Contact Transformations for ODEs

1. General Form of Contact Transformations

A contact transformation has the form

$$x = F(X, Y, Y'_X),\quad y = G(X, Y, Y'_X), \tag{1}$$

where the functions $F(X, Y, U)$ and $G(X, Y, U)$ are chosen so that the derivative y'_x does not depend on Y''_{XX}:

$$y'_x = \frac{y'_X}{x'_X} = \frac{G_X + G_Y Y'_X + G_U Y''_{XX}}{F_X + F_Y Y'_X + F_U Y''_{XX}} = H(X, Y, Y'_X). \tag{2}$$

The subscripts X, Y, and U after F and G denote the respective partial derivatives (it is assumed that $F_U \not\equiv 0$ and $G_U \not\equiv 0$).

It follows from (2) that the relation

$$\frac{\partial G}{\partial U}\left(\frac{\partial F}{\partial X} + U\frac{\partial F}{\partial Y}\right) - \frac{\partial F}{\partial U}\left(\frac{\partial G}{\partial X} + U\frac{\partial G}{\partial Y}\right) = 0 \tag{3}$$

holds; the derivative is calculated by

$$y'_x = \frac{G_U}{F_U}, \tag{4}$$

where $G_U/F_U \not\equiv \text{const}$.

The application of contact transformations preserves the order of differential equations. The inverse of a contact transformation can be obtained by solving system (1) and (4) for X, Y, Y'_X.

2. Method for the Construction of Contact Transformations

Suppose the function $F = F(X, Y, U)$ in the contact transformation (1) is specified. Then relation (3) can be viewed as a linear partial differential equation for the second function G. The corresponding characteristic system of ordinary differential equations (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002),

$$\frac{dX}{1} = \frac{dY}{U} = -\frac{F_U\,dU}{F_X + UF_Y}$$

admits the obvious first integral:

$$F(X, Y, U) = C_1, \tag{5}$$

where C_1 is an arbitrary constant. It follows that, to obtain the general representation of the function $G = G(X, Y, U)$, one has to deal with the ordinary differential equation

$$Y'_X = U, \tag{6}$$

whose right-hand side is defined in implicit form by (5). Let the first integral of equation (6) has the form

$$\Phi(X, Y, C_1) = C_2.$$

Then the general representation of $G = G(X, Y, U)$ in transformation (1) is given by:

$$G = \Psi(F, \widetilde{\Phi}),$$

where $\Psi(F, \widetilde{\Phi})$ is an arbitrary function of two variables, $F = F(X, Y, U)$, and $\widetilde{\Phi} = \Phi(X, Y, F)$.

3. Examples of Contact Transformations

Example 1. Legendre transformation:

$$x = Y'_X, \quad y = XY'_X - Y, \quad y'_x = X \quad \text{(direct transformation)};$$
$$X = y'_x, \quad Y = xy'_x - y, \quad Y'_X = x \quad \text{(inverse transformation)}.$$

Example 2. Contact transformation ($a \neq 0$):

$$x = Y'_X + aY, \quad y = be^{aX}Y'_X, \quad y'_x = be^{aX} \quad \text{(direct transformation)};$$
$$X = \frac{1}{a}\ln\frac{y'_x}{b}, \quad Y = \frac{1}{a}\left(x - \frac{y}{y'_x}\right), \quad Y'_X = \frac{y}{y'_x} \quad \text{(inverse transformation)}.$$

Example 3. Contact transformation:

$$x = Y'_X + aX, \quad y = \tfrac{1}{2}(Y'_X)^2 + aY, \quad y'_x = Y'_X \quad \text{(direct transformation)};$$
$$X = \frac{1}{a}\big(x - y'_x\big), \quad Y = \frac{1}{2a}\big[2y - (y'_x)^2\big], \quad Y'_X = y'_x \quad \text{(inverse transformation)}.$$

Example 4. Contact transformation:

$$x = (Y'_X)^2 - Y^2, \quad y = Y'_X \cosh X - Y \sinh X, \quad y'_x = \frac{\cosh X}{2Y'_X}.$$

Example 5. Contact transformation ($ab \neq 0$):

$$x = a(Y'_X)^2 - bX, \quad y = 2a(Y'_X)^3 - 3bY, \quad y'_x = 3Y'_X \quad \text{(direct transformation)};$$
$$X = \frac{a}{9b}(y'_x)^2 - \frac{1}{b}x, \quad Y = \frac{2a}{81b}(y'_x)^3 - \frac{1}{3b}y, \quad Y'_X = \frac{1}{3}y'_x \quad \text{(inverse transformation)}.$$

References

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for ODEs, Second Edition*, Chapman & Hall/CRC, Boca Raton, 2003.

Polyanin, A. D., Zaitsev, V. F., and Moussiaux, A., *Handbook of First Order PDEs*, Taylor & Francis, London, 2002.

Zwillinger, D., *Handbook of Differential Equations*, Academic Press, San Diego, 1989.