A Note on solutions of the Korteweg – de Vries hierarchy

Nikolay A. Kudryashov

Department of Applied Mathematics, National Research Nuclear University MEPhI, 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

Abstract

Solutions of the Korteweg – de Vries hierarchy are discussed. It is shown that results by Wazwaz [Wazwaz A.M. Multiple – soliton solutions of the perturbed KdV equation, Commun Nonlinear Sci Numer Simulat, 2010; 15911: 3270 – 3273] are the well – known consequences of the full integrability for the Korteweg – de Vries hierarchy.

The Korteweg – de Vries hierarchy can be written as [1, 2]

$$u_t + \frac{\partial}{\partial x}\sum_{n=0}^{N} t_n\, L_{n+1}\,[u] = 0, \tag{1}$$

where t_n, $(n = 0, \ldots, N)$ are real parameters of equation and the Lenard operator $L_{n+1}[u]$ is determined by the following recursion formula [3, 4]

$$\frac{\partial L_{n+1}[u]}{\partial x} = \left(\frac{\partial^3}{\partial^3 x} + 4\,u\,\frac{\partial}{\partial x} + 2\,\frac{\partial u}{\partial x}\right) L_n[u], \quad L_0[u] = \frac{1}{2}. \tag{2}$$

Assuming $n = 0$, $n = 1$ and $n = 2$ in Eq. (2) we have the following formulae

$$L_1[u] = u, \quad L_2[u] = u_{xx} + 3\,u^2, \tag{3}$$

$$L_3[u] = u_{xxxx} + 10uu_{xx} + 5\,u_x^2 + 10\,u^3 \tag{4}$$

and so on.

There is the Lax pair for Eq.(1) in the form

$$\psi_{xx} + (\lambda + u)\psi = 0, \tag{5}$$

$$\psi_t = \left(C + \sum_{n=0}^{N} t_n \sum_{j=0}^{n} (-4\lambda)^{n-j}\, L_{j,x}[u]\right)\psi - 2\left(\sum_{n=0}^{N} t_n \sum_{j=0}^{n} (-4\lambda)^{n-j}\, L_j[u]\right)\psi_x, \tag{6}$$

where we denote

$$L_{j,x}[u] = \frac{\partial}{\partial x}\, L_j[u].$$

In the case $N = 2$ the system equation (5) – (6) takes the form

$$\psi_{xx} + (\lambda + u)\psi = 0, \tag{7}$$

$$\begin{aligned} \psi_t = \left[C + (t_1 - 4\,\lambda\,t_2)\,u_x + t_2\,(u_{xxx} + 6\,u\,u_x)\right]\,\psi - \\ -2\,\left[0.5\,t_0 - 2\,\lambda\,(t_1 - 4\,\lambda\,t_2) + (t_1 - 4\,\lambda\,t_2)\,u + t_2\,(u_{xx} + 3\,u^2)\right]\,\psi_x. \end{aligned} \tag{8}$$

Using the compatibility condition

$$(\psi_{xx})_t = (\psi_t)_{xx}$$

for the system of equation (5)–(6) we obtain the Korteweg–de Vries hierarchy (1). In the case $N = 2$ we have the fifth order Korteweg – de Vries equation in the form

$$u_t + t_0\,u_x + t_1\,(u\,u_x + u_{xxx}) + t_2\,(30\,u^2\,u_x + 10\,u\,u_{xxx} + 20\,u_x\,u_{xx} + u_{xxxxx}) = 0 \tag{9}$$

Assuming $t_0 = 0$, $t_1 = 1$ and $t_2 = \varepsilon$ in Eq.(9) we have the "perturbed KdV equation" by Wazwaz [5]. We can see that that Eq.(9) is the partial case of the Korteweg – de Vries hierarchy (1).

Every member of the Korteweg–de Vries hierarchy (1) (case $t_0 = t_1 = t_2 = \ldots = t_{N-1} = 0$ and $t_n = 1$) and the Lax pair for them was firstly introduced in [3]. The Korteweg–de Vries hierarchy (1) and Lax pair (5) and (6) is trivial generalization of the results by Lax.

It is well known that the Cauchy problem for every member of the Korteweg–de Vries hierachy can be solved by means of the inverse scattering transform method. It is known that there are rational, special and soliton solutions of these equations [1, 2, 6–13].

It is clear that the Cauchy problem for the Korteweg – de Vries hierarchy (1) can be solved by means of the inverse scattering transform method as well taking Lax pair (5)–(6) into account.

Assuming $\lambda = -k^2$ and $u = u_x = u_{xx} = \ldots = 0$ we have from Eq.(5)

$$\psi(x, k, t) = C_1(k, t)\,e^{k\,x} + C_2(k, t)\,e^{-k\,x}. \tag{10}$$

Substituting Eq.(10) into Eq.(6) we obtain the system of equations in the form

$$\frac{d\ln C_2}{dt} - C - \sum_{n=1}^{N} 2^{2\,n}\,t_n\,k^{2\,n+1} \tag{11}$$

$$\frac{d\ln C_1}{dt} - C + \sum_{n=1}^{N} 2^{2\,n}\,t_n\,k^{2\,n+1} \tag{12}$$

From the system of equations (11) and (12) we find the time evolution of the scattering data for solving the Cauchy problem to the Korteweg – de Vries hierarchy

$$\begin{aligned} \kappa_j = Const, \quad C_j(t) = C_j(0)\,\exp\left(t\sum_{n=1}^{N} 2^{2\,n}\,t_n\,\kappa_n^{2\,n+1}\right), \\ (j = 0, 1, \ldots, J), \quad b(k, t) = b(k, 0)\,\exp\left(i\,t\sum_{n=1}^{N} 2^{2\,n+1}\,t_n\,k^{2\,n+1}\right), \end{aligned} \tag{13}$$

where $i^2=-1$, κ_j are values of the discrete spectrum, $C_j(0)$, $(j=0,1,\ldots,J)$ are the normalisation constants determined at $t=0$ and $b(k,0)$ is the reflection coefficient.

The Cauchy problem for the Korteweg – de Vries hierarchy (1) can be solved taking into account the scattering data (13) and the well – known inverse scattering transform method for the Korteweg – de Vries equation [4, 8, 14]. Multi – soliton solutions of the Korteweg – de Vries hierarchy (1) can be found using the scattering data for the reflectionless potentials at $b(k,t)=0$ [4, 8, 14].

Multi – soliton solution of the Korteweg – de Vries hierarchy (1) can also be found using the well – known formula by Hirota

$$u=2\,(\ln F)_{xx}\,. \tag{14}$$

Assuming $F(x,t)$ in the form [15]

$$F_1=1+C_1\,e^{\theta_1},\quad \theta_1=k_1\,x-\sum_{n=1}^{N}t_n\,k_1^{2\,n+1}\,t+k_1\,x_1^{(0)} \tag{15}$$

where C_1 and $x_1^{(0)}$ are arbitrary constant, we obtain one soliton solution.

Taking into account $F(x,t)$ in the form [15]

$$F_1=1+C_1\,e^{\theta_1}+C_2\,e^{\theta_2}+C_1\,C-2\,e^{\theta_1+\theta_2+A_{12}},\quad e^{A_{12}}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{16}$$

where C_1 and C_2 are arbitrary constants and

$$\theta_j=k_j\,x-\sum_{n=1}^{N}t_n\,k_j^{2\,n+1}\,t+k_j\,x_j^{(0)}, \tag{17}$$

we have two soliton solution and so on.

Rational solutions of hierarchy (1) can be found taking the generalized Yablonskii – Vorob'ev polynomials [11, 12]. There are solutions expressed via the higher Painlevé transcendent functions [9, 10].

In discussion author [5] say that "Multiple – soliton solutions and multiple singular soliton solutions were derived. It was formally proved that this equation is completly integrable equation". We agree with author completely but we saw that his results are consequence of the integrability for the Korteweg – de Vries hierarchy (1). In this paper we have pointed out additionally to works [16–20] the danger in finding exact solutions of the well – known equations.

References

[1] Kudryashov N.A. , Analitical theory of nonlinear differential equations, Moskow - Igevsk, Institute of computer investigations 2004: 360 (in Russian).

[2] Kudryashov N.A. Soliton, rational and special solutions of the Korteweg–de vries hierarchy, Appl. Math. Comput. 2009; doi:10.1016/j.amc.2009.11.024

[3] Lax P.D., Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math., 1968; 21: 467 - 490.

[4] Ablowitz M.J. and Clarkson P.A., Solitons Nonlinear Evolution Equations and Inverse Scattering, Cambridge university press, 1991; 516 p.

[5] Wazwaz A.M. Multiple – soliton solutions of the perturbed KdV equation equation, Commun Nonlinear Sci Simul 2010; 15(11): 3270 – 3273

[6] Korteweg D.J., de Vries G. On the change of form of long waves advancing in a rectangular canal and on a new tipe of long stationary waves. Phil. Mag. 1895; 39: 422 – 43

[7] Zabuski N.J., Kruskal M.D., Integration of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 1965; 15: 240 – 42

[8] Gardner C.S., Greene J.M., Kruskal M.D., Miura R.M., Method for solving the Korteweg - de Vries equation, J. Math. Phys., 1967; 19: 1095 – 1097

[9] Kudryashov N. A. The first and the second Painleve equations of higher order and some relations between them, Physics Letters A 1997; 224: 353 – 360

[10] Kudryashov N. A. On new transcendents defined by nonlinear ordinary differential equations, Journal of Physics A. Math. and Gen. 1998; 31(6): L129 - L137

[11] Kudryashov N.A., Demina M.V. The generalized Yablonskii - Vorob'ev polynomials and their properties, Physics Letters A, 2008; 372: 4885 – 4890

[12] Kudryashov N. A. and Pickering A. Rational solutions for Schwarzian integrable hierarchies, Journal of Physics A. Math. And Gen., 1998; 31: 9505 – 9518

[13] Kudryashov N A Fourth - order analogies to the Painleve equations, Journal of Pysics A.: Mathematical and General, 2002; 35: 4617 – 4632

[14] Drazin P.G., Johnson R.S., Soliton: an introduction, Caambridge University Press, 1989; 226 p

[15] Hirota R., Exact solution of the Korteweg-de Vries for multiple collisions of solutions, Phys. Rev. Lett., 1971; 27: 1192 – 94

[16] Kudryashov N.A., Loguinova N.B. Be careful with the Exp-function method, Communications in Nonlinear Science and Numerical Simulation, 2009; 14(5): 1881 – 1890

[17] Kudryashov N.A. On "new travelling wave solutions" of the KdV and the KdV - Burgers equtions, Communications in Nonlinear Science and Numerical Simulation, 2009;14(5): 1891 – 1900

[18] Kudryashov N.A. Seven common errors in finding exact solutions of nonlinear differential equations, Communications in Nonlinear Science and Numerical Simulation, 2009;14: 3507 – 3529

[19] Parkes E.J., A note on travelling - wave solutions to Lax's seventh - order KdV equation, Appl. Math. Comput. 2009; 215:864 - 865

[20] Parkes E.J., A note on solitary travelling-wave solutions to the transformed reduced Ostrovsky equation, Commun. Nonlinear. Sci. Numer. Simul. 2010;15: 2769 – 2771