Popular ansatz methods and solitary waves solutions of the Kuramoto—Sivashinsky equation

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Abstract

Some methods to look for exact solutions of nonlinear differential equations are discussed. It is shown that many popular methods are equivalent to each other. Several recent publications with "new" solitary wave solutions for the Kuramoto—Sivashinsky equation are analyzed. We demonstrate that all these solutions coincide with the known ones.

1 Introduction

Nonlinear differential equations and their solutions play an important role in modern science and we can see constantly increasing number of publications in this area in the last years.

There are a lot of methods for finding exact solutions for nonlinear equations. For example, the inverse scattering transform [1], dressing method [2, 3], Hirota method [4], group methods [5] and some others demonstrate a lot of advantages in the case of exactly solvable nonlinear differential equations.

However most of these methods do not give any new results for the nonlinear nonintegrable equations. In such a case researchers often use ansatz methods. During last decades we observe almost explosion in the number of scientific papers using these methods. Modern computer algebra systems Mathematica and Maple play the main role in this explosion. Using these powerful programs the researcher can make a lot of cumbersome analytical calculations in a short space of time.

It is well known that any expression containing exponents, trigonometric or hyperbolic functions can be rewritten in different forms. In the case of large expressions the equivalence of these forms is not obvious. Therefore it is very easy to think that different forms of one expression are the different solutions. Using the computer algebra programs many researchers find exact solutions of nonlinear differential equations and do not analyze the results obtained. They often believe that a new ansatz can give new solutions, but they often are wrong. In most cases new ansatz gives a new form of solution but not new solution. Moreover, we observe that a lot of "new" solutions can be found from the known solutions with some choice of arbitrary constants and parameters. A lot of such examples and the list of common errors is given in the work [6].

The aim of this paper is to show that different ansatz methods often lead to the same results. We also show that the large amount of "new solutions" of the Kuramoto—Sivashinsky equation presented in three recent papers can be reduced by the algebraic transformations to two known solutions of this equation.

In this paper we consider the solitary wave solutions of the Kuramoto—Sivashinsky equation. This equation takes the form

$$u_t + auu_x + bu_{xx} + ku_{xxxx} = 0. \tag{1}$$

Nonlinear evolution equation (1) has been studied by a number of authors from various viewpoints. This equation has drown much attention not only because it is interesting as a simple one-dimensional nonlinear evolution equation including effects of instability and dissipation but also it is important for description of engineering and scientific problems. Equation (1) was used in work [7] for explanation of the origin of persistent wave propagation through medium of reaction-diffusion type. In paper [8] equation (1) was derived for description of the nonlinear evolution of the disturbed flame front. We can meet the application of equation (1) for studying of motion of a viscous incompressible fluid flowing down an inclined plane [9–11]. Mathematical modeling of dissipative waves in plasma physics by means of equation (1) was presented in [12]. Elementary particles as the solutions of the Kuramoto—Sivashinsky equation were studied in [13]. Equation (1) also can be used for description of nonlinear long waves in viscous-elastic tube [14].

The exact solutions of the Kuramoto—Sivashinsky equation are well known. The solutions of Eq. (1) were first found by Kuramoto [7]. Later Eq. (1) and its generalizations were considered many times. For example, the exact solutions of these equations were obtained and re-discovered in works [15–30].

We do not have any possibility to give the analysis of all works with solutions of the Kuramoto—Sivashinsky equation in this paper so we consider only three recent publications [31–33] with "new" solitary wave solutions of the Kuramoto—Sivashinsky equation.

Let us present well-known solutions of equation (1). Using the traveling wave reduction $\xi = \mu(x - c\,t)$ we can write the Kuramoto—Sivashinsky equation in the form of nonlinear ordinary differential equation

$$k\mu^3 u''' + b\,\mu\,u' + \frac{a}{2}\,u^2 - c\,u + A = 0, \tag{2}$$

where A is an arbitrary constant of integration. Equation (2) is autonomous (not explicitly depends on ξ) so its solution also depends on $\xi - \xi_0$, where ξ_0 is an arbitrary constant.

Solutions of Eq. (2) can be written in the form

$$u(\xi) = \frac{c}{a} + \frac{60\mu(b - 38k\mu^2)}{19a}\tanh(\xi - \xi_0) + \frac{120k\mu^3}{a}\tanh^3(\xi - \xi_0), \tag{3}$$

with

$$\mu^2 = -\frac{b}{76k} \quad \text{or} \quad \mu^2 = \frac{11b}{76k}. \tag{4}$$

Here c is an arbitrary constant. In fact it depends on arbitrary constant A, so we take c arbitrary to avoid using of A in (3).

In this paper we use the following notation:

$$\mu_1 = \frac{1}{2}\sqrt{\frac{-b}{19k}}, \quad \mu_2 = \frac{1}{2}\sqrt{\frac{11b}{19k}}, \quad c_1 = \frac{30b}{19}\sqrt{\frac{-b}{19k}}, \quad c_2 = \frac{30b}{19}\sqrt{\frac{11b}{19k}}. \tag{5}$$

Taking these notations into account, two known solutions of the Kuramoto—Sivashinsky equation can be written as

$$u^{(1)} = \frac{c}{a} + \frac{3c_1}{2a}\tanh\left(\mu_1(x - ct) - \xi_0\right) - \frac{c_1}{2a}\tanh^3\left(\mu_1(x - ct) - \xi_0\right), \tag{6}$$

if we take $\mu = \mu_1$ in (3) and

$$u^{(2)} = \frac{c}{a} - \frac{9c_2}{2a}\tanh\left(\mu_2(x - ct) - \xi_0\right) + \frac{11c_2}{2a}\tanh^3\left(\mu_2(x - ct) - \xi_0\right), \tag{7}$$

if we take $\mu = \mu_2$ in (3). Note that c and ξ_0 are arbitrary constants in these expressions.

The outline of this paper is as follows. In section 2 we obtain known solutions (3) of the Kuramoto—Sivashinsky equation by the truncated expansion method. Then we show the equivalence of truncated expansion method and some ansatz methods using these solutions. In section 3 we give the analysis of six "new" solitary wave solutions of the Kuramoto—Sivashinsky equation by Wazwaz [31] and prove that all his solutions are not new. In section 4 we consider eight "new" exact solutions of the Kuramoto—Sivashinsky equation by Chen and Zhang [32] and demonstrate that their exact solutions are not new as well. Section 5 contains the analysis and discussion of sixteen solitary wave solutions of the Kuramoto—Sivashinsky equation by Wazzan [33]. In this section we show that all solutions by Wazzan can be reduced to two known solutions (6) and (7).

2 Equivalence of different ansatz methods in finding solitary waves of the Kuramoto—Sivashinsky equation

Let us obtain solution of the Kuramoto—Sivashinsky equation by means of the truncated expansion method [34]. We look for solution in the form of the infinite series

$$u(x,t)=\frac{u_0}{\varphi^p}+\frac{u_1}{\varphi^{p-1}}+\ldots+u_p+\ldots. \tag{8}$$

In the case $p=\text{const}>0$ we have that the coefficients $u_i(x,t)$, $i=0,1,\ldots$ do not contain any singularities and equation $\varphi(x,t)=0$ gives the position of movable pole, φ_x and φ_t are not equals to zero. Substituting (8) into equation (1) we get that $p=3$ and $u_0=120k\varphi_x^3/a$. This means that we have the only movable singularity in solution of the Kuramoto—Sivashinsky equation, that is the third order movable pole.

One can calculate other coefficients $u_i(x,t)$ step by step, but this process is infinite and sum of this series is not known nowadays, so the solution in closed form cannot be obtained by this approach in the general case.

The main idea of the truncated expansion method [34] is to cut this series off to nonpositive powers of $\varphi(x,t)$. This truncated expansion is taken

$$u(x,t)=\frac{u_0}{\varphi^3}+\frac{u_1}{\varphi^2}+\frac{u_2}{\varphi}+u_3 \tag{9}$$

for the third order pole in solution of the Kuramoto—Sivashinsky equation. Substituting this finite sum into the the Kuramoto—Sivashinsky equation we obtain a finite number of equations to calculate coefficients $u_i(x,t)$ and the unknown function $\varphi(x,t)$.

Calculating $u_0(x,t)$ $u_1(x,t)$ $u_2(x,t)$, we have

$$u(x,t)=60\frac{k}{a}\frac{\partial^3\log\varphi}{\partial x^3}+\frac{60b}{19a}\frac{\partial\log\varphi}{\partial x}+u_3(x,t) \tag{10}$$

with nonsingular part $u_3(x,t)$. The other result of these calculations is four differential equations with respect to $\varphi(x,t)$ and $u_3(x,t)$. General solution of these four differential equations is not known, but one can find the particular solutions.

This particular solution takes the form [15]

$$\varphi(x,t) = 1 + \beta e^{\gamma(x-ct)} \tag{11}$$

where β and c are arbitrary constants. The value of constant γ is determined from biquadratic equation and gives

$$\gamma^2 = -\frac{b}{19k} \quad \text{or} \quad \gamma^2 = \frac{11b}{19k}. \tag{12}$$

Comparing (4) with (12) we can note that $\gamma = 2\mu$ and use the same notation as in the previous section, i.e. $\xi = \mu(x - ct)$ and $\varphi(x,t) = \varphi(\xi) = 1 + \beta e^{2\xi}$. So the solitary wave solutions are

$$u(\xi) = 60\frac{k\mu^3}{a}\frac{d^3}{d\xi^3}\log\left(1 + \beta e^{2\xi}\right) + \frac{60b\mu}{19a}\frac{d}{d\xi}\log\left(1 + \beta e^{2\xi}\right) + \frac{c}{a} - \frac{60b\mu}{19a} \tag{13}$$

with μ given by (4). We can see that the only known solutions are traveling waves, so the one can look for solutions of equation (2) instead of (1) and get the same results.

This situation is common in the case of nonlinear nonintegrable equations because it is difficult to find nontraveling wave solutions for them. That is why the most researchers start using the traveling wave of the partial differential equation. Thus they deal with the nonlinear ordinary differential equation, in our case — with equation (2). Note that the value of constant A in this ODE is crucial for the number of arbitrary constants in solution. Taking $A = 0$ it is impossible to obtain two arbitrary constants.

Some generalization of equation (2) was studied recently by Eremenko [35]. He proved that there are no other meromorphic solutions besides those found by Kuramoto [7] and Kudryashov [15, 17]. In the case of equation (2) it means that any solution containing only poles (with no more complicated singularities like essential singularities, branching points, etc.) must be represented by formula (3).

Let us discuss three popular approaches in finding exact solutions of nonlinear differential equations: the (G'/G)-expansion method, the exp-function method and the tanh-method.

The (G'/G)-expansion method was introduced in [36] (here G' stands for $dG(\xi)/d\xi$).

First of all let us note that taking into account expressions

$$\frac{d\log\varphi}{d\xi} = \frac{\varphi_\xi}{\varphi}, \quad \frac{d^3\log\varphi}{d\xi^3} = 2\frac{\varphi_\xi^3}{\varphi^3} - 6\frac{\varphi_\xi^2}{\varphi^2} + 4\frac{\varphi_\xi}{\varphi}. \tag{14}$$

we can write the expression (13) in the form

$$u(\xi) = 120\frac{k\mu^3}{a}\left(\frac{\varphi_\xi}{\varphi}\right)^3 - 360\frac{k\mu^3}{a}\left(\frac{\varphi_\xi}{\varphi}\right)^2 + \frac{60\mu}{a}\left(\frac{b}{19} + 4k\mu^2\right)\frac{\varphi_\xi}{\varphi} + \frac{c}{a} - \frac{60b\mu}{19a}. \tag{15}$$

The solution of ODE under study (eq. (2) in our case) by the (G'/G)-expansion method is looked for in the form

$$u(\xi) = \sum_{j=0}^{m} \alpha_j \left(\frac{G'}{G}\right)^j. \tag{16}$$

Here $G(\xi)$ is the solution of linear second order differential equation with unknown constant coefficients $G'' + a_1 G' + a_2 G = 0$.

We can see that known solution (15) have the same form as if it have been obtained by the (G'/G)-expansion method. Is there any other solution of this form?

It is well known that any solution of linear second order homogeneous differential equation with constant coefficients does not have any singularities. So the fraction (G'/G) has only first order poles in the positions where function $G(\xi)$ has zeroes.

The degree m of polynomial (16) "can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE" [36]. The highest order derivative in equation (2) is u''', the highest order nonlinear term is u^2, so we get $m = 3$ and consequently

$$u(\xi) = \alpha_3\left(\frac{G'}{G}\right)^3 + \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\frac{G'}{G} + \alpha_0. \tag{17}$$

Hence the solution has triple pole and no other singularities. So due to the Eremenko theorem [35] expression (17) cannot give new solution.

The next attempt to obtain new solutions can be done by *the* exp-*function method* first appeared in [37]. Performing all differentiation and reducing all fractions to the same denomination in expression (13) we have

$$\begin{gathered} u(\xi) = \frac{f(\xi)}{g(\xi)}, \quad g(\xi) = 19a\left(1 + \beta e^{2\xi}\right)^3, \\ f(\xi) = (19c - 60b\mu) + 3\beta(19c - 20b\mu + 3040k\mu^3)e^{2\xi} + \\ + 3\beta^2(19c + 20b\mu - 3040k\mu^3)e^{4\xi} + \beta^3(19c + 60b\mu)e^{6\xi}. \end{gathered} \tag{18}$$

The similar form of solution is used in the exp-function method. Namely, solutions are looked for in the form

$$u(\xi) = \frac{\sum_{n=-h}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{19}$$

where h, d, p and q are unknown positive integers, a_n and b_m are unknown constants.

The relations between constants h, d, p and q can be found by the same balancing procedure as in the case of (G'/G)-expansion method. The result of this calculation is $h = p$ and $d = q$. Any other choice of these constants cannot lead to balance of linear (highest order derivative) and nonlinear term, but this curious calculation takes place in many papers (we saw in all papers) dealing with the exp-function method.

Canceling common factor $\exp(-p\xi)$ we can rewrite (19) in the form

$$u(\xi) = \frac{a_{-p} + \ldots + a_q \exp((p+q)\xi)}{b_{-p} + \ldots + b_q \exp((p+q)\xi)}. \tag{20}$$

The researchers using this method cannot say anything about values of p and q. Therefore they try to substitute different values expecting to obtain different solutions. So the question is: can the ansatz (20) give any solution different from (18)?

To give the answer we note that $\exp(\xi)$ has no singularities in the complex plane. So the only singularities of fraction (20) are poles at positions where denominator has zeroes. The other possible case takes place when numerator and denominator have zeroes at the same positions. But in this case the fraction (20) is a constant. Hence the nonconstant solution is meromorphic and Eremenko theorem [35] holds. The result is that no new solutions different from (18) can be found by the exp-function method.

Note that in the case of the exp-function method the amount of unknown parameters is greater then in most other ansatz methods, so the difficulty of calculations increases. That is why the popularity of this ansatz method seems mysterious. Due to the cumbersome calculations the authors often obtain incorrect results with the exp-function method, some references may be found in recent reviews of Kudryashov [6, 38].

Now let us consider *the* tanh-*method* [39]. First of all let us rewrite the fraction φ_ξ/φ as follows

$$\begin{aligned} \frac{\varphi_\xi}{\varphi} &= \frac{2\beta e^{2\xi}}{1+\beta e^{2\xi}} = \frac{2e^{\xi+(\log\beta)/2}}{e^{\xi+(\log\beta)/2} + e^{-\xi-(\log\beta)/2}} = \\ &= 1 + \tanh\left(\xi + \frac{\log\beta}{2}\right) = 1 + \tanh(\xi - \xi_0), \end{aligned} \tag{21}$$

where $(\log\beta)/2 = -\xi_0$. Substituting this result into (15) we obtain solution of the Kuramoto—Sivashinsky equation in the form (3).

The tanh-method (the solution is looked for as polynomial in $\tanh\xi$) is used quite often due to its simplicity. Function $\tanh\xi$ has simple poles (and no other singularities), so the Eremenko results [35] can be applied. Therefore the tanh-method cannot give solutions other than already known ones (3).

The one should bear in mind that we are writing $\tanh\xi$ (i.e. omitting the constant phase ξ_0) only for simplicity. Sometimes the researchers actually forget this arbitrary constant. The result is a large amount of particular solutions instead of several more general ones. Some examples of this situation will be presented in the following sections.

3 "New solutions" of the Kuramoto—Sivashinsky equation by Wazwaz

Wazwaz [31] used "the tanh method and the extended tanh method for analytic treatment for two equations": the Kuramoto—Sivashinsky equation and the Kawahara equation. By means of these methods, "new solitary wave solutions for each equation" were found by him.

In previous section we have already seen that the tanh-method cannot give new solutions of the Kuramoto—Sivashinsky equation. In the framework of extended tanh-method solutions are represented as polynomials in $\tanh\xi$ and $\coth\xi$. Function $\coth\xi$ has simple poles in the complex plain, so the polynomial in $\tanh\xi$ and $\coth\xi$ is meromorphic. Therefore this new ansatz cannot lead to new solutions.

Let us show that all "new" solutions obtained by Wazwaz are the special cases of the known solutions (6) and (7) ($u^{(1)}$ and $u^{(2)}$ correspondingly) of the Kuramoto—Sivashinsky equation.

Wazwaz has given the following six solutions of Eq. (1)

$$u_1 = \frac{c_1}{2a}\left(2 + 3\tanh\zeta_1 - \tanh^3\zeta_1\right), \tag{22}$$

$$u_2 = \frac{c_1}{2a}\left(2 + 3\coth\zeta_1 - \coth^3\zeta_1\right), \tag{23}$$

$$u_3 = \frac{c_2}{2a}\left(2 - 9\tanh\zeta_2 + 11\tanh^3\zeta_2\right), \tag{24}$$

$$u_4 = \frac{c_2}{2a}\left(2 - 9\coth\zeta_2 + 11\coth^3\zeta_2\right), \tag{25}$$

$$u_5 = \frac{c_1}{16a}\left(16 + 9\tanh\frac{\zeta_1}{2} - \tanh^3\frac{\zeta_1}{2} + 9\coth\frac{\zeta_1}{2} - \coth^3\frac{\zeta_1}{2}\right), \tag{26}$$

$$u_6 = \frac{c_2}{16a}\left(16 - 3\tanh\frac{\zeta_2}{2} + 11\tanh^3\frac{\zeta_2}{2} - 3\coth\frac{\zeta_2}{2} + 11\coth^3\frac{\zeta_2}{2}\right), \tag{27}$$

where $\zeta_1 = \mu_1(x - c_1 t)$, $\zeta_2 = \mu_2(x - c_2 t)$. Solutions (22), (23), (24), (25), (26) and (27) correspond to formulae (16), (17), (19), (20), (25) and (26) in paper [31].

Analyzing expressions (22) and (24) one can see that they coincide with solutions $u^{(1)}$ with $c = c_1$ and $u^{(2)}$ with $c = c_2$ at $\xi_0 = 0$.

Taking into account the identity

$$\coth \zeta_k = \tanh\left(\zeta_k - i\pi/2\right), \quad k = 1, 2. \tag{28}$$

we can see that expression (23) is transformed to $u^{(1)}$ with $\xi_0 = i\pi/2$ and $c = c_1$. In the same way expression (25) is transformed to $u^{(2)}$ with $\xi_0 = i\pi/2$ and $c = c_2$.

Formulae (26) and (27) can be reduced to $u^{(1)}$ with $c = c_1$ and $u^{(2)}$ with $c = c_2$ at $\xi_0 = i\pi/2$ by means of formulae

$$\tanh \frac{\zeta_k}{2} + \coth \frac{\zeta_k}{2} = 2\tanh\left(\zeta_k - i\pi/2\right), \quad k = 1, 2, \tag{29}$$

$$\tanh^3 \frac{\zeta_k}{2} + \coth^3 \frac{\zeta_k}{2} = 8\tanh^3\left(\zeta_k - i\pi/2\right) - 6\tanh\left(\zeta_k - i\pi/2\right), \quad k = 1, 2. \tag{30}$$

Indeed, for solution (26) we have

$$u_5 = \frac{c_1}{16a}\left(16 + 9\tanh\frac{\zeta_1}{2} + 9\coth\frac{\zeta_1}{2} - \tanh^3\frac{\zeta_1}{2} - \coth^3\frac{\zeta_1}{2}\right) =$$
$$= \frac{c_1}{16a}\left(16 + 18\tanh\left(\zeta_1 - i\pi/2\right) - \left(8\tanh^3\left(\zeta_1 - i\pi/2\right) - 6\tanh\left(\zeta_1 - i\pi/2\right)\right)\right) =$$
$$= \frac{c_1}{2a}\left(2 + 3\tanh\left(\zeta_1 - i\pi/2\right) - \tanh^3\left(\zeta_1 - i\pi/2\right)\right). \tag{31}$$

We also obtain the following set of equalities for solution (27)

$$u_6 = \frac{c_2}{16a}\left(16 - 3\tanh\frac{\zeta_2}{2} - 3\coth\frac{\zeta_2}{2} + 11\tanh^3\frac{\zeta_2}{2} + 11\coth^3\frac{\zeta_2}{2}\right) =$$
$$= \frac{c_2}{16a}\left(16 - 6\tanh\left(\zeta_2 - i\pi/2\right) + 11\left(8\tanh^3\left(\zeta_2 - i\pi/2\right) - 6\tanh\left(\zeta_2 - i\pi/2\right)\right)\right) =$$
$$= \frac{c_2}{2a}\left(2 - 9\tanh\left(\zeta_2 - i\pi/2\right) + 11\tanh^3\left(\zeta_2 - i\pi/2\right)\right). \tag{32}$$

So all solutions by Wazwaz are transformed to solutions (6) and (7). The statement by Wazwaz that he has found new solutions of the Kuramoto—Sivashinsky is not correct.

Note that Wazwaz obtained six expressions instead of two solutions $u^{(1)}$ and $u^{(2)}$ because he forgot the arbitrary constant ξ_0. The wave velocities in (22)–(27) are not arbitrary as in $u^{(1)}$ and $u^{(2)}$ because Wazwaz has taken $A = 0$ in eq. (2).

4 "New" solutions of the Kuramoto—Sivashinsky equation by Chen and Zhang

Chen and Zhang [32] applied a generalized tanh-function method to find "new multiple soliton solutions" of the general Burgers—Fisher and the Kuramoto—Sivashinsky equations. The main idea of generalized tanh method is to represent solution as polynomial in function $F(\xi)$. Here $F(\xi)$ stands for special solution of the Riccati equation $F' = A + BF + CF^2$ with constant coefficients A, B, C. Chen and Zhang believed that taking different special solutions like $\coth\xi \pm \operatorname{csch}\xi$, $\sec\xi \pm \tan\xi$, $\tanh\xi$, etc., they can obtain new solutions of the Kuramoto—Sivashinsky equation.

It is well known that general solution of the Riccati equation with constant coefficients has simple poles and no other singularities. So the solutions are meromorphic and Eremenko theorem holds. Therefore this new ansatz does not give any new solution of the Kuramoto—Sivashinsky equation. In this section we will show that all solutions in [32] are the special cases of (3).

Note that Chen and Zhang used the special solutions of Riccati equation with omitted arbitrary constant ξ_0. Therefore they obtain large number of solutions of the Kuramoto—Sivashinsky equation instead expression (3).

Chen and Zhang take the Kuramoto—Sivashinsky equation in the form

$$u_t + uu_x + pu_{xx} - qu_{xxxx} = 0 \tag{33}$$

and use the traveling wave variable $\xi = k(x - \omega t)$. One can see that this equation coincides with (1) if we take $a = 1$ in eq. (1). Then they obtain eight solutions (or twelve if we will take into account upper and lower signs in their expressions).

To avoid intersection with notation introduced in first section we have changed their notation as follows. We have changed (p, q, ω) in all solutions given by authors [32] into $(b, -k, c)$ in formulae below. Also we have changed k to 2μ in u_1, u_2, u_7, u_8 and k to μ in u_3, u_4, u_5, u_6. Therefore these solutions can be written as

$$\begin{gathered} u_1 = c + \frac{60}{19}\mu\left(b - 38k\mu^2\right)(\tanh 2\xi \pm i\operatorname{sech} 2\xi) + \\ +120k\mu^3(\tanh 2\xi \pm i\operatorname{sech} 2\xi)^3, \end{gathered} \tag{34}$$

$$\begin{gathered} u_2 = c + \frac{60}{19}\mu\left(b - 38k\mu^2\right)(\coth 2\xi \pm \operatorname{csch} 2\xi) + \\ +120k\mu^3(\coth 2\xi \pm \operatorname{csch} 2\xi)^3, \end{gathered} \tag{35}$$

$$u_3 = c + \frac{60}{19}\mu\left(b - 38k\mu^2\right)\tanh\xi + 120k\mu^3\tanh^3\xi, \tag{36}$$

$$u_4 = c + \frac{60}{19}\mu\left(b - 38k\mu^2\right)\coth\xi + 120k\mu^3\coth^3\xi, \tag{37}$$

$$u_5 = c - \frac{60}{19}\mu\left(b + 38k\mu^2\right)\tan\xi - 120k\mu^3\tan^3\xi, \tag{38}$$

$$u_6 = c - \frac{60}{19}\mu\left(b + 38k\mu^2\right)\cot\xi - 120k\mu^3\cot^3\xi, \tag{39}$$

$$\begin{gathered} u_7 = c - \frac{60}{19}\mu\left(b + 38k\mu^2\right)(\tan 2\xi \pm \sec 2\xi) - \\ -120k\mu^3(\tan 2\xi \pm \sec 2\xi)^3, \end{gathered} \tag{40}$$

$$\begin{gathered} u_8 = c + \frac{60}{19}\mu\left(b + 38k\mu^2\right)(\cot 2\xi \pm \csc 2\xi) + \\ +120k\mu^3(\cot 2\xi \pm \csc 2\xi)^3. \end{gathered} \tag{41}$$

Here $\xi = \mu(x - ct)$ in all expressions where c is an arbitrary constant. Note that wave vector μ in formulae (34)–(41) can have two values: $\mu = \mu_1$ or $\mu = \mu_2$ in solutions u_1, u_2, u_3, u_4, and $\mu = i\mu_1$ or $\mu = i\mu_2$ in solutions u_5, u_6, u_7, u_8.

The paper [32] contains some misprints: the meanings of wave vectors in solutions u_1, u_2, u_7, u_8 are wrong and solutions u_7 and u_8 contain wrong signs of coefficients. We corrected these misprints in expressions (34)–(41).

Taking into account formulae

$$\begin{gathered} \tanh i\alpha = i\tan\alpha, \quad \operatorname{sech} i\alpha = \sec\alpha, \\ \coth i\alpha = -i\cot\alpha, \quad \operatorname{csch} i\alpha = -i\csc\alpha \end{gathered} \tag{42}$$

we can note that solutions u_5, u_6, u_7 and u_8 are the copies of solutions u_3, u_4, u_1 and u_2. We can see this fact if we will choose $\mu = \mu_1$ (or $\mu = \mu_2$) in u_1, u_2, u_3, u_4 and $\mu = i\mu_1$ (or $\mu = i\mu_2$) in u_5, u_6, u_7, u_8. So we have $u_3 = u_5$, $u_4 = u_6$ $u_1 = u_7$ and $u_2 = u_8$. That is why we consider expressions (34)–(37) further (i.e. solutions u_1, u_2, u_3 and u_4 only).

Comparing u_3 with solution (3) we can see that they are the same if we take $\xi_0 = 0$ in (3).

Taking into account the identity

$$\tanh(\xi + i\pi/2) = \coth\xi \tag{43}$$

one can see that u_4 coincides with (3) if $\xi_0 = -i\pi/2$.

Using the formula

$$\tanh 2\xi \pm i\operatorname{sech} 2\xi = \tanh(\xi \pm i\pi/4) \tag{44}$$

we obtain that u_1 with upper sign is equal to (3) if $\xi_0 = -i\pi/4$. In the same way u_1 with lower sign is equal to (3) if $\xi_0 = i\pi/4$.

With the aid of the identity

$$\coth 2\xi + \operatorname{csch} 2\xi = \tanh(\xi + i\pi/2) \qquad (45)$$

we have that u_2 with upper sign is equal to (3) if $\xi_0 = -i\pi/2$.

Taking into account the formula

$$\coth 2\xi - \operatorname{csch} 2\xi = \tanh \xi \qquad (46)$$

we get that u_2 with lower sign is equal to (3) if $\xi_0 = 0$.

Hence we obtain that all solutions by Chen and Zhang [32] are the special cases of the known solution (3) of the Kuramoto—Sivashinsky equation. Chen and Zhang did not present any new solutions of the Kuramoto—Sivashinsky equation.

5 "New" solutions of the Kuramoto—Sivashinsky equation by Wazzan

Wazzan [33] used "a modified tanh-coth method to solve the general Burgers—Fisher and the Kuramoto—Sivashinsky equations". He believed that "new multiple travelling wave solutions were obtained for the general Burgers—Fisher and the Kuramoto—Sivashinsky equations".

The idea of modified tanh-coth method is very close to the generalized tanh method discussed in previous section. The only difference is that the solution is represented as polynomial in $F(\xi)$ and $1/F(\xi)$ simultaneously. Here $F(\xi)$ is special solution of the Riccati equation $F' = A + BF + CF^2$ with constant coefficients. Therefore this ansatz adds the poles of $1/F(\xi)$ (i.e. the zeroes of $F(\xi)$) to the poles of $F(\xi)$. Singularities of other type are absent because of the nature of $F(\xi)$ so Eremenko results can be applied again. Hence this new ansatz cannot give new solutions of the Kuramoto—Sivashinsky equation.

Wazzan takes $A = 0$ in eq. (2) so the wave velocities in his solutions are fixed contrary to known solutions (6) and (7). He also omit the arbitrary constant ξ_0 and this is the main reason that sixteen solutions (instead of two) of eq. (1) are given in paper [33]. The Wazzan solutions are the following

$$u_1 = \pm\frac{2c_1}{a}\frac{3e^{\mp 2\zeta_3} - 1}{\left(e^{\mp 2\zeta_3} - 1\right)^3}, \qquad (47)$$

$$u_2 = \pm\frac{2c_1}{a}\left(1 - \frac{3e^{\pm 2\zeta_3} - 1}{\left(e^{\pm 2\zeta_3} - 1\right)^3}\right), \qquad (48)$$

$$u_3 = \mp \frac{2c_2}{a} \frac{12e^{\mp 4\zeta_4} + 9e^{\mp 2\zeta_4} + 1}{\left(e^{\mp 2\zeta_4} - 1\right)^3}, \tag{49}$$

$$u_4 = \pm \frac{2c_2}{a} \frac{e^{\pm 2\zeta_4}\left(e^{\pm 4\zeta_4} + 9e^{\pm 2\zeta_4} + 12\right)}{\left(e^{\pm 2\zeta_4} - 1\right)^3}, \tag{50}$$

$$u_5 = \frac{c_1}{2a}\left(\pm 2 + 3(\tanh 2\zeta_3 - i\,\mathrm{sech}\,2\zeta_3) - (\tanh 2\zeta_3 - i\,\mathrm{sech}\,2\zeta_3)^3\right), \tag{51}$$

$$u_6 = \frac{c_1}{2a}\left(\pm 2 - 3(\mathrm{csch}\,2\zeta_3 - \coth 2\zeta_3) + (\mathrm{csch}\,2\zeta_3 - \coth 2\zeta_3)^3\right), \tag{52}$$

$$u_7 = \frac{c_2}{2a}\left(\pm 2 - 9(\tanh 2\zeta_4 - i\,\mathrm{sech}\,2\zeta_4) + 11(\tanh 2\zeta_4 - i\,\mathrm{sech}\,2\zeta_4)^3\right), \tag{53}$$

$$u_8 = \frac{c_2}{2a}\left(\pm 2 + 9(\mathrm{csch}\,2\zeta_4 - \coth 2\zeta_4) - 11(\mathrm{csch}\,2\zeta_4 - \coth 2\zeta_4)^3\right), \tag{54}$$

$$u_9 = \frac{c_1}{2a}\left(\pm 2 + 3(\tanh 2\zeta_3 - i\,\mathrm{sech}\,2\zeta_3)^{-1} - (\tanh 2\zeta_3 - i\,\mathrm{sech}\,2\zeta_3)^{-3}\right), \tag{55}$$

$$u_{10} = \frac{c_1}{2a}\left(\pm 2 - 3(\mathrm{csch}\,2\zeta_3 - \coth 2\zeta_3)^{-1} + (\mathrm{csch}\,2\zeta_3 - \coth 2\zeta_3)^{-3}\right), \tag{56}$$

$$u_{11} = \frac{c_2}{2a}\left(\pm 2 - 9(\tanh 2\zeta_4 - i\,\mathrm{sech}\,2\zeta_4)^{-1} + 11(\tanh 2\zeta_4 - i\,\mathrm{sech}\,2\zeta_4)^{-3}\right), \tag{57}$$

$$u_{12} = \frac{c_2}{2a}\left(\pm 2 + 9(\mathrm{csch}\,2\zeta_4 - \coth 2\zeta_4)^{-1} - 11(\mathrm{csch}\,2\zeta_4 - \coth 2\zeta_4)^{-3}\right), \tag{58}$$

$$\begin{aligned} u_{13} = \frac{c_1}{16a}\Big(&\pm 16 + 9(\tanh \zeta_3 - i\,\mathrm{sech}\,\zeta_3) - (\tanh \zeta_3 - i\,\mathrm{sech}\,\zeta_3)^3 \\ &+ 9(\tanh \zeta_3 - i\,\mathrm{sech}\,\zeta_3)^{-1} - (\tanh \zeta_3 - i\,\mathrm{sech}\,\zeta_3)^{-3}\Big), \end{aligned} \tag{59}$$

$$\begin{aligned} u_{14} = \frac{c_1}{16a}\Big(&\pm 16 - 9(\mathrm{csch}\,\zeta_3 - \coth \zeta_3) + (\mathrm{csch}\,\zeta_3 - \coth \zeta_3)^3 \\ &- 9(\mathrm{csch}\,\zeta_3 - \coth \zeta_3)^{-1} + (\mathrm{csch}\,\zeta_3 - \coth \zeta_3)^{-3}\Big), \end{aligned} \tag{60}$$

$$\begin{aligned} u_{15} = \frac{c_2}{16a}\Big(&\pm 16 - 3(\tanh \zeta_4 - i\,\mathrm{sech}\,\zeta_4) + 11(\tanh \zeta_4 - i\,\mathrm{sech}\,\zeta_4)^3 \\ &- 3(\tanh \zeta_4 - i\,\mathrm{sech}\,\zeta_4)^{-1} + 11(\tanh \zeta_4 - i\,\mathrm{sech}\,\zeta_4)^{-3}\Big), \end{aligned} \tag{61}$$

$$\begin{aligned} u_{16} = \frac{c_2}{16a}\Big(&\pm 16 + 3(\mathrm{csch}\,\zeta_4 - \coth \zeta_4) - 11(\mathrm{csch}\,\zeta_4 - \coth \zeta_4)^3 \\ &+ 3(\mathrm{csch}\,\zeta_4 - \coth \zeta_4)^{-1} - 11(\mathrm{csch}\,\zeta_4 - \coth \zeta_4)^{-3}\Big). \end{aligned} \tag{62}$$

Here we use

$$\zeta_3 = \mu_1(x \mp c_1 t), \quad \zeta_4 = \mu_2(x \mp c_2 t). \tag{63}$$

Note that the upper and the lower signs in ζ_3 and ζ_4 correspond to upper and lower signs in expressions (51)–(62).

The paper by Wazzan [33] contains a number of misprints: $\mu\left(x - \frac{30}{19}b\mu t\right)$ instead of $\mu\left(x + \frac{30}{19}b\mu t\right)$ in arguments of u_1 and u_3, rearranged upper and lower signs in arguments of u_5–u_{16} and wrong signs in terms with negative powers in expressions u_8–u_{16}. Only two Wazzan solutions of the Kuramoto—Sivashinsky equation (u_2 and u_4) are absolutely correct. We have corrected all misprints in the list of solutions (47)–(62).

Now let us show that all cited solutions are the special cases of $u^{(1)}$ (formula (6)) and $u^{(2)}$ (formula (7)).

Consider solution (6) of the Kuramoto—Sivashinsky equation at $c = \pm c_1$, $\xi_0 = i\pi/2$. Then taking into account formula

$$\tanh(\zeta_j - i\pi/2) = \mp\frac{e^{\mp 2\zeta_j}+1}{e^{\mp 2\zeta_j}-1}, \quad j = 3, 4 \tag{64}$$

solution $u^{(1)}$ can be written as

$$\begin{aligned} u^{(1)} &= \pm\frac{c_1}{a} + \frac{3c_1}{2a}\tanh(\zeta_3 - i\pi/2) - \frac{c_1}{2a}\tanh^3(\zeta_3 - i\pi/2) = \\ &= \pm\frac{c_1}{a} \mp \frac{3c_1}{2a}\frac{e^{\mp 2\zeta_3}+1}{e^{\mp 2\zeta_3}-1} \pm \frac{c_1}{2a}\left(\frac{e^{\mp 2\zeta_3}+1}{e^{\mp 2\zeta_3}-1}\right)^3 = \\ &= \pm\frac{c_1}{2a}\frac{2\left(e^{\mp 2\zeta_3}-1\right)^3 - 3\left(e^{\mp 2\zeta_3}+1\right)\left(e^{\mp 2\zeta_3}-1\right)^2 + \left(e^{\mp 2\zeta_3}+1\right)^3}{\left(e^{\mp 2\zeta_3}-1\right)^3} = \\ &= \pm\frac{2c_1}{a}\frac{3e^{\mp 2\zeta_3}-1}{\left(e^{\mp 2\zeta_3}-1\right)^3} = u_1. \end{aligned} \tag{65}$$

Therefore solution of the Kuramoto—Sivashinsky equation by Wazzan (47) coincides with known solution (6) at $c = \pm c_1$ and $\xi_0 = i\pi/2$.

Let us rewrite solution u_2 by Wazzan as follows

$$\begin{aligned} u_2 &= \pm\frac{2c_1}{a}\left(1 - \frac{3e^{\pm 2\zeta_3}-1}{\left(e^{\pm 2\zeta_3}-1\right)^3}\right) = \pm\frac{2c_1}{a}\frac{\left(e^{\pm 2\zeta_3}-1\right)^3 - 3e^{\pm 2\zeta_3}+1}{e^{\pm 6\zeta_3}\left(1-e^{\mp 2\zeta_3}\right)^3} = \\ &= \pm\frac{2c_1}{a}\frac{3e^{\mp 2\zeta_3}-1}{\left(e^{\mp 2\zeta_3}-1\right)^3} = u_1. \end{aligned} \tag{66}$$

Hence $u_1 = u_2$ and solution by Wazzan (48) coincides with the known solution (6) at $c = \pm c_1$ and $\xi_0 = i\pi/2$.

Now let us consider solution (7) of the Kuramoto—Sivashinsky equation with $c = \pm c_2$, $\xi_0 = i\pi/2$. Taking into account formula (64) we have

$$\begin{gathered}
u^{(2)} = \pm\frac{c_2}{a} - \frac{9c_2}{2a}\tanh(\zeta_4 - i\pi/2) + \frac{11c_2}{2a}\tanh^3(\zeta_4 - i\pi/2) = \\
= \pm\frac{c_2}{a} \pm \frac{9c_2}{2a}\frac{e^{\mp 2\zeta_4}+1}{e^{\mp 2\zeta_4}-1} \mp \frac{11c_2}{2a}\left(\frac{e^{\mp 2\zeta_4}+1}{e^{\mp 2\zeta_4}-1}\right)^3 = \\
= \pm\frac{c_2}{2a}\frac{2\left(e^{\mp 2\zeta_4}-1\right)^3 + 9\left(e^{\mp 2\zeta_4}+1\right)\left(e^{\mp 2\zeta_4}-1\right)^2 - 11\left(e^{\mp 2\zeta_4}+1\right)^3}{\left(e^{\mp 2\zeta_4}-1\right)^3} = \\
= \mp\frac{2c_2}{a}\frac{12e^{\mp 4\zeta_4} + 9e^{\mp 2\zeta_4} + 1}{\left(e^{\mp 2\zeta_4}-1\right)^3} = u_3.
\end{gathered} \tag{67}$$

We obtain that solution u_3 of the Kuramoto—Sivashinsky equation by Wazzan coincides with known solution (7) at $c = \pm c_2$ and $\xi_0 = i\pi/2$.

Solution u_4 can be transformed using the set of equalities

$$\begin{gathered}
u_4 = \pm\frac{2c_2}{a}\frac{e^{\pm 2\zeta_4}\left(e^{\pm 4\zeta_4} + 9e^{\pm 2\zeta_4} + 12\right)}{\left(e^{\pm 2\zeta_4}-1\right)^3} = \\
= \pm\frac{2c_2}{a}\frac{e^{\pm 2\zeta_4}\left(e^{\pm 4\zeta_4} + 9e^{\pm 2\zeta_4} + 12\right)}{e^{\pm 6\zeta_4}\left(1-e^{\mp 2\zeta_4}\right)^3} = \mp\frac{2c_2}{a}\frac{12e^{\mp 4\zeta_4} + 9e^{\mp 2\zeta_4} + 1}{\left(e^{\mp 2\zeta_4}-1\right)^3} = u_3
\end{gathered} \tag{68}$$

We get that $u_3 = u_4$. So Wazzan solution (50) coincides with the known solution $u^{(2)}$ at $c = \pm c_2$ and $\xi_0 = i\pi/2$.

Taking into account the identity

$$\tanh 2\zeta_j - i\,\mathrm{sech}\,2\zeta_j = \tanh(\zeta_j - i\pi/4), \quad j = 3, 4 \tag{69}$$

we obtain that solution u_5 by Wazzan coincides with solution $u^{(1)}$ if we take $c = \pm c_1$ and $\xi_0 = i\pi/4$.

Using the formula

$$\mathrm{csch}\,2\zeta_j - \coth 2\zeta_j = -\tanh\zeta_j, \quad j = 3, 4 \tag{70}$$

we get that solution u_6 by Wazzan coincides with solution $u^{(1)}$ at $c = \pm c_1$ and $\xi_0 = 0$.

Taking the expression (69) into consideration we have that solution u_7 by Wazzan coincides with solution $u^{(2)}$ if $c = \pm c_2$ and $\xi_0 = i\pi/4$. With the aid of formula (70) we also obtain that solution u_8 by Wazzan coincides with solution $u^{(2)}$ at $c = \pm c_2$ and $\xi_0 = 0$.

Using the identity

$$(\tanh 2\zeta_j - i\,\mathrm{sech}\,2\zeta_j)^{-1} = \tanh(\zeta_j + i\pi/4), \quad j = 3, 4 \tag{71}$$

we have that solution u_9 by Wazzan coincides with solution $u^{(1)}$ if we take $c = \pm c_1$ and $\xi_0 = -i\pi/4$.

By means of the formulae

$$(\operatorname{csch} 2\zeta_j - \coth 2\zeta_j) = -\tanh(\zeta_j - i\pi/2), \quad j = 3, 4 \tag{72}$$

we obtain that solution u_{10} by Wazzan coincides with solution $u^{(1)}$ at $c = \pm c_1$ and $\xi_0 = i\pi/2$.

Taking expression (71) into account we get that solution u_{11} coincides with solution $u^{(2)}$ if we take $c = \pm c_2$ and $\xi_0 = -i\pi/4$. Using the identity (72) we obtain that solution u_{12} by Wazzan coincides with solution $u^{(2)}$ in the case $c = \pm c_2$ and $\xi_0 = i\pi/2$.

Taking into account the identities

$$(\tanh \zeta_j - i \operatorname{sech} \zeta_j) + (\tanh \zeta_j - i \operatorname{sech} \zeta_j)^{-1} = 2 \tanh \zeta_j, \quad j = 3, 4, \tag{73}$$

$$\begin{aligned}(\tanh \zeta_j - i \operatorname{sech} \zeta_j)^3 + (\tanh \zeta_j - i \operatorname{sech} \zeta_j)^{-3} = \\ = 8 \tanh^3 \zeta_j - 6 \tanh \zeta_j, \quad j = 3, 4\end{aligned} \tag{74}$$

we have that solution u_{13} by Wazzan coincides with $u^{(1)}$ if we take $c = \pm c_1$ and $\xi_0 = 0$.

Taking into account the identities

$$(\operatorname{csch} \zeta_j - \coth \zeta_j) + (\operatorname{csch} \zeta_j - \coth \zeta_j)^{-1} = -2 \tanh(\zeta_j + i\pi/2), \quad j = 3, 4, \tag{75}$$

$$\begin{aligned}(\operatorname{csch} \zeta_j - \coth \zeta_j)^3 + (\operatorname{csch} \zeta_j - \coth \zeta_j)^{-3} = \\ = -8 \tanh(\zeta_j - i\pi/2) + 6 \tanh(\zeta_j - i\pi/2), \quad j = 3, 4\end{aligned} \tag{76}$$

we obtain that solution u_{14} coincides with $u^{(1)}$ at $c = \pm c_1$ and $\xi_0 = i\pi/2$.

Using the expressions (73)–(74) we find that solution u_{15} by Wazzan is equal to solution $u^{(2)}$ if we take $c = \pm c_2$ and $\xi_0 = 0$.

By means of formulae (75)–(76) we obtain that solution u_{16} by Wazzan is equal to solution $u^{(2)}$ in the case $c = \pm c_2$ and $\xi_0 = i\pi/2$.

So we have proved that all Wazzan solutions are not new. All his solutions are the special cases of the known solutions $u^{(1)}$ and $u^{(2)}$ with different values of free parameters c and ξ_0.

Note that some Wazzan solutions are equal to each other. From the above mentioned formulae we can see that there are obvious equalities $u_1 = u_2 = u_{10} = u_{14}$, $u_3 = u_4 = u_{12} = u_{16}$, $u_6 = u_{13}$, $u_8 = u_{15}$.

6 Conclusion

Let us shortly formulate the results of this paper. We can see that the ansatz methods used by different authors [31–33] do not give any new solutions of the Kuramoto—Sivashinsky equation. The main reason is that all these methods use the same meromorphic structure of solutions. Therefore due to Eremenko theorem these methods cannot give new results.

The authors of papers discussed in our work were unable to reconstruct known solutions with all necessary arbitrary constants. This leads to the great number of special solutions either without arbitrary parameters. One can construct infinitely many particular solutions in such a way. But this "achievement" shows misunderstanding of fundamental properties of differential equations.

We shall also note that all discussed methods (may be except the truncated expansion method) work well only in the case of ordinary differential equations. Theory of ordinary differential equations has more then three hundred years history. So the expectations of new results obtained by simple algebraic substitutions in the case of well-known equations are at least naive.

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