A note on Abundant new exact solutions for the (3+1)-dimensional Jimbo-Miwa equation

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Abstract

We demonstrate that eight from twenty seven solutions obtained by Li and Dai [Abundant new exact solutions for the (3+1)-dimensional Jimbo-Miwa equation, J. Math. Anal. Appl. 361 (2010). 587–590] are wrong and do not satisfy the equation. The other nineteen exact solutions are not new solutions and can be found from the well known solution.

Li and Dai [1] looked for exact solutions of the $(3+1)$ Jimbo—Miwa equation

$$u_{xxxy} + 3\,u_x\,u_{xy} + 3\,u_y u_{xx} + 2\,u_{yt} - 3\,u_{xz} = 0. \tag{1}$$

Authors [1] used the traveling wave ansatz $u(x,y,z,t) = U(\xi)$, $\xi = k\,x + l\,y + m\,z + \omega\,t$ in Eq.(1). After integrating with respect to ξ they obtained the nonlinear ordinary differential equation in the form

$$k^3\,l\,U^{'''} + 3\,k^2\,l\,(U')^2 + (2\,l\,\omega - 3\,k\,m)U^{'} = C. \tag{2}$$

Here C is an constant of integration. Authors [1] applied the generalized Riccati equation method to look for exact solution of Eq.(2). However we know very well the general solution of Eq.(2).

Let us demonstrate this fact. Denoting $U' = V(\xi)$ in Eq.(2) we have the following equation

$$k^3\,l\,V^{''} + 3\,k^2\,l\,V^2 + (2\,l\,\omega - 3\,k\,m)V = C \tag{3}$$

Multiplying Eq. (3) on V' and integrating the equation with respect to ξ we obtain the equation

$$\frac{k^3\,l}{2}\,(V')^2 + k^2\,l\,V^3 + \frac{2\,l\,\omega - 3\,k\,m}{2}\,V^2 = C\,V + C_1, \tag{4}$$

where C_1 is an arbitrary constant.

Using transformation

$$V = -2\,k\,\left(\wp + \frac{2\,l\,\omega - 3\,k\,m}{12\,k^3\,l}\right) \tag{5}$$

from Eq. (4) we obtain

$$(\wp')^2 = 4\,\wp^3 - g_2\,\wp - g_3, \tag{6}$$

where

$$g_2 = \frac{(2\,l\,\omega - 3\,k\,m)^2 + 12\,k^2\,l\,C}{12\,k^6\,l^2},$$

$$g_3 = \frac{(2\,l\,\omega - 3\,k\,m)^3 - 108\,k^4\,l^2\,C_1 + 18\,k^2\,l\,C(2\,l\,\omega - 3\,k\,m)}{220\,k^9\,l^3}. \tag{7}$$

We can see that the general solution of Eq.(6) and consequently Eq.(2) is expressed via the Weierstrass elliptic function [2]. As for the rational, periodic and solitary wave solutions of (2) they can be found from Eq. (6) or Eq. (4) [3–8].

In the case of $C_1 = C = 0$ solution of Eq. (4) takes the form

$$V(\xi) = -\frac{2\,l\,\omega - 3\,k\,m}{2\,k^2\,l}\left(1 + \tan^2\left\{\frac{1}{2}\sqrt{\frac{2\,l\,\omega - 3\,k\,m}{k^3\,l}}(\xi + C_2)\right\}\right), \tag{8}$$

where C_2 in an arbitrary constant.

Integrating solution (8) once with respect to ξ, we obtain the solution of Eq. (2)

$$U(\xi) = C_3 - k\sqrt{\frac{2\,l\omega - 3\,k\,m}{k^3\,l}}\,\tan\left\{\frac{1}{2}\sqrt{\frac{2\,l\,\omega - 3\,k\,m}{k^3\,l}}(\xi + C_2)\right\}. \tag{9}$$

Here C_3 in an constant of integration.

If $C_1 = C = 0$ and $\omega = \frac{3\,k\,m}{2\,l}$ we have rational solution of Eq. (4)

$$V(\xi) = -\frac{2k}{(\xi + C_2)^2}, \tag{10}$$

where C_2 in an arbitrary constant.

Integrating (10) once with respect to ξ, we obtain the rational solution of Eq. (2)

$$U(\xi) = C_3 + \frac{2k}{\xi + C_2}. \tag{11}$$

Here C_3 in an integration constant.

The paper [1] contains the important misprint: the value of ω presented by Li and Dai in formula (2.8) is wrong.

The right value of ω is the following

$$\omega = \frac{4\,k^3\,l\,q\,r + 3\,k\,m - p^2\,k^3\,l}{2l}. \tag{12}$$

We checked all solutions by Li and Dai [1] with value of ω (12) and obtained that expressions $u_7, u_{10}, u_{11}, u_{12}, u_{17} - u_{19}, u_{22}, u_{23}$ and u_5 with lower sign do not satisfy Eq. (2). Solutions $u_1 - u_4, u_6, u_8, u_9, u_{13} - u_{16}, u_{20}, u_{21}, u_{24} - u_{26}$ and u_5 with upper sign are the same and can be obtained from formula (9).

Substituting (12) in solution (9) we have

$$U(z) = C_3 - k\sqrt{4\,q\,r - p^2}\,\tan\left\{\frac{1}{2}\sqrt{4\,q\,r - p^2}(\xi + C_2)\right\}. \tag{13}$$

Taking into account formula

$$i\tan i\alpha = -\tanh\alpha \tag{14}$$

at $4\,q\,r - p^2 < 0$ from Eq. (13) we have

$$U(z) = C_3 + k\,\sqrt{p^2 - 4\,q\,r}\,\tanh\left\{\frac{1}{2}\,\sqrt{p^2 - 4\,q\,r}(\xi + C_2)\right\}. \tag{15}$$

We can see that (15) coincides with u_1 at $C_3 = a_0 + k\,p$, $C_2 = 0$.

Using the identity

$$\tanh(\xi + i\pi/2) = \coth\xi \tag{16}$$

in the case of $C_3 = a_0 + k\,p$, $C_2 = i\pi$ from (15) we obtain solution u_2 by Li and Dai.

Taking into account the identity

$$\tanh\,\xi \pm i\mathrm{sech}\xi = \tanh(\frac{\xi}{2} \pm \frac{i\pi}{4}) \tag{17}$$

one can see that u_3 is equal to (15) at $C_3 = a_0 + k\,p$, $C_2 = \pm\frac{i\pi}{2}$.

In the case of $C_3 = a_0 + k\,p$, $C_2 = i\pi$ and using formula

$$\coth\xi + \mathrm{csch}\xi = \tanh(\frac{\xi}{2} + \frac{i\pi}{2}) \tag{18}$$

we can see that (15) is equal to u_4 with upper sign.

Using formula

$$\coth\xi - \mathrm{csch}\xi = \tanh\frac{\xi}{2} \tag{19}$$

and suppose that $C_3 = a_0 + k\,p$, $C_2 = 0$ in (15) we have solution u_4 with lower sign.

Let us show that solution u_8 is particular case of (15). Denoting $C_3 = a_0 + k\,p$, $C_2 = i\pi + \phi'$, $\phi' = -\frac{2}{\sqrt{p^2 - 4\,q\,r}}\,\phi$, $\phi = \mathrm{arccosh}\frac{p^2 - 4\,q\,r}{-4\,q\,r}$ from (15) we obtain

$$U(\xi) = a_0 + k\,p + k\,\sqrt{p^2 - 4\,q\,r}\,\coth\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\} = a_0 +$$

$$+k\,\frac{\sqrt{p^2-4\,q\,r}\cosh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\} + p\,\sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}}{\sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}} =$$

$$= a_0 + k\,\frac{\sqrt{-4qr}\left(\cosh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}\,\cosh\phi + \sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}\,\sinh\phi\right)}{\sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}} =$$

$$= a_0 + k\,\frac{\sqrt{-4qr}\,\cosh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi\right\}}{\sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi - \phi\right\}} = a_0 -$$

$$-\frac{4kqr\,\cosh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi\right\}}{\sqrt{p^2-4\,q\,r}\sinh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi\right\} - p\,\cosh\left\{\frac{\sqrt{p^2-4\,q\,r}}{2}\xi\right\}} \tag{20}$$

In the same way one can show that u_9 is equal to (15). Using the same the double angle formulas for the hyperbolic functions we can see that u_{12} is equal to u_9 and consequently to Eq. (15).

By means of formulae

$$\tanh\frac{\xi}{2} + \coth\frac{\xi}{2} = 2\,\tanh(\xi - \frac{i\pi}{2}) \tag{21}$$

and in the case of $C_3 = a_0 + k\,p$, $C_2 = 0$ from (15) we have solution u_5 with upper sign.

In the case of $C_3 = a_0 + k\,p$, $C_2 = -i\,\pi$ from (13) we obtain solution u_{13} from [1].

Denoting $C_3 = a_0 + k\,p$, $C_2 = \pi$ in (13) we have solution u_{14} from [1].

Using formula

$$\tan\xi \pm \sec\xi = \tan(\frac{\xi}{2} \pm \frac{\pi}{4}) \tag{22}$$

we obtain that u_{15} is equal to (13) at $C_3 = a_0 + k\,p$, $C_2 = \pm\frac{\pi}{2}$.

Let us show that solution u_{20} is particular case of Eq. (13). Denoting $C_3 = a0 + k\,p$, $C_2 = \pi + \phi'$, $\phi' = \frac{2}{\sqrt{4\,q\,r - p^2}}\,\phi$, $\phi = \arccos\sqrt{\frac{4qr-p^2}{4qr}}$ from Eq. (13)

we obtain

$$
\begin{gathered}
U(\xi)=a_0+k\,p+k\,\sqrt{4\,q\,r-p^2}\,\cot\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}=a_0+\\
+k\,\frac{\sqrt{4\,q\,r-p^2}\cos\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}+p\,\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}}{\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}}=\\
=a_0+k\,\frac{\sqrt{4qr}\left(\cos\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}\,\cos\phi+\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}\,\sin\phi\right)}{\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}}=\\
=a_0+k\,\frac{\sqrt{4qr}\,\cos\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi\right\}}{\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi+\phi\right\}}=a_0+\\
+\frac{4kqr\,\cos\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi\right\}}{\sqrt{4\,q\,r-p^2}\sin\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi\right\}+p\,\cos\left\{\frac{\sqrt{4\,q\,r-p^2}}{2}\xi\right\}}
\end{gathered}
\tag{23}
$$

In the same way one can obtain that solution u_{21} is particular case of Eq. (13).

Using the double angle formulas for trigonometric functions one can see that u_{24} is equal to u_{21} and consequently to Eq. (15).

By means of following identities

$$
\begin{gathered}
u_{25}=a_0+\frac{2\,k\,p\,d}{d+\cosh(p\,\xi)-\sinh(p\,\xi)}=a_0+k\,p\,\frac{2d}{d+2\,e^{-p\,\xi}}=\\
=a_0+k\,p\left(\frac{1-e^{-p\,\xi+\phi}}{1+e^{-p\,\xi+\phi}}+1\right)=a_0+k\,p+k\,p\,\tanh\left\{\frac{p\,\xi-\phi}{2}\right\}\\
\phi=\ln\frac{2}{d}
\end{gathered}
\tag{24}
$$

we have that u_{25} is equal to Eq. (15) at $r=0$, $C_3=a0+k\,p$, $C_2=-\phi$.

Using identity

$$
\frac{2(\cosh p\,\xi+\sinh p\,\xi)}{d+\cosh p\,\xi+\sinh p\,\xi}=\tanh\left(\frac{p\,\xi+\phi}{2}\right)+1,\quad \phi=\ln\frac{1}{d}
\tag{25}
$$

we obtain that u_{26} is equal to Eq. (15) at $r=0$, $C_3=a0+k\,p$, $C_2=\phi$.

One can see that u_{27} coincide with Eq. (10) if we take $C_3=a_0$ and $C_2=\frac{c_1}{q}$.

So Li and Dai [1] found twenty seven solutions of Eq. (2). However as we have seen above there is the general solution of Eq. (2). We can also see that Li and Dai in paper [1] made the second common error from the list of errors given by Kudryashov in [9]: "Some authors do not use the known general solutions of ordinary differential equations".

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