# Comment on: "Multi soliton solution, rational solution of the Boussinesq-Burgers equations"

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**Abstract**

We demonstrate that all "new" exact solutions of the Boussinesq - Burgers equations by Rady A.S.A., Osman, E.S., Khalfallah M., Communications in Nonlinear Science and Numerical Simulation (2009) doi:10.10016/j.cnsns.2009.05.053] are well known and were obtained many years ago.

In work [1] Rady, Osman and Khalfallah have found multi soliton solution, rational solution and "new trigonometric function periodic solutions" of the Boussinesq-Burgers equations

$$u_t + 2uu_x - \frac{1}{2}\,v_x = 0, \tag{1}$$

$$v_t + 2(uv)_x - \frac{1}{2}\,u_{xxx} = 0. \tag{2}$$

It is known [2] that the system of equations (1), (2) has the Lax pair and the Cauchy problem for this system can be solved by the inverse scattering transform. Nevertheless the authors [1] decided to consider this system again using the peculiar approach. They do not study the system of Eqs.(1),(2) but use the additional condition

$$v = -u_x. \tag{3}$$

Assuming Eq.(3) the authors believe that they still considered the system of equations (1) - (2) but this is not the case. As result of the condition (3) is that the authors obtained the system of equations which is equivalent to the well known Burgers equation with negative viscosity [3, 4]

$$u_t + 2uu_x + \frac{1}{2}\,u_{xx} = 0. \tag{4}$$

So Rady, Osman and Khalfallah in the work [1] studied the Burgers equation and presented all results for this equation. However the authors [1] did not give the name of this equation and they did not present references for many results corresponding to this equation.

Eq.(4) was first studied by Batemann [3] but we know this equation in periodic literature as the Burgers equation [4]. There is the remarkable transformation that is the Cole-Hopf transformation for the Eq.(4) [5, 6]

$$u = \frac{1}{2}\,\frac{\partial \ln F}{\partial x}. \tag{5}$$

Taking the transformation (5) into account we have for Eq.(4) the following relation

$$u_t + 2uu_x + \frac{1}{2}\,u_{xx} = \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{1}{F}(F_t + \frac{1}{2}\,F_{xx})\right). \qquad (6)$$

The impressive result of relation (6) is that all solutions of the heat equation

$$F_t + \frac{1}{2}\,F_{xx} = 0 \qquad (7)$$

give solutions of Eq.(4) by means of formula (5). Many solutions of the linear heat equation (7) can be found in textbooks on mathematical physics. Therefore finding special solutions or solving the Cauchy problem for the Burgers equation is not too difficult (see, for example, chapter IV in the book of Whitham [7]).

Let us demonstrate that the paper [1] does not contain new results. The authors [1] claim that they "obtain an auto-Backlund transformation and abundant new exact solutions, including the multi-solitary wave solution and the rational series solutions. Besides the new trigonometric function periodic solutions are obtained by using the generalized tanh method".

The first result in [1] is the auto-Backlund transformation for the Burgers equation

$$u = \frac{1}{2}\,\frac{\partial \ln w}{\partial x} + u_0, \qquad (8)$$

$$u_{0t} + 2\,u_0\,u_{0x} + \frac{1}{2}\,u_{0xx} = 0, \qquad (9)$$

$$w_t + 2\,u_0\,w_x + \frac{1}{2}\,w_{xx} = 0. \qquad (10)$$

However transformations (8) - (10) were found thirty years ago in [8] and rediscovered in [9–11] using other approaches for the Burgers hierarchy. The authors [1] do not give any references.

Using transformations (8) - (10) the authors [1] want to obtain "abundant new exact solutions" but they assume $u_0 = a = Const$. However they did not note that in this case the the auto-Baclund transformation (8) - (10) is transformed by means of $w(x,t) = F(x,t)\,e^{2a(at-x)}$ into formulae (5) and (7).

Solving the linear equation with respect to $w$ at $u_0 = a$ Abdel Rady, Osman and Khalfallah [1] obtain "the one-soliton solutions" (17), "the multi-soliton solution" (19) and "the rational solution" (23) (all number for formulae are taken from [1]). All these solutions are not new and it was pointed out in [1].

The last result by Rady, Osman and Khalfallah is "the new trigonometric function periodic solution" (formula (29) [1]). This solution is given in book [12]. To obtain this solution the authors [1] use the traveling wave $\zeta = k\,x - c\,t$ and solve the equation

$$-cu^{'} + 2kuu^{'} + \frac{1}{2}\,k^2\,u^{''} = 0. \qquad (11)$$

Instead integration of equation (11) with respect to $\zeta$ (we obtain

$$c_2 - c\,u + k\,u^2 + \frac{k^2}{2}\,u^{'} = 0, \qquad (12)$$

where $c_2$ is a constant of integration) the authors [1] apply the homogeneous balance method and look for solution of (11) as solution in the form of polynomial

$u = a_0 + a_1\,\varphi$ for the Riccati equation

$$\varphi^{'} = l(1+\varphi^2). \qquad (13)$$

It should be noted that the the authors [1] looked for solutions of the Riccati equation (12) using the Riccati equation (13). Results of calculations is expressed by the tangent but the authors [1] call this function "a sinusoidal type periodical solution" (formula (29) in [1]). They also presented these solutions on figures 1 and 2 as periodical but we can see that there are not periodicity on figures at all.

We can see that the authors [1] have done the typical errors (see [13–15]).

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