\mathrm{e}^{\eta}}{c^2 b_{-1}} + b_0 + b_{-1}\,\mathrm{e}^{-\eta}} =$$

$$= -\frac{c\,(a_0 - b_{-1}\,c\,\mathrm{e}^{-\eta})}{(1+\frac{b_0\,c+a_0}{b_{-1}\,c}\,\mathrm{e}^{\eta})(a_0 - b_{-1}\,c\,\mathrm{e}^{-\eta})} = -\frac{c}{1+\frac{b_0\,c+a_0}{b_{-1}\,c}\,\mathrm{e}^{\eta}} = -\frac{c}{1+C_1\,\mathrm{e}^{\eta}}, \qquad (18)$$

where $C_1 = \frac{b_0\,c+a_0}{b_{-1}\,c}$ is an arbitrary constant. So, we can see that the solution u_1 is essentially simplified. Moreover we note that this solution is the partial case of solution (10) at $\mu = \pm c$.

The solution u_2 can be simplified and is the partial case of solution (10) at $\mu = \mp\frac{c}{2}$. In this case we have

$$u_2(\eta) = -\frac{b_1\,c\mathrm{e}^{\eta}}{b_1\,\mathrm{e}^{\eta} + b_{-1}\,\mathrm{e}^{-\eta}} = -\frac{c}{1 + C_1\,e^{-2\,\eta}}, \qquad (19)$$

where $C_1 = \frac{b_{-1}}{b_1}$ is an arbitrary constant.

The solution u_3 by Wazwaz and Mehanna do not satisfy (7) but if we substitute b_{-1} for b_1 we will have

$$u_3(\eta) = \frac{-b_1\, c\mathrm{e}^{\eta} + a_0}{b_1\, \mathrm{e}^{\eta} + b_0 - \frac{a_0\,(a_0+b_0\, c)\mathrm{e}^{-\eta}}{c^2 b_1}} =$$

$$= -\frac{c(a_0 - b_1\, c\, \mathrm{e}^{\eta})}{(1 + \frac{a_0+b_0}{b_1\, c}\, \mathrm{e}^{-\eta})(a_0 - b_1\, c\, \mathrm{e}^{\eta})} = -\frac{c}{1 + \frac{a_0+b_0}{b_1\, c}\, \mathrm{e}^{-\eta}} = -\frac{c}{1 + C_1\, e^{-\,\eta}}, \tag{20}$$

where $C_1 = \frac{a_0+b_0}{b_1\, c}$ is an arbitrary constant. In this case we have that u_3 is the partial case of solution (10) at $\mu = \mp c$ again.

The solution u_4 can be simplified by transformations

$$u_4(\eta) = \frac{-\frac{{a_0}^2\mathrm{e}^{\eta}}{4b_{-1}\, c} + a_0 - \frac{cb_{-1}}{2}\, \mathrm{e}^{-\eta}}{\frac{a_0^2\mathrm{e}^{\eta}}{2c^2 b_{-1}} + b_{-1}\, \mathrm{e}^{-\eta}} = -\frac{2\, c^2\, C_2\, \mathrm{e}^{\eta}}{2\, c^2 + C_2^2\, \mathrm{e}^{2\,\eta}} - \frac{c}{2}, \tag{21}$$

where $C_2 = \frac{a_0}{b_{-1}}$ is an arbitrary constant. We see that (21) is the partial case of solution (12).

The solution u_5 is the partial case of (10) at $\mu = \mp\frac{c}{3}$. It follows from equalities

$$u_5(\eta) = -\frac{b_2\, c\mathrm{e}^{2\,\eta}}{b_2\, \mathrm{e}^{2\,\eta} + b_{-1}\, \mathrm{e}^{-\eta}} = -\frac{c}{1 + C_1\, e^{-3\,\eta}}, \tag{22}$$

where $C_1 = \frac{b_{-1}}{b_2}$ is an arbitrary constant.

We observe that u_6 is the partial case of solution (10) at $\mu = \mp\frac{c}{2}$ if we take into consideration the following transformations

$$u_6(\eta) = \frac{-b_2\, c\mathrm{e}^{2\,\eta} - \frac{cb_2\, b_{-1}\, \mathrm{e}^{\eta}}{b_0}}{b_2\, \mathrm{e}^{2\,\eta} + \frac{b_2\, b_{-1}\, \mathrm{e}^{\eta}}{b_0} + b_0 + b_{-1}\, \mathrm{e}^{-\eta}} =$$

$$= -\frac{\frac{c\, b_2}{b_0}\, \mathrm{e}^{2\,\eta}(b_0 + b_{-1}\mathrm{e}^{-\eta})}{(b_0 + b_{-1}\mathrm{e}^{-\eta})(1 + \frac{b_2}{b_0}\, \mathrm{e}^{2\,\eta})} = -\frac{\frac{c\, b_2}{b_0}\, \mathrm{e}^{2\,\eta}}{1 + \frac{b_2}{b_0}\, \mathrm{e}^{2\,\eta}} = -\frac{c}{1 + \frac{b_0}{b_2}\, e^{-2\,\eta}} = \tag{23}$$

$$= -\frac{c}{1 + C_1\, \mathrm{e}^{-2\,\eta}},$$

where $C_1 = \frac{b_0}{b_2}$ is an arbitrary constant.

We obtain that u_7 is the partial case of solution (10) at $\mu = \mp c$ using the following transformations

$$u_7(\eta) = \frac{-b_2\, c\mathrm{e}^{2\,\eta} + a_1\,\mathrm{e}^{\eta} + a_0}{b_2\,\mathrm{e}^{2\,\eta} + b_1\,\mathrm{e}^{\eta} - \frac{a_1\,b_1\,c + a_0\,b_2\,c + a_1{}^2}{b_2\,c^2} - \frac{a_0\,(a_1 + b_1\,c)\mathrm{e}^{-\eta}}{b_2\,c^2}} =$$

$$= \frac{-b_2\,c^2\,(b_2\,c\,e^{\eta} - a_1 - a_0\,e^{-\eta})\,e^{\eta}}{(b_1\,c + a_1 + b_2\,c\,e^{\eta})\,(b_2\,c\,e^{\eta} - a_1 - \ a_0\,e^{-\eta})} = \tag{24}$$

$$= -\frac{c}{1 + \frac{b_1\,c + a_1}{b_2\,c}\,e^{-\eta}} = -\frac{c}{1 + C_1\,e^{-\eta}},$$

where $C_1 = \frac{b_1\,c + a_1}{b_2\,c}$ is an arbitrary constant.

We can see that the solution u_8 is the partial case of (10) at $\mu = \pm\frac{c}{3}$ if we reduce u_8 to the form

$$u_8(\eta) = -\frac{cb_{-1}\,\mathrm{e}^{-\eta}}{b_2\,\mathrm{e}^{2\,\eta} + b_{-1}\,\mathrm{e}^{-\eta}} = -\frac{c}{1 + \frac{b_2}{b_{-1}}\,\mathrm{e}^{3\,\eta}} = -\frac{c}{1 + C_1\,\mathrm{e}^{3\,\eta}}, \tag{25}$$

where $C_1 = \frac{b_2}{b_{-1}}$ is an arbitrary constant.

The solution u_9 can be simplified as well. As a result we can see that u_9 is the partial case of solution (10) at $\mu = \pm\frac{c}{2}$ by using transformations

$$u_9(\eta) = \frac{-b_0\,c - cb_{-1}\,\mathrm{e}^{-\eta}}{b_2\,\mathrm{e}^{2\,\eta} + \frac{b_2\,b_{-1}\,\mathrm{e}^{\eta}}{b_0} + b_0 + b_{-1}\,\mathrm{e}^{-\eta}} = -\frac{c(b_0 + b_{-1}\,\mathrm{e}^{-\eta})}{\left(1 + \frac{b_2}{b_0}\,\mathrm{e}^{2\eta}\right)(b_0 + b_{-1}\,\mathrm{e}^{-\eta})} =$$

$$= -\frac{c}{1 + \frac{b_2}{b_0}\,\mathrm{e}^{2\eta}} = -\frac{c}{1 + C_1\,\mathrm{e}^{2\,\eta}}, \tag{26}$$

where $C_1 = \frac{b_2}{b_0}$ is an arbitrary constant.

The solution u_{10} can be simplified by performing the transformations

$$u_{10}(\eta) = \frac{a_1\,\mathrm{e}^{\eta} - \frac{c(cb_2\,b_{-1} + b_0\,a_1)}{a_1} - cb_{-1}\,\mathrm{e}^{-\eta}}{b_2\,\mathrm{e}^{2\,\eta} - \frac{(c^3 b_2{}^2 b_{-1} + a_1\,b_0\,b_2\,c^2 + a_1{}^3)\mathrm{e}^{\eta}}{ca_1{}^2} + b_0 + b_{-1}\,\mathrm{e}^{-\eta}} =$$

$$= \frac{c\left(a_1^3\,e^{\eta} - c\,a_1\,(c\,b_2\,b_{-1} + b_0\,a_1) - c\,a_1^2\,b_{-1}\,e^{-\eta}\right)}{\left(\frac{b_2 c}{a_1} e^{\eta} - 1\right)(a_1^3\,e^{\eta} - c\,a_1\,(c\,b_2\,b_{-1} + b_0\,a_1) - c\,a_1^2\,b_{-1}\,e^{-\eta})} = \tag{27}$$

$$= -\frac{c}{1 - \frac{b_2\,c}{a_1}\,e^{\eta}} = -\frac{c}{1 + C_1\,e^{\eta}},$$

where $C_1 = -\frac{b_2\,c}{a_1}$ is an arbitrary constant. We can see that the solution u_{10} is the partial case of solution (10) at $\mu = \pm c$.

It is easy to see that the solution u_{11} is the partial case of solution (10) at $\mu = \mp c$

$$u_{11}(\eta) = -\frac{cb_{-1}\,\mathrm{e}^{-\eta}}{b_{-1}\,\mathrm{e}^{-\eta} + b_{-2}\,\mathrm{e}^{-2\,\eta}} = -\frac{c}{1 + \frac{b_{-2}}{b_{-1}}e^{-\eta}} = -\frac{c}{1 + C_1\mathrm{e}^{-\eta}}, \tag{28}$$

where $C_1=\frac{b_{-2}}{b_{-1}}$ is an arbitrary constant.

Taking into account the following transformations for u_{12}

$$u_{12}(\eta)=\frac{-b_0\,c+a_{-1}\,\mathrm{e}^{-\eta}}{b_0+b_{-1}\,\mathrm{e}^{-\eta}-\frac{a_{-1}\,(b_{-1}\,c+a_{-1})\mathrm{e}^{-2\eta}}{c^2 b_0}}=$$

$$=-\frac{c(b_0-\frac{a_{-1}}{c}\,\mathrm{e}^{-\eta})}{(b_0-\frac{a_{-1}}{c}\,\mathrm{e}^{-\eta})(1+\frac{c\,b_{-1}+a_{-1}}{c\,b_0}\,\mathrm{e}^{-\eta})}=-\frac{c}{1+\frac{c\,b_{-1}+a_{-1}}{c\,b_0}\,\mathrm{e}^{-\eta}}= \tag{29}$$

$$=-\frac{c}{1+C_1\,\mathrm{e}^{-\eta}},$$

where $C_1=\frac{c\,b_{-1}+a_{-1}}{c\,b_0}$ is an arbitrary constant, we note that u_{12} is partial case of solution (10) at $\mu=\mp c$.

We can simplify u_{13} using the transformations

$$u_{13}(\eta)=\frac{-c\,b_{-2}\,\mathrm{e}^{-2\,\eta}}{b_0+b_{-2}\,\mathrm{e}^{-2\,\eta}}=-\frac{c}{1+\frac{b_0}{b_{-2}}\mathrm{e}^{2\,\eta}}=-\frac{c}{1+C_1\,\mathrm{e}^{2\,\eta}}, \tag{30}$$

where $C_1=\frac{b_0}{b_{-2}}$ is an arbitrary constant. We can see that u_{13} is the partial case of (10) at $\mu=\pm\frac{c}{2}$.

We obtain that the solution u_{14} is the partial case of solution (10) at $\mu=\pm c$ using the following transformations

$$u_{14}(\eta)=\frac{a_{-1}\,\mathrm{e}^{-\eta}-cb_{-2}\,\mathrm{e}^{-2\,\eta}}{-\frac{a_{-1}\,(b_{-1}\,c+a_{-1})}{c^2 b_{-2}}+b_{-1}\,\mathrm{e}^{-\eta}+b_{-2}\,\mathrm{e}^{-2\,\eta}},=$$

$$=-\frac{c\,\mathrm{e}^{-\eta}\left(\frac{a_{-1}}{c}-b_{-2}\,\mathrm{e}^{-\eta}\right)}{\left(\frac{b_{-1}}{b_{-2}}+\frac{a_{-1}}{c\,b_{-2}}+\mathrm{e}^{-\eta}\right)\left(\frac{a_{-1}}{c}-b_{-2}\,\mathrm{e}^{-\eta}\right)}=-\frac{c\,\mathrm{e}^{-\eta}}{\left(\frac{b_{-1}}{b_{-2}}+\frac{a_{-1}}{c\,b_{-2}}+\mathrm{e}^{-\eta}\right)}= \tag{31}$$

$$=-\frac{c}{1+\left(\frac{b_{-1}}{b_{-2}}+\frac{a_{-1}}{c\,b_{-2}}\right)\mathrm{e}^{\eta}}=-\frac{c}{1+C_1\,\mathrm{e}^{\eta}},$$

where $C_1=\left(\frac{b_{-1}}{b_{-2}}+\frac{a_{-1}}{c\,b_{-2}}\right)$ is an arbitrary constant.

We have that the solution u_{15} is the partial case of (10) at $\mu=\pm\frac{c}{2}$ by taking into account the following transformations

$$u_{15}(\eta)=\frac{-c\,b_{-1}\,\mathrm{e}^{-\eta}-cb_{-2}\,\mathrm{e}^{-2\,\eta}}{b_1\,\mathrm{e}^{\eta}+\frac{b_1\,b_{-2}}{b_{-1}}+b_{-1}\mathrm{e}^{-\eta}+b_{-2}\mathrm{e}^{-2\,\eta}}=$$

$$=-\frac{c\,\mathrm{e}^{-\eta}\;(b_{-1}+b_{-2}\,\mathrm{e}^{-\eta})}{\left(\frac{b_1}{b_{-1}}\,\mathrm{e}^{\eta}+\mathrm{e}^{-\eta}\right)(b_{-1}+b_{-2}\mathrm{e}^{-\eta})}=-\frac{c\,e^{-\eta}}{\frac{b_1}{b_{-1}}\mathrm{e}^{\eta}+\mathrm{e}^{-\eta}}= \tag{32}$$

$$=-\frac{c}{1+C_1\,\mathrm{e}^{2\,\eta}},$$

where $C_1 = \frac{b_1}{b_{-1}}$ is an arbitrary constant.

We obtain that the solution u_{16} is the partial case of solution (10) at $\mu = \pm c$ using the following transformations

$$u_{16}(\eta) = \frac{a_0 - \frac{c(a_0\, b_{-1} + c b_1\, b_{-2})\mathrm{e}^{-\eta}}{a_0} - c\, b_{-2}\, \mathrm{e}^{-2\,\eta}}{b_1 \mathrm{e}^{\eta} - \frac{a_0\, b_1\, c^2 b_{-1} + b_1^2 c^3 b_{-2} + a_0^3}{c\, a_0^2} + b_{-1}\, \mathrm{e}^{-\eta} + b_{-2} \mathrm{e}^{-2\,\eta}} =$$

$$= \frac{c\,\left(a_0^3 - c\, a_0\, (a_0\, b_{-1} + c\, b_1 b_{-2})\, \mathrm{e}^{-\eta} - c\, a_0^2\, b_{-2} \mathrm{e}^{-2\,\eta}\right)}{\left(\frac{b_1 c}{a_0}\, \mathrm{e}^{\eta} - 1\right)\, (a_0^3 - c\, a_0\, (a_0\, b_{-1} + c\, b_1\, b_{-2}) \mathrm{e}^{-\eta} - c\, a_0^2\, b_{-2} \mathrm{e}^{-2\,\eta})} = \tag{33}$$

$$= \frac{c}{\frac{b_1\, c}{a_0}\, \mathrm{e}^{\,\eta} - 1} = -\frac{c}{1 + C_1\, \mathrm{e}^{\,\eta}},$$

where $C_1 = -\frac{b_1\, c}{a_0}$ is an arbitrary constant.

So, we can see that 15 out of 16 solutions by Wazwaz and Mehanna can be simplified and these simplified forms are partial cases of solution (10) at $\mu = \mp c$; $\mu = \pm c$, $\mu = \mp \frac{c}{2}$; $\mu = \pm \frac{c}{2}$; $\mu = \mp \frac{c}{3}$; $\mu = \pm \frac{c}{3}$. So, the authors of [7] also made the fifth error from the list of errors of Ref. [8].

We would like to ask the authors: why did they find only 16 solutions at these values μ? It is obvious that using other values of μ they could find 100, 1000 or even more solutions. They can fill out all pages of the journal by these "solutions" of Eq.(7) at various values of the parameter μ.

There is also another good question. Why did the authors use only multipliers with 2 monomials in the solutions u_1, u_3, u_6, u_9, u_{12}, u_{14}, u_{15}, with 3 monomials in the solution u_7, with 4 monomials in the solutions u_{10} and u_{16}? They could obtain much more solutions for Eq.(7) if they used 5, 6 and more monomials. Many other solutions could be made up and presented as some "new solutions" of Eq.(7).

5 Redundant exact solutions of nonlinear differential equations

In section 4 we analyzed the exact solutions of Eq.(7) suggested in Ref. [7]. We have observed that all these solutions can be simplified to simple forms and all these solutions are partial cases of the well known general solution (9). We believe that these exact solutions by Wazwaz and Mehanna make no sense.

We can observe different types of such solutions. Many examples of these solutions were given in Refs. [8–13].

Suppose that we have a nonlinear differential equation in the form

$$E(w, w_z, \ldots, z) = 0. \tag{34}$$

Assume that there is an exact solution of this equation in the form of a fraction

$$w = \frac{\varphi(z)}{\psi(z)}. \tag{35}$$

Let us also suppose that the functions $\varphi(z)$ and $\psi(z)$ can be presented in the form: $\varphi(z) = \varphi_1(z)\, f(z)$ and $\psi(z) = \psi_1(z)\, f(z)$. In this case exact solution (35) can be reduced to the form

$$w = \frac{\varphi_1(z)\, f(z)}{\psi_1(z)\, f(z)} = \frac{\varphi_1(z)}{\psi_1(z)}. \tag{36}$$

We can see that the exact solution (35) can be simplified. Thus the exact solution (35) is the redundant exact solution because it can be reduced to a simpler form. Let us introduce the following definition of an redundant exact solution.

Definition. *The exact solution* (35) *is called the redundant exact solution of the differential equation* (34) *if we can reduce this exact solution to the a simpler form with a necessary amount of arbitrary constants.*

Let us explain what we mean when we say about a necessary amount of arbitrary constants. We mean that if we study the ordinary differential equation of the $N-$th order we can have the general solution with N arbitrary constants. As special cases we can obtain solution with less amount of arbitrary constants. For the second order Eq.(7) we have the general solution with two arbitrary constants.

However if we obtain exact solutions without arbitrary constants we can also have redundant exact solutions in many cases. From the formal point of view exact solutions (13), (14) and (15) are exact solutions of Eq.(7) but these solutions are redundant solutions as well. Due to arbitrary constant in solution (11) we can find infinite quantity of exact solutions of Eq.(7). Unfortunately many popular methods of finding exact solutions can often lead to obtaining some redundant exact solutions.

As we can see from the previous section, all the solutions of Eq.(7) obtained by Wazwaz and Mehanna using the Exp $-$ function method are the redundant exact solutions. It is clear that we can obtain a lot of solutions of nonlinear differential equation if we multiple nominator and denominator on any expression but these solutions can be simplified. All these solutions are not of no interest. Moreover, the authors of [7] do not take care of paper usage and set a bad example for many young people.

It was shown in Ref. [9] that the application of the Exp-function method in finding exact solutions is not effective and can lead to redundant solutions. The work of Wazwaz and Mehanna illustrates all disadvantages of the Exp-function method. First, Wazwaz and Mehanna obtained two incorrect solutions. Second, the exact solutions from Ref. [7] contain superfluous arbitrary constants. Third, these exact solutions have cumbersome form and, as we have shown above, can be simplified. And the last point is that all the correct exact solutions by Wazwaz and Mehanna can be simplified and they are partial cases of the well-known exact solution. So we must say to the scientific community again: be careful with the Exp $-$ function method.

In conclusion, we strongly recommend authors to look through the papers [8–20] carefully before they start looking for exact solutions of nonlinear differential equations.

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