Contact Transformations for ODEs

1. General Form of Contact Transformations

A contact transformation has the form

\[
\begin{align*}
    x &= F(X, Y, Y_X'), \\
    y &= G(X, Y, Y_X'),
\end{align*}
\]

where the functions \( F(X, Y, U) \) and \( G(X, Y, U) \) are chosen so that the derivative \( y'_x \) does not depend on \( Y''_{XX} \):

\[
y'_x = \frac{y'_X}{x'_X} = \frac{G_X + G_Y Y'_X + G_U Y''_{XX}}{F_X + F_Y Y'_X + F_U Y''_{XX}} = H(X, Y, Y'_X).
\]

The subscripts \( X, Y, \) and \( U \) after \( F \) and \( G \) denote the respective partial derivatives (it is assumed that \( F_U \neq 0 \) and \( G_U \neq 0 \)).

It follows from (2) that the relation

\[
\frac{\partial G}{\partial U} \left( \frac{\partial F}{\partial X} + U \frac{\partial F}{\partial Y} \right) - \frac{\partial F}{\partial U} \left( \frac{\partial G}{\partial X} + U \frac{\partial G}{\partial Y} \right) = 0
\]

holds; the derivative is calculated by

\[
y'_x = \frac{G_U}{F_U},
\]

where \( G_U/F_U \neq \text{const} \).

The application of contact transformations preserves the order of differential equations. The inverse of a contact transformation can be obtained by solving system (1) and (4) for \( X, Y, Y'_X \).

2. Method for the Construction of Contact Transformations

Suppose the function \( F = F(X, Y, U) \) in the contact transformation (1) is specified. Then relation (3) can be viewed as a linear partial differential equation for the second function \( G \). The corresponding characteristic system of ordinary differential equations (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002),

\[
\frac{dX}{1} = \frac{dY}{U} = -\frac{F_U}{F_X} \frac{dU}{F_Y}
\]

admits the obvious first integral:

\[
F(X, Y, U) = C_1,
\]

where \( C_1 \) is an arbitrary constant. It follows that, to obtain the general representation of the function \( G = G(X, Y, U) \), one has to deal with the ordinary differential equation

\[
Y'_X = U,
\]

whose right-hand side is defined in implicit form by (5). Let the first integral of equation (6) has the form

\[
\Phi(X, Y, C_1) = C_2.
\]

Then the general representation of \( G = G(X, Y, U) \) in transformation (1) is given by:

\[
G = \Psi(F, \tilde{\Phi}),
\]

where \( \Psi(F, \tilde{\Phi}) \) is an arbitrary function of two variables, \( F = F(X, Y, U) \), and \( \tilde{\Phi} = \Phi(X, Y, F) \).
3. Examples of Contact Transformations

Example 1. Legendre transformation:
\[ x = Y_X, \quad y = X Y'_X - Y, \quad y'_x = X \] (direct transformation);
\[ X = y'_x, \quad Y = x y'_x - y, \quad Y'_X = x \] (inverse transformation).

Example 2. Contact transformation \((a \neq 0)\):
\[ x = Y_X + a Y, \quad y = b e^{a X} Y'_X, \quad y'_x = b e^{a X} \] (direct transformation);
\[ X = \frac{1}{a} \ln \frac{y'_x}{b}, \quad Y = \frac{1}{a} \left( x - \frac{y}{y'_x} \right), \quad Y'_X = \frac{y}{y'_x} \] (inverse transformation).

Example 3. Contact transformation:
\[ x = Y_X + a X, \quad y = \frac{1}{4} (Y_X)^2 + a Y, \quad y'_x = Y'_X \] (direct transformation);
\[ X = \frac{1}{a} (x - y'_x), \quad Y = \frac{1}{4a} \left[ 2y - (y'_x)^2 \right], \quad Y'_X = y'_x \] (inverse transformation).

Example 4. Contact transformation:
\[ x = (Y'_X)^2 - Y^2, \quad y = Y'_X \cos X - Y \sinh X, \quad y'_x = \frac{\cosh X}{2Y'_X} \]

Example 5. Contact transformation \((a b \neq 0)\):
\[ x = a(Y'_X)^2 - b X, \quad y = 2a(Y'_X)^3 - 3b Y, \quad y'_x = 3Y'_X \] (direct transformation);
\[ X = \frac{a}{9b} (y'_x)^2 - \frac{1}{b} x, \quad Y = \frac{2a}{81b} (y'_x)^3 - \frac{1}{3b} y, \quad Y'_X = \frac{1}{3} y'_x \] (inverse transformation).

References