Mathematical Excalibur

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Olympiad Corner

The Fifth Hong Kong (China) Mathematical Olympiad was held on December 21, 2002. The problems are as follow.

Problem 1. Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

(a) Prove that the tangent to the second circle at R is parallel to AK.

(b) Prove that these two tangents are concurrent with KM.

Problem 2. Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of the n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Functional Equations

Kin Y. Li

A *functional equation* is an equation whose variables are ranging over functions. Hence, we are seeking all possible functions satisfying the equation. We will let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}^+$ or $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0$ denote the nonnegative integers, $\mathbb{Q}$ denote the rational numbers, $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{C}$ denote the complex numbers.

In simple cases, a functional equation can be solved by introducing some substitutions to yield more information or additional equations.

Example 1. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1-x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution. Replacing x by $1 - x$, we have

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4.$$

Since $f(1-x) = 2x - x^4 - x^2 f(x)$ by the given equation, substituting this into the last equation and solving for $f(x)$, we get $f(x) = 1 - x^2$.

Check: For $f(x) = 1 - x^2$,

$$x^2 f(x) + f(1-x) = x^2(1-x^2) + (1-(1-x)^2) = 2x - x^4.$$

For certain types of functional equations, a standard approach to solving the problem is to determine some special values (such as $f(0)$ or $f(1)$), then inductively determine $f(n)$ for $n \in \mathbb{N}_0$, follow by the values $f(1/n)$ and use density to find $f(x)$ for all $x \in \mathbb{R}$. The following are examples of such approach.

Example 2. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that the *Cauchy equation*

$$f(x+y) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{Q}$.

Solution. *Step 1* Taking $x = 0 = y$, we get $f(0) = f(0) + f(0) + f(0)$, which implies $f(0) = 0$.

Step 2 We will prove $f(kx) = k f(x)$ for $k \in \mathbb{N}$, $x \in \mathbb{Q}$ by induction. This is true for $k = 1$. Assume this is true for k. Taking $y = kx$, we get

$$f((k+1)x) = f(x + kx) = f(x) + f(kx) = f(x) + k f(x) = (k+1) f(x).$$

Step 3 Taking $y = -x$, we get

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x),$$

which implies $f(-x) = -f(x)$. So

$$f(-kx) = -f(kx) = -k f(x) \text{ for } k \in \mathbb{N}.$$

Therefore, $f(kx) = k f(x)$ for $k \in \mathbb{Z}$, $x \in \mathbb{Q}$.

Step 4 Taking $x = 1/k$, we get

$$f(1) = f(k(1/k)) = k f(1/k),$$

which implies $f(1/k) = (1/k) f(1)$.

Step 5 For $m \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$f(m/n) = m f(1/n) = (m/n) f(1).$$

Therefore, $f(x) = cx$ with $c = f(1)$.

Check: For $f(x) = cx$ with $c \in \mathbb{Q}$,

$$f(x+y) = c(x+y) = cx + cy = f(x) + f(y).$$

In dealing with functions on $\mathbb{R}$, after finding the function on $\mathbb{Q}$, we can often finish the problem by using the following fact.

Density of Rational Numbers *For every real number x, there are rational numbers $p_1, p_2, p_3, \ldots$ increase to x and there are rational numbers $q_1, q_2, q_3, \ldots$ decrease to x.*

This can be easily seen from the decimal representation of real numbers. For example, the number $\pi = 3.1415\ldots$ is the limits of 3, 31/10, 314/100, 3141/1000, 31415/10000, … and also 4, 32/10, 315/100, 3142/1000, 31416/10000, ….

(In passing, we remark that there is a similar fact with rational numbers replaced by irrational numbers.)

Example 3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in \mathbb{R}$ and $f(x)\ge 0$ for $x\ge 0$.

Solution. Step 1 By example 2, we have $f(x)=xf(1)$ for $x\in\mathbb{Q}$.

Step 2 If $x\ge y$, then $x-y\ge 0$. So

$$f(x)=f((x-y)+y)=f(x-y)+f(y)\ge f(y).$$

Hence, f is increasing.

Step 3 If $x\in\mathbb{R}$, then by the density of rational numbers, there are rational p_n, q_n such that $p_n\le x\le q_n$, the p_n's increase to x and the q_n's decrease to x. So by step 2, $p_n f(1)=f(p_n)\le f(x)\le f(q_n)=q_n f(1)$. Taking limits, the sandwich theorem gives $f(x)=xf(1)$ for all x. Therefore, $f(x)=cx$ with $c\ge 0$. The checking is as in example 2.

Remarks. (1) In example 3, if we replace the condition that "$f(x)\ge 0$ for $x\ge 0$" by "f is monotone", then the answer is essentially the same, namely $f(x)=cx$ with $c=f(1)$. Also if the condition that "$f(x)\ge 0$ for $x\ge 0$" is replaced by "f is continuous at 0", then steps 2 and 3 in example 3 are not necessary. We can take rational p_n's increase to x and take limit of $p_n f(1)=f(p_n)=f(p_n-x)+f(x)$ to get $xf(1)=f(x)$ since p_n-x increases to 0.

(2) The Cauchy equation $f(x+y)=f(x)+f(y)$ for all $x, y\in\mathbb{R}$ has noncontinuous solutions (in particular, solutions not of the form $f(x)=cx$). This requires the concept of a *Hamel basis* of the vector space $\mathbb{R}$ over $\mathbb{Q}$ from linear algebra.

The following are some useful facts related to the Cauchy equation.

Fact 1. *Let $A=\mathbb{R}$, $[0,\infty)$ or $(0,\infty)$. If $f:A\to\mathbb{R}$ satisfies $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$ for all $x, y\in A$, then either $f(x)=0$ for all $x\in A$ or $f(x)=x$ for all $x\in A$.*

Proof. By example 2, we have $f(x)=f(1)x$ for all $x\in\mathbb{Q}$. If $f(1)=0$, then $f(x)=f(x\cdot 1)=f(x)f(1)=0$ for all $x\in A$.

Otherwise, we have $f(1)\ne 0$. Since $f(1)=f(1)f(1)$, we get $f(1)=1$. Then $f(x)=x$ for all $x\in A\cap\mathbb{Q}$.

If $y\ge 0$, then $f(y)=f(y^{1/2})^2\ge 0$ and

$$f(x+y)=f(x)+f(y)\ge f(x),$$

which implies f is increasing. Now for any $x\in A\setminus\mathbb{Q}$, by the density of rational numbers, there are $p_n, q_n\in\mathbb{Q}$ such that $p_n<x<q_n$, the p_n's increase to x and the q_n's decrease to x. As f is increasing, we have $p_n=f(p_n)\le f(x)\le f(q_n)=q_n$. Taking limits, the sandwich theorem gives $f(x)=x$ for all $x\in A$.

Fact 2. *If a function $f:(0,\infty)\to\mathbb{R}$ satisfies $f(xy)=f(x)f(y)$ for all $x, y>0$ and f is monotone, then either $f(x)=0$ for all $x>0$ or there exists c such that $f(x)=x^c$ for all $x>0$.*

Proof. For $x>0$, $f(x)=f(x^{1/2})^2\ge 0$. Also $f(1)=f(1)f(1)$ implies $f(1)=0$ or 1. If $f(1)=0$, then $f(x)=f(x)f(1)=0$ for all $x>0$. If $f(1)=1$, then $f(x)>0$ for all $x>0$ (since $f(x)=0$ implies $f(1)=f(x(1/x))=f(x)f(1/x)=0$, which would lead to a contradiction).

Define $g:\mathbb{R}\to\mathbb{R}$ by $g(w)=\ln f(e^w)$. Then

$$\begin{aligned} g(x+y) &= \ln f(e^{x+y}) = \ln f(e^x e^y)\\ &= \ln f(e^x)f(e^y)\\ &= \ln f(e^x)+\ln f(e^y)\\ &= g(x)+g(y). \end{aligned}$$

Since f is monotone, it follows that g is also monotone. Then $g(w)=cw$ for all w. Therefore, $f(x)=x^c$ for all $x>0$.

As an application of these facts, we look at the following example.

Example 4. (*2002 IMO*) Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\begin{aligned} &(f(x)+f(z))(f(y)+f(t))\\ &= f(xy-zt)+f(xt+yz) \end{aligned}$$

for all x, y, z, t in $\mathbb{R}$.

Solution. (*Due to Yu Hok Pun, 2002 Hong Kong IMO team member, gold medalist*) Suppose $f(x)=c$ for all x. Then the equation implies $4c^2=2c$. So c can only be 0 or 1/2. Reversing steps, we can also check $f(x)=0$ for all x or $f(x)=1/2$ for all x are solutions.

Suppose the equation is satisfied by a nonconstant function f. Setting $x=0$ and $z=0$, we get $2f(0)(f(y)+f(t))=2f(0)$, which implies $f(0)=0$ or $f(y)+f(t)=1$ for all y, t. In the latter case, setting $y=t$, we get the constant function $f(y)=1/2$ for all y. Hence we may assume $f(0)=0$.

Setting $y=1, z=0, t=0$, we get $f(x)f(1)=f(x)$. Since $f(x)$ is not the zero function, $f(1)=1$. Setting $z=0$, $t=0$, we get $f(x)f(y)=f(xy)$ for all x,y. In particular, $f(w)=f(w^{1/2})^2\ge 0$ for $w>0$.

Setting $x=0$, $y=1$ and $t=1$, we have $2f(1)f(z)=f(-z)+f(z)$, which implies $f(z)=f(-z)$ for all z. So f is even.

Define the function $g:(0,\infty)\to\mathbb{R}$ by $g(w)=f(w^{1/2})\ge 0$. Then for all $x,y>0$,

$$\begin{aligned} g(xy) &= f((xy)^{1/2}) = f(x^{1/2}y^{1/2})\\ &= f(x^{1/2})f(y^{1/2}) = g(x)g(y). \end{aligned}$$

Next f is even implies $g(x^2)=f(x)$ for all x. Setting $z=y$, $t=x$ in the given equation, we get

$$\begin{aligned} (g(x^2)+g(y^2))^2 &= g((x^2+y^2)^2)\\ &= g(x^2+y^2)^2 \end{aligned}$$

for all x,y. Taking square roots and letting $a=x^2$, $b=y^2$, we get $g(a)+g(b)=g(a+b)$ for all $a, b>0$.

By fact 1, we have $g(w)=w$ for all $w>0$. Since $f(0)=0$ and f is even, it follows $f(x)=g(x^2)=x^2$ for all x.

Check: If $f(x)=x^2$, then the equation reduces to

$$(x^2+z^2)(y^2+t^2)=(xy-zt)^2+(xt+yz)^2,$$

which is a well known identity and can easily be checked by expansion or seen from $|p|^2|q|^2=|pq|^2$, where $p=x+iz$, $q=y+it\in\mathbb{C}$.

The concept of fixed point of a function is another useful idea in solving some functional equations. Its definition is very simple. We say w is a fixed point of a function f if and only if w is in the domain of f and $f(w)=w$. Having information on the fixed points of functions often help to solve certain types of functional equations as the following examples will show.

Example 5. (*1983 IMO*) Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f(xf(y))=yf(x)$ for all $x, y\in\mathbb{R}^+$ and as $x\to+\infty$, $f(x)\to 0$.

Solution. Step 1 Taking $x=1=y$, we get $f(f(1))=f(1)$. Taking $x=1$ and $y=f(1)$, we get $f(f(f(1)))=f(1)^2$. Then $f(1)^2=f(f(f(1)))=f(f(1))=f(1)$, which implies $f(1)=1$. So 1 is a fixed point of f.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***February 28, 2003***.

Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right]-\left[\frac{c}{a}\right]\left[\frac{c}{b}\right]\le c\min\left\{\frac{1}{a},\frac{1}{b}\right\}.$$

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n + 1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[(n + 1) / 2]$ triangles, each containing at least one frog.

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Solutions

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In the last issue, problems 166, 167 and 169 were stated incorrectly. They are revised as problems 171, 172, 173, respectively. As the problems became easy due to the mistakes, we received many solutions. Regretfully we do not have the space to print the names and affiliations of all solvers. We would like to apologize for this.

Problem 166. Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\,[a/b] - [c/a]\,[c/b] \le c\min\{1/a, 1/b\}.$$

Solution. Over 30 solvers disproved the inequality by providing different counter-examples, such as $(a, b, c) = (3, 2, 1)$.

Problem 167. Find all positive integers such that they are equal to the sum of their digits in base 10 representation.

Solution. Over 30 solvers sent in solutions similar to the following. For a positive integer N with digits $a_n, \ldots, a_0$ (from left to right), we have

$$N = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0 \ge a_n + a_{n-1} + \cdots + a_0$$

because $10^k > 1$ for $k > 0$. So equality holds if and only if $a_n = a_{n-1} = \cdots = a_1 = 0$. Hence, $N = 1, 2, \ldots, 9$ are the only solutions.

Problem 168. Let AB and CD be nonintersecting chords of a circle and let K be a point on CD. Construct (with straightedge and compass) a point P on the circle such that K is the midpoint of the part of segment CD lying inside triangle ABP. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7)

Draw the midpoint M of AB. If $AB \parallel CD$, then draw ray MK to intersect the circle at P. Let AP, BP intersect CD at Q,R, respectively. Since $AB \parallel QR$, $\Delta ABP \sim \Delta QRP$. Then M being the midpoint of AB will imply K is the midpoint of QR.

If AB intersects CD at E, then draw the circumcircle of EMK meeting the original circle at S and S'. Draw the circumcircle of BES meeting CD at R. Draw the circumcircle of AES meeting CD at Q. Let AQ, BR meet at P. Since $\angle PBS = \angle RBS = \angle RES = \angle QES = \angle QAS = \angle PAS$, P is on the original circle.

Next, $\angle SMB = \angle SME = \angle SKE = \angle SKR$ and $\angle SBM = 180^\circ - \angle SBE = 180^\circ - \angle SRE = \angle SRK$ imply $\Delta SMB \sim \Delta SKR$ and $MB/KR = BS/RS$. Replacing M by A and K by Q, similarly $\Delta SAB \sim \Delta SQR$ and $AB/QR = BS/RS$. Since $AB = 2MB$, we get $QR = 2KR$. So K is the midpoint of QR.

Problem 169. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 11/2 times any other group.

Solution. Almost all solvers used the following argument. Let m and M be the weights of the lightest and heaviest apple(s). Then $3m \ge M$. If the problem is false, then there are two groups A and B with weights w_A and w_B such that $(11/2)\,w_B < w_A$. Since $4m \le w_B$ and $w_A \le 4M$, we get $(11/2)4m < 4M$ implying $3m \le (11/2)m < M$, a contradiction.

Problem 170. (*Proposed by Abderrahim Ouardini, Nice, France*) For any (nondegenerate) triangle with sides a, b, c, let $\sum' h(a, b, c)$ denote the sum $h(a, b, c) + h(b, c, a) + h(c, a, b)$. Let $f(a, b, c) = \sum' (a/(b + c - a))^2$ and $g(a, b, c) = \sum' j(a, b, c)$, where $j(a,b,c) = (b + c - a)/\sqrt{(c+a-b)(a+b-c)}$. Show that $f(a, b, c) \ge \max\{3, g(a, b, c)\}$ and determine when equality occurs. (Here $\max\{x,y\}$ denotes the maximum of x and y.)

Solution. **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 6), **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (SKH Lam Kau Mow Secondary School) and **WONG Wing Hong** (La Salle College, Form 5).

Let $x = b + c - a$, $y = c + a - b$ and $z = a + b - c$. Then $a = (y + z)/2$, $b = (z + x)/2$ and $c = (x + y)/2$.

Substituting these and using the *AM-GM* inequality, the rearrangement inequality and the *AM-GM* inequality again, we find

$$f(a,b,c)$$

$$= \left(\frac{y+z}{2x}\right)^2 + \left(\frac{z+x}{2y}\right)^2 + \left(\frac{x+y}{2z}\right)^2$$

$$\ge \left(\frac{\sqrt{yz}}{x}\right)^2 + \left(\frac{\sqrt{zx}}{y}\right)^2 + \left(\frac{\sqrt{xy}}{z}\right)^2$$

$$\geq \frac{\sqrt{yz}\sqrt{zx}}{xy}+\frac{\sqrt{zx}\sqrt{xy}}{yz}+\frac{\sqrt{xy}\sqrt{yz}}{zx}$$

$$=\frac{x}{\sqrt{yz}}+\frac{y}{\sqrt{zx}}+\frac{z}{\sqrt{xy}}=g(a,b,c)$$

$$\geq 3\sqrt[3]{\frac{xyz}{\sqrt{yz}\sqrt{zx}\sqrt{xy}}}=3.$$

So $f(a,b,c)\geq g(a,b,c)=\max\{3,g(a,b,c)\}$ with equality if and only if $x=y=z$, which is the same as $a=b=c$.

Olympiad Corner

(continued from page 1)

Problem 3. If $a \geq b \geq c \geq 0$ and $a+b+c=3$, then prove that $ab^2+bc^2+ca^2 \leq 27/8$ and determine the equality case(s).

Problem 4. Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate (with reason)

$$\sum_{k=1}^{\frac{p-1}{2}}\left\{\frac{k^2}{p}\right\},$$

where $\{x\}=x-[x]$, $[x]$ being the greatest integer not exceeding x.

Functional Equations

(continued from page 2)

Step 2 Taking $y=x$, we get $f(xf(x))=xf(x)$. So $w=xf(x)$ is a fixed point of f for every $x\in\mathbb{R}^+$.

Step 3 Suppose f has a fixed point $x>1$. By step 2, $xf(x)=x^2$ is also a fixed point, $x^2f(x^2)=x^4$ is also a fixed point and so on. So the x^m's are fixed points for every m that is a power of 2. Since $x>1$, for m ranging over the powers of 2, we have $x^m\to\infty$, but $f(x^m)=x^m\to\infty$, not to 0. This contradicts the given property. Hence, f cannot have any fixed point $x>1$.

Step 4 Suppose f has a fixed point x in the interval (0,1). Then

$$1=f((1/x)x)=f((1/x)f(x))=xf(1/x),$$

which implies $f(1/x)=1/x$. This will lead to f having a fixed point $1/x>1$, contradicting step 3. Hence, f cannot have any fixed point x in (0,1).

Step 5 Steps 1, 3, 4 showed the only fixed point of f is 1. By step 2, we get $xf(x)=1$ for all $x\in\mathbb{R}^+$. Therefore, $f(x)=1/x$ for all $x\in\mathbb{R}^+$.

Check: For $f(x)=1/x$, $f(xf(y))=f(x/y)=y/x=yf(x)$. As $x\to\infty$, $f(x)=1/x\to 0$.

Example 6. (1996 IMO) Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that

$$f(m+f(n))=f(f(m))+f(n)$$

for all $m,n\in\mathbb{N}_0$.

Solution. Step 1 Taking $m=0=n$, we get $f(f(0))=f(f(0))+f(0)$, which implies $f(0)=0$. Taking $m=0$, we get $f(f(n))=f(n)$, i.e. $f(n)$ is a fixed point of f for every $n\in\mathbb{N}_0$. Also the equation becomes

$$f(m+f(n))=f(m)+f(n).$$

Step 2 If w is a fixed point of f, then we will show kw is a fixed point of f for all $k\in\mathbb{N}_0$. The cases $k=0,1$ are known. If kw is a fixed point, then $f((k+1)w)=f(kw+w)=f(kw)+f(w)=kw+w=(k+1)w$ and so $(k+1)w$ is also a fixed point.

Step 3 If 0 is the only fixed point of f, then $f(n)=0$ for all $n\in\mathbb{N}_0$ by step 1. Obviously, the zero function is a solution.

Otherwise, f has a least fixed point $w>0$. We will show the only fixed points are kw, $k\in\mathbb{N}_0$. Suppose x is a fixed point. By the division algorithm, $x=kw+r$, where $0\leq r<w$. We have

$$\begin{aligned}x&=f(x)=f(r+kw)=f(r+f(kw))\\&=f(r)+f(kw)=f(r)+kw.\end{aligned}$$

So $f(r)=x-kw=r$. Since w is the least positive fixed point, $r=0$ and $x=kw$.

Since $f(n)$ is a fixed point for all $n\in\mathbb{N}_0$ by step 1, $f(n)=c_nw$ for some $c_n\in\mathbb{N}_0$. We have $c_0=0$.

Step 4 For $n\in\mathbb{N}_0$, by the division algorithm, $n=kw+r$, $0\leq r<w$. We have

$$\begin{aligned}f(n)&=f(r+kw)=f(r+f(kw))\\&=f(r)+f(kw)=c_rw+kw\\&=(c_r+k)w=(c_r+[n/w])w.\end{aligned}$$

Check: For each $w>0$, let $c_0=0$ and let $c_1,\ldots,c_{w-1}\in\mathbb{N}_0$ be arbitrary. The function $f(n)=(c_r+[n/w])w$, where r is the remainder of n divided by w, (and the zero function) are all the solutions. Write $m=kw+r$ and $n=lw+s$ with $0\leq r,s<w$. Then

$$\begin{aligned}f(m+f(n))&=f(r+kw+(c_s+l)w)\\&=c_rw+kw+c_sw+lw\\&=f(f(m))+f(n).\end{aligned}$$

Other than the fixed point concept, in solving functional equations, the injectivity and surjectivity of the functions also provide crucial informations.

Example 7. (*1987 IMO*) Prove that there is no function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(f(n))=n+1987$.

Solution. Suppose there is such a function f. Then f is injective because $f(a)=f(b)$ implies

$$a=f(f(a))-1987=f(f(b))-1987=b.$$

Suppose $f(n)$ misses exactly k distinct values $c_1,\ldots,c_k$ in $\mathbb{N}_0$, i.e. $f(n)\neq c_1,\ldots,c_k$ for all $n\in\mathbb{N}_0$. Then $f(f(n))$ misses the $2k$ distinct values $c_1,\ldots,c_k$ and $f(c_1),\ldots,f(c_k)$ in $\mathbb{N}_0$. (The $f(c_j)$'s are distinct because f is injective.) Now if $w\neq c_1,\ldots,c_k, f(c_1),\ldots,f(c_k)$, then there is $m\in\mathbb{N}_0$ such that $f(m)=w$. Since $w\neq f(c_j)$, $m\neq c_j$, so there is $n\in\mathbb{N}_0$ such that $f(n)=m$, then $f(f(n))=w$. This shows $f(f(n))$ misses only the $2k$ values $c_1,\ldots,c_k,f(c_1),\ldots,f(c_k)$ and no others. Since $n+1987$ misses the 1987 values 0, 1, …, 1986 and $2k\neq 1987$, this is a contradiction.

Example 8. (*1999 IMO*) Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x-f(y))=f(f(y))+xf(y)+f(x)-1$$

for all $x,y\in\mathbb{R}$.

Solution. Let $c=f(0)$. Setting $x=y=0$, we get $f(-c)=f(c)+c-1$. So $c\neq 0$. Let A be the range of f, then for $x=f(y)\in A$, we get $c=f(0)=f(x)+x^2+f(x)-1$. Solving for $f(x)$, this gives $f(x)=(c+1-x^2)/2$.

Next, if we set $y=0$, we get

$$\begin{aligned}&\{f(x-c)-f(x):x\in\mathbb{R}\}\\&=\{cx+f(c)-1:x\in\mathbb{R}\}=\mathbb{R}\end{aligned}$$

because $c\neq 0$. Then $A-A=\{y_1-y_2: y_1,y_2\in A\}=\mathbb{R}$.

Now for an arbitrary $x\in\mathbb{R}$, let $y_1,y_2\in A$ be such that $y_1-y_2=x$. Then

$$\begin{aligned}f(x)&=f(y_1-y_2)=f(y_2)+y_1y_2+f(y_1)-1\\&=(c+1-y_2^2)/2+y_1y_2+(c+1-y_1^2)/2-1\\&=c-(y_1-y_2)^2/2=c-x^2/2.\end{aligned}$$

However, for $x\in A$, $f(x)=(c+1-x^2)/2$. So $c=1$. Therefore, $f(x)=1-x^2/2$ for all $x\in\mathbb{R}$.

Check: For $f(x)=1-x^2/2$, both sides equal $1/2+y^2/2-y^4/8+x-xy^2/2-x^2/2$.