Cauchy's and Pexider's Functional Equations in Restricted Domains
Part A - Cauchy's Equations

Nadia Zoabi, P.O.B 146, Kfar Masser 19345, Israel, nadoosh_z@hotmail.com

Shani Ardazi, 7 Hahagana Street, Holon 58272, Israel, shania@myself.com

Emilie Mamane, 77, chemin de Planta, 1223 Cologny-CH, Switzerland, dig18_825@hotmail.com

Vital Kharash, 24 Kziv Street, Ma'a lot 24952, Israel, vital_k@hotmail.com

Mentor: Mr. Orr Shalit, Department of Mathematics

Abstract

In this paper, Cauchy's and Pexider's functional equations are studied. Our aim is to find out whether the solutions of those equations remain the same or new ones are added when the domain of validity is restricted. We will observe that on certain curves the solutions of the equations remain exactly the same as in the original equation, while in others the set of solutions grows.

1. Introduction

A functional equation is an equation in which a function (or a set of functions) satisfying a certain relationship has to be found. The solution of functional equations is one of the oldest topics of mathematical analysis. D'Alembert, Cauchy, Pexider, Abel and Hilbert are among the mathematicians who have been concerned with functional equations and methods of solving them. It's interesting to know that functional equations arise in many fields of applied science, such as Statistics, Economics, Engineering, Artificial Intelligence and more.

For example, let us take a look at Cauchy's additive functional equation:

$$(1.1) \quad f(x+y) = f(x) + f(y) \qquad (1.2) \quad (x,y) \in D$$

(1.1) is the functional equation, while (1.2) points out that f satisfies (1.1) for all (x,y) that belongs to D, which is a subset of $\mathbb{R}^2$ ($\to$ the x,y plane).

D is called the *domain of validity* of the equation.

The solutions depends on (1.1), as well as on (1.2) and on other smoothness conditions of the function, such as differentiability, continuity etc.

If the domain of validity is restricted, then the solutions may remain as they originally were, or new solutions may appear.

There are functional equations in which even if the domain is restricted, the set of solutions remains as it was, and no new solutions are added.

These equations are called *over-determined*.

That means that the restricted domain is enough to determine the solutions of the equation in a certain class of functions, and that there is no need in a wider domain.

We shall explore Cauchy's and Pexider's functional equations:

$$(1.1)\quad f(x+y)=f(x)+f(y)$$

$$(1.3)\quad f(x+y)=f(x)\cdot f(y)$$

$$(1.4)\quad f(x+y)=g(x)+h(y)$$

$$(1.5)\quad f(x+y)=g(x)\cdot h(y)$$

For $f,g,h:[-1,1]\to\mathbb{R}$.

We will study those equations in their natural domains, as well as in restricted domains. We shall show how different domains and smoothness assumptions will affect the set of solutions of the above equations. In part A we shall study Cauchy's equations, and in part B we shall study Pexider's equations.

2. Notations and Definitions

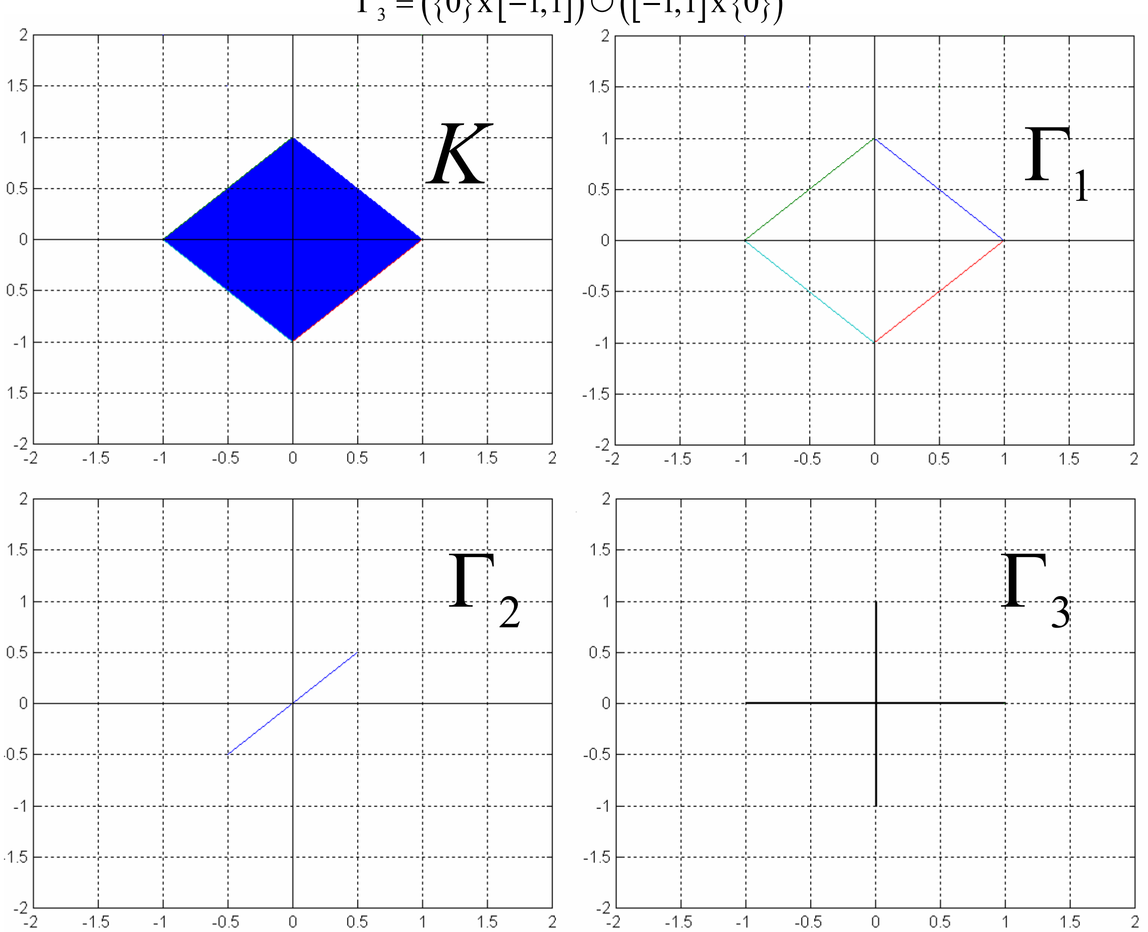
2.1 For working with the functional equations in restricted domains, we need to define the following subsets of $\mathbb{R}^2$:

$$K=\{(a,b)\ \mid\ |a|+|b|\le 1\}$$

$$\Gamma_1=\{(a,b)\ \mid\ |a|+|b|=1\}$$

$$\Gamma_2=\{(a,b)\ \mid\ b=a,\ a\in[-1/2,1/2]\}$$

$$\Gamma_3=(\{0\}\text{x}[-1,1])\cup([-1,1]\text{x}\{0\})$$



Now, some terms need to be explained:

2.2 An important concept that will be used in the sequel is the notion of a dynamical system. A dynamical system is a combination of an interval I and some continuous functions which their domain of definition and their range is I.

Example: $I = [0,1]$ and $\alpha(x) = \sqrt{x}$

Let us see how $\alpha(x)$ will move an initial point x_0 in the interval when $x_0 = \frac{1}{3}$:

$$x_1 = \alpha\left(\frac{1}{3}\right) = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3^{\frac{1}{2}}}$$

$$x_2 = \alpha^2\left(\frac{1}{3}\right) = \alpha\left(\alpha\left(\frac{1}{3}\right)\right) = \frac{1}{3^{\frac{1}{4}}}$$

$$x_n = \alpha^n(x_0) = \frac{1}{3^{\frac{1}{2^n}}}$$

$$\lim_{n\to\infty} x_n = 1 \quad \text{where} \quad \{\alpha^n(x_0)\} = \{x_n\}_{n=1}^{\infty}$$

We see that the motion of an initial point x_0 can have two different behaviors in the interval $[0,1]$:

1) If $x_0 = 0$ or $x_0 = 1$ so $\alpha(x_0) = x_0$
2) If $0 < x_0 < 1$ we obtain a sequence which grows up and gets closer to 1 as n tends to infinity.

In this paper we will use a dynamical system composed of two functions $\alpha, \beta : I \to I$ and an interval I and we will note it (I, α, β).

2.3 In a dynamical system (I, α, β) the Orbit $O(x_0)$ of a point x_0 is the set of the points which can be reached under iterations of α, β:

$$O(x_0) = \{x_0,\ \alpha(x_0),\ \beta(x_0),\ \alpha(\beta(x_0)),\ \alpha(\alpha(x_0)),\ \beta(\alpha(x_0)),\ \beta(\beta(x_0)),\ \ldots\}$$

2.4 We will use different classes of functions: $C, C^1, \ldots, C^\infty$. Their definition are:

C is the class of the continuous functions.

C^1 is the class of the continuously differentiable functions.

C^∞ is the class of the functions which are infinitely differentiable.

3. The Additive Cauchy Functional Equation

$$(3.1) \quad f(x+y) = f(x) + f(y)$$

It is well known (see [1], [2], or [4]) that the continuous solutions of this equation on the domain K (or $\mathbb{R}^2$) are of the form:

$$(3.2)\quad f(x)=c\cdot x,\quad \text{where } c=f(1).$$

We will show that if the domain of validity of the functional equation is just the boundary of K (Γ_1), then the solutions of this functional equation are exactly the same as when the domain of validity is K.

Taking Γ_1 to be the domain of validity of the functional equation, we get four equations:

$$(3.3)\quad f(1)=f(x)+f(-x+1)\quad ,\quad 0\le x\le 1$$

$$(3.4)\quad f(2x-1)=f(x)+f(x-1)\quad ,\quad 0\le x\le 1$$

$$(3.5)\quad f(-1)=f(x)+f(-x-1)\quad ,\quad -1\le x\le 0$$

$$(3.6)\quad f(2x+1)=f(x)+f(x+1)\quad ,\quad -1\le x\le 0$$

By placing x=0 in (3.3) we obtain $f(0)=0$.

We change the variable in (3.3) $x=\frac{t+1}{2}$:

$$(3.7)\quad f(1)=f\left(\frac{t+1}{2}\right)+f\left(-\frac{t-1}{2}\right)$$

Similarly with the other three equations:

$$(3.8)\quad f(t)=f\left(\frac{t+1}{2}\right)+f\left(\frac{t-1}{2}\right)$$

$$(3.9)\quad f(-1)=f\left(\frac{t-1}{2}\right)+f\left(-\frac{t+1}{2}\right)$$

$$(3.10)\quad f(t)=f\left(\frac{t-1}{2}\right)+f\left(\frac{t+1}{2}\right)$$

We can ignore the (3.10) equation due to the fact that it's exactly the same as (3.8).
If we choose $t=-t$ in (3.8) we obtain:

$$(3.11)\quad f(-t)=f\left(-\frac{t-1}{2}\right)+f\left(-\frac{t+1}{2}\right)$$

If we add the two equations (3.7) and (3.9), and separately the two equations (3.8) and (3.11) we obtain

$$(3.12)\quad f(t)+f(-t)=f(1)+f(-1)$$

Taking $t=0$ we obtain

$$f(1)+f(-1)=f(0)+f(0)=0$$

$$\Downarrow$$

$$(3.13)\quad f(t)+f(-t)=0$$

Let us define two maps:

$$\alpha(t)=\frac{t+1}{2}$$

$$\beta(t)=\frac{t-1}{2}$$

Using (3.13) and placing the two maps we obtain:

$$(3.14)\quad f(1) = f(\alpha(t)) - f(\beta(t))$$

$$(3.15)\quad f(t) = f(\alpha(t)) + f(\beta(t))$$

$$(3.16)\quad -f(1) = f(\beta(t)) - f(\alpha(t))$$

Solving for $f(\alpha(t))$ and $f(\beta(t))$ we obtain:

$$(3.17)\quad f(\beta(t)) = \frac{f(t) - f(1)}{2}$$

$$(3.18)\quad f(\alpha(t)) = \frac{f(t) + f(1)}{2}$$

We shall now find all continuous solutions of the system of equations (3.17), (3.18).
Let us examine a dynamical system: $(I, \alpha, \beta),\quad I = [-1,1]$
α and β have the following properties:

$$(3.19)\quad |\alpha(x) - \alpha(y)| = \left|\frac{x+1}{2} - \frac{y+1}{2}\right| = \frac{1}{2}|x - y|$$

$$(3.20)\quad |\beta(x) - \beta(y)| = \frac{1}{2}|x - y|$$

$\alpha(x_0) \in [0,1]$ for all x_0 in I

$\beta(x_0) \in [-1,0]$ for all x_0 in I

Proposition 3.1: *For any $x_0 \in I$ the orbit of x_0, $O(x_0)$, is dense in I.*

Proof: We will show by induction that for any $y \in I$ there is $z \in O(x_0)$ such that $|y - z| \le \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$.

For $n = 0$, let $y \in I$. It's clear that for any $z \in O(x_0)$

$$|z - y| \le \frac{1}{2^{-1}} = 2$$

Now assume that $n \ge 0$ and:

$$(*)\quad \forall y \in I,\ \exists z \in O(x_0).\quad |y - z| \le \frac{1}{2^{n-1}}$$

We shall show that $(*)$ holds with n replacing $n-1$.
Assume that $y \in [0,1]$. There is $y' \in I$ such that $\alpha(y') = y$.

According to the inductive hypothesis there is $z' \in O(x_0)$ such that $|z' - y'| \le \frac{1}{2^{n-1}}$.

We choose $z = \alpha(z')$. So $z \in O(x_0)$ and

$$|z - y| = |\alpha(z') - \alpha(y')| = \frac{1}{2}|z' - y'| \le \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}$$

as required.
If $y \in [-1,0]$ we do the same with α replaced by β. This proves $(*)$ which implies that $O(x_0)$ is dense.

Proposition 3.2: *If $t \in O(1)$ then $f(t) = f(1) \cdot t$*

Proof:
It's clear that for $t=1$: $f(1)=f(1)\cdot 1$.
Assume that $f(z_0)=g(1)\cdot z_0$. Then:

$$f(\alpha(z_0))=\frac{f(z_0)+f(1)}{2}=\frac{z_0\cdot f(1)+f(1)}{2}=\frac{f(1)\cdot(z_0+1)}{2}=f(1)\cdot\alpha(z_0)$$

$$f(\beta(z_0))=\frac{f(z_0)-f(1)}{2}=\frac{z_0\cdot f(1)-f(1)}{2}=\frac{f(1)\cdot(z_0-1)}{2}=f(1)\cdot\beta(z_0)$$

The first equality in each of the above lines follows from (3.17) and (3.18) and the last equality follows from the definition of α and β.
That proves the proposition.

Theorem 3.1*: The only solutions of the Additive Cauchy Functional Equation when the domain of validity is* Γ_1 *are of the form (3.2).*
Proof: According to proposition 3.2 the functions $f(1)\cdot t$ and $f(t)$ coincide on $O(1)$. According to proposition 3.1 $O(1)$ is dense in I. Because $f(x)$ is a continuous function the only solution is of the form $f(x)=f(1)\cdot x$.

Considering another domain of validity - Γ_2 shows some more interesting results.
When we look for solutions in the class C and the domain of validity is
$y=x,\quad x\in[-1/2,1/2]$, then the functional equation becomes

$$(3.21)\quad f(2x)=2f(x)$$

we find out that there are additional solutions such as $f(x)=|x|,\quad \mathrm{x}\in[\text{-}1,1]$ and also functions like
$f(x)=x\sin(2\pi\log_2|x|),\ \text{ if } x\neq 0,\ \text{and} f(0)=0.$
Checking these functions proves that they are solutions:

$f(x)=|x|:\ f(2x)=|2x|=2|x|=2f(x)$

$$f(x)=x\sin(2\pi\log_2|x|):\ f(2x)=2x\sin(2\pi\log_2(2|x|))=2x\sin(2\pi(\log_2 2+\log_2|x|))=$$
$$=2x\sin(2\pi+2\pi\log_2|x|)=2x\sin(2\pi\log_2|x|)=2f(x)$$

As is shown in [5] and [6], when we look for solution in the class C^1 and the domain of validity is $y=x,\quad x\in[-1/2,1/2]$ then the only solution is of the form $f(x)=f(1)\cdot x$.

4. The Cauchy Multiplicative Functional Equation

$$(4.1)\quad f(x+y)=f(x)\cdot f(y)\quad ,(x,y)\in K$$

It is proved in [3] that if f is a continuous function that satisfies the above equation, it is of the form $f(x)=0$ or $f(x)=c^x$.
We will now show that if the domain of validity of the equation is only a part of the boundary of K then new solutions appear.
As we have observed in section 3, when the domain of validity of (4.1) is restricted to Γ_1 we obtain three equations in one variable t:

$$(4.2)\quad f(1)=f\left(-\frac{t-1}{2}\right)\cdot f\left(\frac{t+1}{2}\right)\qquad t\in[-1,1]$$

$$(4.3)\quad f(t)=f\left(\frac{t+1}{2}\right)\cdot f\left(\frac{t-1}{2}\right)\qquad t\in[-1,1]$$

$$(4.4)\quad f(-1)=f\left(\frac{t-1}{2}\right)\cdot f\left(-\frac{t+1}{2}\right)\qquad t\in[-1,1]$$

For equation (4.3) we find new solutions such as:

$$(4.5)\quad f(x)=2\cos\left(\left(\frac{\pi}{2}+\pi\cdot k\right)x\right)\qquad k\in\mathbb{N}$$

Let us verify that for $k=0$ (4.5) is a solution to (4.3).

By using the formula: $\cos x\cos y=\frac{1}{2}\cos(x+y)+\frac{1}{2}\cos(x-y)$ we obtain:

$$2\cos\left(\frac{(x+1)\pi}{4}\right)\cdot 2\cos\left(\frac{(x-1)\pi}{4}\right)=4\cdot\cos\left(\frac{x\pi}{4}+\frac{\pi}{4}\right)\cdot\cos\left(\frac{x\pi}{4}-\frac{\pi}{4}\right)=$$

$$2\left[\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi x}{2}\right)\right]=2\cos\left(\frac{\pi}{2}\right)+2\cos\left(\frac{\pi x}{2}\right)=2\cos\left(\frac{\pi x}{2}\right).$$

We checked that this specific solution satisfies the equation on the curves $y=x-1$, $x\in[0,1]$ and $y=x+1$, $x\in[-1,0]$. (Let us note that this solution is C^∞, whereas in the additive Cauchy's equation all the C^1 solutions on these curves are of the form $f(t)=c\cdot t$ by [5]).

We will now take our domain of validity to be $\Gamma_1\cup\Gamma_2$.

Theorem 4.1: *The only continuous solutions of the functional equation*

$$f(x+y)=f(x)\cdot f(y)\quad,\ (x,y)\in\Gamma_1\cup\Gamma_2$$

are of the form: $f(x)=c^x$ or $f(x)=0$.

Proof:

First, we will prove that either $f(x)\equiv 0$ or $f>0$.

By restricting (4.1) to Γ_2 we obtain: $f(2x)=f^2(x)\geq 0$, $x\in[-1/2,1/2]$

By making a change of variables $2x=t$, we obtain the equation:

$$(4.8)\quad f(t)=\left(f\left(\frac{t}{2}\right)\right)^2\quad,\ t\in[-1,1]$$

We suppose that there is a t_0 such that: $f(t_0)=0$

By (4.8) we deduce that:

$$f\left(\frac{t_0}{2}\right)=0$$

If we continue inductively, we find that:

$$f\left(\frac{t_0}{4}\right)=0 \ , \ f\left(\frac{t_0}{8}\right)=0 \ , \quad \text{etc...}$$

So we deduce by induction that:

$$f\left(\frac{t_0}{2^n}\right)=0$$

We define:

$$x_n=\frac{t_0}{2^n}$$

Then $x_n \to 0$, and so we obtain

$$f(0)=\lim_{n\to\infty} f(x_n)=0$$

because f is continuous.
The conclusion is that if $\exists t_0 : f(t_0)=0$ then $f(0)=0$.
Suppose there is a t_1 for which $f(t_1)\neq 0$, we will prove that $f(0)\neq 0$.
Let us define $a=f(t_1)$.
By (4.8), we can see that:

$$a=f\left(\frac{t_1}{2}\right)^2$$

By induction, we deduce that:

$$a=f\left(\frac{t_1}{2^n}\right)^{2^n} \quad \Rightarrow \quad \sqrt[2^n]{a}=f\left(\frac{t_1}{2^n}\right)$$

We can calculate the limit when $n\to\infty$:

$$1=\lim_{n\to\infty}\sqrt[2^n]{a}=\lim_{n\to\infty} f\left(\frac{t_1}{2^n}\right)=f(0)$$

The conclusions are: if $f(0)=0$, then the function is constant and equals 0 and if $f(0)\neq 0$, then $f(t)>0$ for every t.

As we have already observed, (4.1) on Γ_1 gives us the three equations (4.2), (4.3), (4.4). We will now reduce those three equations to the additive one (3.1) that we solved in section 3. Due to the fact that $f>0$ (we shall ignore the trivial solution $f(x)\equiv 0$) we can take the natural logarithm of both sides of the equation.

$$\text{(4.9)} \qquad \ln\big(f(1)\big)=\ln\left(f\left(-\frac{t-1}{2}\right)\right)+\ln\left(f\left(\frac{t+1}{2}\right)\right) \qquad t\in[-1,1]$$

$$\text{(4.10)} \qquad \ln\big(f(t)\big)=\ln\left(f\left(\frac{t+1}{2}\right)\right)+\ln\left(f\left(\frac{t-1}{2}\right)\right) \qquad t\in[-1,1]$$

$$(4.11)\quad \ln\left(f(-1)\right)=\ln\left(f\left(\frac{t-1}{2}\right)\right)+\ln\left(f\left(-\frac{t+1}{2}\right)\right)\qquad t\in[-1,1]$$

We define:

$$(4.12)\quad g(t)=\ln(f(t))$$

$$\Downarrow$$

$$g(t)=g\left(-\frac{t-1}{2}\right)+g\left(\frac{t+1}{2}\right)$$

$$g(t)=g\left(\frac{t+1}{2}\right)+g\left(\frac{t-1}{2}\right)$$

$$g(t)=g\left(\frac{t-1}{2}\right)+g\left(-\frac{t+1}{2}\right)$$

It is clear that those new equations are of the form of the additive Cauchy's equation on Γ_1 that we solved before. Thus, the solutions of those equations are of the form: $g(t)=c\cdot t$.

According to (4.12): $c\cdot t=\ln(f(t)) \;\Rightarrow\; f(t)=e^{ct}$.

We conclude there are no new continuous solutions to this equation, and this concludes the proof of the theorem.

If we take the Γ_2 to be the domain of validity of the function we obtain this equation:

$$(4.13)\quad f(2t)=f^2(t)$$

Now we will see that there are continuous, not differentiable, new solutions that satisfy this equation.

A possible solution to this equation is: $f(t)=e^{t\sin(2\pi\log_2 t)}$. This is a non differentiable solution for the equation. However, if we want the solutions that are in the class C^1, we find out that they are of the set of the solutions for the equation in K (By taking the $\ln$ function on (4.13) we obtain (3.21)).

Cauchy's and Pexider's Functional Equations in Restricted Domains
Part B – Pexider's Equations

Nadia Zoabi
P.O.B 146
Kfar Masser 19345
Israel
nadoosh_z@hotmail.com

Shani Ardazi
7 Hahagana Street
Holon 58272
Israel
shania@myself.com

Emilie Mamane
77, chemin de Planta
1223 Cologny-CH
Switzerland
dig18_825@hotmail.com

Vital Kharash
24 Kziv Street
Ma'a lot 24952
Israel
vital_k@hotmail.com

Mentor: Mr. Orr Shalit, Department of Mathematics

This is a continuation of Cauchy's and Pexider's Functional Equations in Restricted Domains part A.

5. The Additive Pexider Functional Equation

$$(5.1)\quad f(x+y)=h(x)+g(y) \quad (x,y)\in K$$

It is well known ([1], [3]) that the continuous solutions of this functional equation on the domain K are of the form:

$$(5.2)\quad f(t)=c\cdot t+a+b$$
$$h(t)=c\cdot t+b$$
$$g(t)=c\cdot t+a$$

where a=g(0) , b=h(0)

We want to get the solutions for this equation on the curve $\Gamma_1\cup\Gamma_3$.

By placing (x,y) that are in the domain $\Gamma_1\cup\Gamma_3$ in (5.1) (as in section 3) we get seven equations:

(5.3.1) $f(t)=g\left(\beta(t)\right)+h\left(\alpha(t)\right)$

(5.3.2) $f(1)=g(-\beta(t))+h(\alpha(t))$

(5.3.3) $f(t)=g(\alpha(t))+h(\beta(t))$

(5.3.4) $f(-1)=g(\beta(t))+h(-\alpha(t))$

(5.3.5) $f(t)=g(t)+h(0)$

(5.3.6) $f(t)=g(0)+h(t)$

(5.3.7) $f(0)=g(0)+h(0)$

where : $\alpha(t)=\dfrac{t+1}{2} \qquad \beta\left(\text{t}\right)=\dfrac{t-1}{2}$

By placing (5.3.5), (5.3.6) and (5.3.7) in the first four equations we obtain four new equations:

$$(5.3.1)\quad f(t)=f(\beta(t))+f(\alpha(t))-f(0)$$
$$(5.3.2)\quad f(1)=f(-\beta(t))+f(\alpha(t))-f(0)$$
$$(5.3.3)\quad f(t)=f(\alpha(t))+f(\beta(t))-f(0)$$
$$(5.3.4)\quad f(-1)=f(\beta(t))+f(-\alpha(t))-f(0)$$

Now we define a new function $m(t)=f(t)-f(0) \quad \Rightarrow \; f(t)=m(t)+f(0)$.
By placing $f(t)=m(t)+f(0)$ in (5.3.1) to (5.3.4) we get:

$$(5.4.1)\quad m(t)=m(\alpha(t))+m(\beta(t))$$
$$(5.4.2)\quad m(1)=m(-\beta(t))+m(\alpha(t))$$
$$(5.4.3)\quad m(-1)=m(\beta(t))+m(-\alpha(t))$$

This is Cauchy's additive functional equation, and its continuous solutions (as we found in section 3) are of the form: $m(t)=c\cdot t$.

That gives us these solutions of the equation (5.1) on the domain $\Gamma_1 \cup \Gamma_3$:

$$f(t)=c\cdot t+a+b$$
$$h(t)=c\cdot t+b$$
$$g(t)=c\cdot t+a$$

where b=h(0)   a=g(0)

So we can conclude that there are no new continuous solutions on the curve $\Gamma_1 \cup \Gamma_3$.

We seek the solutions for this equation on the curve Γ_2. To this end, place $y=x$ in the equation (5.1) to obtain:

$$f(2x)=h(x)+g(x), \quad x\in[-1/2,1/2]$$

For this equation we didn't find the general solutions but there are new solutions, for example:

1. $f(t)=1$, h(t)=$\cos^2 t$, $g(t)=\sin^2 t$
2. $f(t)=\cos(2t)$, h(t)=$\cos^2 t$, g(t)= -$\sin^2 t$
3. $f(t)=\sin(2t)$, h(t)=g(t)=cos(t)$\cdot$sin(t)

Note that these solutions are C^∞, so no smoothness condition will preserve the set of solutions (5.2). However in Cauchy's additive equation we do not obtain new solutions when we add the condition that the function will be C^1 or differentiable at 0.

6. The Multiplicative Pexider Functional Equation

$$(6.1)\quad f(x+y)=g(x)\cdot h(y)$$

As it is shown in [3] the continuous solutions of this equation on the domain K are of the forms:

(6.2) $f(x) = ab \cdot e^{cx}$, $g(x) = a \cdot e^{cx}$, $h(x) = b \cdot e^{cx}$

or

(6.3) $f(x) = 0$, $g(x) = 0$, $h(x)$ arbitrary

or

(6.4) $f(x) = 0$, $g(x) = \text{arbitrary}$, $h(x) = 0$

We will show that if the domain of validity of the functional equation is $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ then the solutions of this functional equation are exactly the same as when the domain of validity is K.

Taking $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ to be the domain of validity of the functional equation, and changing the variables as in the other equations, we get these equations:

(6.5) $f(1) = g\left(-\frac{t-1}{2}\right) \cdot h\left(\frac{t+1}{2}\right) \quad , \quad t \in [-1,1]$

(6.6) $f(t) = g\left(\frac{t+1}{2}\right) \cdot h\left(\frac{t-1}{2}\right) \quad , \quad t \in [-1,1]$

(6.7) $f(-1) = g\left(\frac{t-1}{2}\right) \cdot h\left(-\frac{t+1}{2}\right) \quad , \quad t \in [-1,1]$

(6.8) $f(t) = g\left(\frac{t-1}{2}\right) \cdot h\left(\frac{t+1}{2}\right) \quad , \quad t \in [-1,1]$

(6.9) $f(0) = g(0) \cdot h(0)$

(6.10) $f(t) = g(t) \cdot h(0) \quad , \quad t \in [-1,1]$

(6.11) $f(t) = g(0) \cdot h(t) \quad , \quad t \in [-1,1]$

(6.12) $f(2p) = g(p) \cdot h(p) \quad p \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Assume that $f(x)$ is not identicaly 0, then $g(0), h(0) \neq 0$ and it is possible to change (6.10) and (6.11):

$$(6.10) \quad g(t) = \frac{f(t)}{h(0)}$$

$$(6.11) \quad h(t) = \frac{f(t)}{g(0)}$$

Using (6.5) to (6.12) we obtain:

$$(6.13)\quad f(1)=\frac{f\left(-\frac{t-1}{2}\right)\cdot f\left(\frac{t+1}{2}\right)}{f(0)}$$

$$(6.14)\quad f(t)=\frac{f\left(\frac{t+1}{2}\right)\cdot f\left(\frac{t-1}{2}\right)}{f(0)}$$

$$(6.15)\quad f(-1)=\frac{f\left(\frac{t-1}{2}\right)\cdot f\left(-\frac{t+1}{2}\right)}{f(0)}$$

$$(6.16)\quad f(t)=\frac{f\left(\frac{t-1}{2}\right)\cdot f\left(\frac{t+1}{2}\right)}{f(0)}$$

$$(6.17)\quad f(2p)=\frac{f(p)\cdot f(p)}{f(0)}\quad p\in\left[-\frac{1}{2},\frac{1}{2}\right]$$

Let us define $m(x)=\frac{f(x)}{f(0)}$.

Now it is possible to rewrite the equations (6.13) to (6.16):

$$(6.18)\quad m(1)=m\left(-\frac{t-1}{2}\right)\cdot m\left(\frac{t+1}{2}\right)$$

$$(6.19)\quad m(t)=m\left(\frac{t+1}{2}\right)\cdot m\left(\frac{t-1}{2}\right)$$

$$(6.20)\quad m(-1)=m\left(\frac{t-1}{2}\right)\cdot m\left(-\frac{t+1}{2}\right)$$

$$(6.21)\quad m(t)=m\left(\frac{t-1}{2}\right)\cdot m\left(\frac{t+1}{2}\right)$$

$$(6.22)\quad m(2p)=(m(p))^2\quad p\in\left[-\frac{1}{2},\frac{1}{2}\right]$$

These are the equations we obtained when solving Cauchy's multiplicative equation. Due to the fact that the continuous solutions for these equations are $m(x)=e^{cx}$ ($m\neq 0$ because $f\neq 0$), the solutions for (6.1) when the domain of validity is $\Gamma_1\cup\Gamma_2\cup\Gamma_3$ are the same as for Pexider's multiplicative equation when the domain of validity is K and when $f(x)$ is not identicaly 0.

Let us consider the option when $g(0)=0$ and $h(0)\neq 0$. According to (6.11) we obtain that $f(x)\equiv 0$, and using (6.10) we obtain that $g(x)\equiv 0$, and $h(x)$ arbitrary. These are not new solutions. The same with $h(0)=0$ and $g(0)\neq 0$. But if $g(0)=0$ and $h(0)=0$ there are more continuous solutions that are not of the forms (6.2) or (6.3) or (6.4). A general class of solutions for any $x_0\in(0,1)$ is:

$f(x) \equiv 0$

$h(x) = 0$ when $x \in [-1, x_0] \cup \{1\}$ and any continuous function when $x \in (x_0, 1)$

$g(x) = 0$ when $x \in [x_0 - 1, 0] \cup \{-1\}$ and any continuous function when $x \in (-1, x_0 - 1)$

It's obvious that these solutions satisfy the equations (6.9) to (6.12). To show that they satisfy (6.5) to (6.8) let us rewrite these equations:

$$(6.23) \quad 0 = g(1-x) \cdot h(x), \quad x \in [0,1]$$

$$(6.24) \quad 0 = g(x+1) \cdot h(x), \quad x \in [-1,0]$$

$$(6.25) \quad 0 = g(-1-x) \cdot h(x), \quad x \in [-1,0]$$

$$(6.26) \quad 0 = g(x-1) \cdot h(x), \quad x \in [0,1]$$

It's now clear that the functions satisfy the equations.

If we change the domain of validity to Γ_2 there are more solutions that are not of the form (6.2) or (6.3) or (6.4). The functional equation on Γ_2 is $f(2x) = g(x) \cdot h(x)$. An example for three functions is:

$$f(x) = \sin(2x), \quad g(x) = 2 \cdot \sin(x), \quad h(x) = \cos(x)$$

These functions are C^∞ so no smoothness assumption will guaranty that the set of solutions is of the form (6.2) or (6.3) or (6.4). This is a different situation then what we encountered Cauchy's multiplicative equation

7. Conclusions and Open Questions

As we have observed there are some results about each equation. For the additive Cauchy equation it's sufficient to take Γ_1 to determine the same set of continuous solutions as when K is the domain of validity. On Γ_2 if we look for solutions in the class C^1 the functional equation determines the same set of continuous solutions, but if we look for solutions in the class C we find some new solutions. For the multiplicative Cauchy equation on $\Gamma_1 \cup \Gamma_2$ the set of continuous solutions is the same as when K is the domain of validity. On Γ_2 the situation is exactly the same as in the additive equation.

For Pexider's equations there are some different results. For the additive Pexider equation it's sufficient to take $\Gamma_1 \cup \Gamma_3$ to determine the same set of continuous solutions as when K is the domain of validity. On Γ_2 there are new solution which are infinitely differentiable. This result is different from the one we obtained for Cauchy's additive equation. For the multiplicative Pexider equation it's sufficient to take $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ to determine the same set of continuous solutions as when K is the domain of validity, and on Γ_2 there are new solution which are infinitely differentiable. Again, this result is different from the result we got in the multiplicative Cauchy equation.

We can conclude that all of these equations are overdetermined. But yet, there are some open questions about those equations, such as: Can we reduce the domain of validity of the equations even further and still obtain the same results? In particular, can we obtain the same results for (4.1), (5.1) and (6.1) using only Γ_1, without any other conditions, or can we find solutions that prove the opposite?

8. Acknowledgements
We would like to thank the Technion - Israel institute of Technology – and to the Mathematics Faculty for giving us the opportunity to gain experience in a Science research. We are especially grateful to our mentor, Mr. Orr Shalit, our Scitech counselors and Scitech management. We would also like to thank “Ort Israel” for giving the scholarship to Nadia Zoabi and Vital Kharash. We must end with a great appreciation to our Scitech friends and the amazing time we all had together those 4 weeks; An unforgettable experience.

9. References
[1] Castillo & Ruis-Cobo , *Functional equations and modeling in science and engineering*, Marcel Dekker, Inc.1992.
[2] J.Aczel , *Functional equations and their applications* ,Academic press 1966.
[3] J.Aczel , *On Application and Theory of Functional Equations* .Birkhauser Verlag Basel,1969.
[4] Jaroslav Smital ,*On functions and functional equations*. Adam Hilger, Bristol and Philadephia .(1988)
[5] Boris Paneah ,On the over determinedness of some functional equations .*Discrete and continuous dynamical systems* , vol. 10, 1-2, (2004),pp. 497-505.
[6] Maceij Sablik, Note on a Cauchy conditional equation*, Radovi Matematicki*, vol.1 (1985),241- 245.