

CONTENTS

Preface to the Second Edition	xxv
Preface to the First Edition	xxvi
Authors	xxix
Notation	xxxix
Part I Exact Solutions	1
1 First-Order Equations with Two Independent Variables	3
1.1 Equations of the Form $f(x, y) \frac{\partial w}{\partial x} + g(x, y) \frac{\partial w}{\partial y} = 0$	3
1.1.1 Equations Containing Power-Law Functions	3
1.1.2 Equations Containing Exponential Functions	22
1.1.3 Equations Containing Hyperbolic Functions	29
1.1.4 Equations Containing Logarithmic Functions	33
1.1.5 Equations Containing Trigonometric Functions	36
1.1.6 Equations Containing Inverse Trigonometric Functions	45
1.1.7 Equations Containing Arbitrary Functions of x	51
1.1.8 Equations Containing Arbitrary Functions of Different Arguments	59
1.2 Equations of the Form $f(x, y) \frac{\partial w}{\partial x} + g(x, y) \frac{\partial w}{\partial y} = h(x, y)$	66
1.2.1 Equations Containing Power-Law Functions	67
1.2.2 Equations Containing Exponential Functions	71
1.2.3 Equations Containing Hyperbolic Functions	74
1.2.4 Equations Containing Logarithmic Functions	77
1.2.5 Equations Containing Trigonometric Functions	78
1.2.6 Equations Containing Inverse Trigonometric Functions	82
1.2.7 Equations Containing Arbitrary Functions	85
1.3 Equations of the Form $f(x, y) \frac{\partial w}{\partial x} + g(x, y) \frac{\partial w}{\partial y} = h(x, y)w$	91
1.3.1 Equations Containing Power-Law Functions	91
1.3.2 Equations Containing Exponential Functions	95
1.3.3 Equations Containing Hyperbolic Functions	97
1.3.4 Equations Containing Logarithmic Functions	100
1.3.5 Equations Containing Trigonometric Functions	102
1.3.6 Equations Containing Inverse Trigonometric Functions	105
1.3.7 Equations Containing Arbitrary Functions	108
1.4 Equations of the Form $f(x, y) \frac{\partial w}{\partial x} + g(x, y) \frac{\partial w}{\partial y} = h_1(x, y)w + h_0(x, y)$	114
1.4.1 Equations Containing Power-Law Functions	114
1.4.2 Equations Containing Exponential Functions	120
1.4.3 Equations Containing Hyperbolic Functions	122
1.4.4 Equations Containing Logarithmic Functions	125
1.4.5 Equations Containing Trigonometric Functions	126
1.4.6 Equations Containing Inverse Trigonometric Functions	131
1.4.7 Equations Containing Arbitrary Functions	133

2	First-Order Equations with Three or More Independent Variables	139
2.1	Equations of the Form $f(x, y, z) \frac{\partial w}{\partial x} + g(x, y, z) \frac{\partial w}{\partial y} + h(x, y, z) \frac{\partial w}{\partial z} = 0$	139
2.1.1	Equations Containing Power-Law Functions	139
2.1.2	Equations Containing Exponential Functions	150
2.1.3	Equations Containing Hyperbolic Functions	154
2.1.4	Equations Containing Logarithmic Functions	158
2.1.5	Equations Containing Trigonometric Functions	159
2.1.6	Equations Containing Inverse Trigonometric Functions	162
2.1.7	Equations Containing Arbitrary Functions	164
2.2	Equations of the Form $f_1 \frac{\partial w}{\partial x} + f_2 \frac{\partial w}{\partial y} + f_3 \frac{\partial w}{\partial z} = f_4$, $f_n = f_n(x, y, z)$	170
2.2.1	Equations Containing Power-Law Functions	171
2.2.2	Equations Containing Exponential Functions	177
2.2.3	Equations Containing Hyperbolic Functions	179
2.2.4	Equations Containing Logarithmic Functions	183
2.2.5	Equations Containing Trigonometric Functions	184
2.2.6	Equations Containing Inverse Trigonometric Functions	188
2.2.7	Equations Containing Arbitrary Functions	191
2.3	Equations of the Form $f_1 \frac{\partial w}{\partial x} + f_2 \frac{\partial w}{\partial y} + f_3 \frac{\partial w}{\partial z} = f_4 w$, $f_n = f_n(x, y, z)$	196
2.3.1	Equations Containing Power-Law Functions	197
2.3.2	Equations Containing Exponential Functions	203
2.3.3	Equations Containing Hyperbolic Functions	205
2.3.4	Equations Containing Logarithmic Functions	209
2.3.5	Equations Containing Trigonometric Functions	210
2.3.6	Equations Containing Inverse Trigonometric Functions	214
2.3.7	Equations Containing Arbitrary Functions	217
2.4	Equations of the Form $f_1 \frac{\partial w}{\partial x} + f_2 \frac{\partial w}{\partial y} + f_3 \frac{\partial w}{\partial z} = f_4 w + f_5$, $f_n = f_n(x, y, z)$...	222
2.4.1	Equations Containing Power-Law Functions	223
2.4.2	Equations Containing Exponential Functions	228
2.4.3	Equations Containing Hyperbolic Functions	230
2.4.4	Equations Containing Logarithmic Functions	234
2.4.5	Equations Containing Trigonometric Functions	236
2.4.6	Equations Containing Inverse Trigonometric Functions	240
2.4.7	Equations Containing Arbitrary Functions	243
2.4.8	Underdetermined Equations Containing Operator div	248
2.4.9	Equations with Four or More Independent Variables	251
3	Second-Order Parabolic Equations with One Space Variable	261
3.1	Constant Coefficient Equations	261
3.1.1	Heat Equation $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2}$	261
3.1.2	Equation of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + \Phi(x, t)$	270
3.1.3	Equation of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + bw + \Phi(x, t)$	275
3.1.4	Equation of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + \Phi(x, t)$	280
3.1.5	Equation of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + cw + \Phi(x, t)$	284

3.2	Heat Equation with Axial or Central Symmetry and Related Equations	288
3.2.1	Equation of the Form $\frac{\partial w}{\partial t} = a \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)$	288
3.2.2	Equation of the Form $\frac{\partial w}{\partial t} = a \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$	294
3.2.3	Equation of the Form $\frac{\partial w}{\partial t} = a \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right)$	298
3.2.4	Equation of the Form $\frac{\partial w}{\partial t} = a \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$	305
3.2.5	Equation of the Form $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1-2\beta}{x} \frac{\partial w}{\partial x}$	308
3.2.6	Equation of the Form $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1-2\beta}{x} \frac{\partial w}{\partial x} + \Phi(x, t)$	311
3.3	Equations Containing Power Functions and Arbitrary Parameters	312
3.3.1	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t)w$	312
3.3.2	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x}$	318
3.3.3	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w + h(x, t)$	321
3.3.4	Equations of the Form $\frac{\partial w}{\partial t} = (ax + b) \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	324
3.3.5	Equations of the Form $\frac{\partial w}{\partial t} = (ax^2 + bx + c) \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	327
3.3.6	Equations of the Form $\frac{\partial w}{\partial t} = f(x) \frac{\partial^2 w}{\partial x^2} + g(x, t) \frac{\partial w}{\partial x} + h(x, t)w$	329
3.3.7	Equations of the Form $\frac{\partial w}{\partial t} = f(x, t) \frac{\partial^2 w}{\partial x^2} + g(x, t) \frac{\partial w}{\partial x} + h(x, t)w$	334
3.3.8	Liquid-Film Mass Transfer Equation $(1 - y^2) \frac{\partial w}{\partial x} = a \frac{\partial^2 w}{\partial y^2}$	335
3.3.9	Equations of the Form $f(x, y) \frac{\partial w}{\partial x} + g(x, y) \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} + h(x, y)$	338
3.4	Equations Containing Exponential Functions and Arbitrary Parameters	338
3.4.1	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t)w$	338
3.4.2	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x}$	341
3.4.3	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	344
3.4.4	Equations of the Form $\frac{\partial w}{\partial t} = ax^n \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	345
3.4.5	Equations of the Form $\frac{\partial w}{\partial t} = ae^{\beta x} \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	346
3.4.6	Other Equations	349
3.5	Equations Containing Hyperbolic Functions and Arbitrary Parameters	349
3.5.1	Equations Containing a Hyperbolic Cosine	349
3.5.2	Equations Containing a Hyperbolic Sine	350
3.5.3	Equations Containing a Hyperbolic Tangent	351
3.5.4	Equations Containing a Hyperbolic Cotangent	352
3.6	Equations Containing Logarithmic Functions and Arbitrary Parameters	354
3.6.1	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	354
3.6.2	Equations of the Form $\frac{\partial w}{\partial t} = ax^k \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	354
3.7	Equations Containing Trigonometric Functions and Arbitrary Parameters	356
3.7.1	Equations Containing a Cosine	356
3.7.2	Equations Containing a Sine	357
3.7.3	Equations Containing a Tangent	358
3.7.4	Equations Containing a Cotangent	359
3.8	Equations Containing Arbitrary Functions	360
3.8.1	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t)w$	360
3.8.2	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x}$	363
3.8.3	Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	368

3.8.4	Equations of the Form $\frac{\partial w}{\partial t} = ax^n \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	370
3.8.5	Equations of the Form $\frac{\partial w}{\partial t} = ae^{\beta x} \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t)w$	372
3.8.6	Equations of the Form $\frac{\partial w}{\partial t} = f(x) \frac{\partial^2 w}{\partial x^2} + g(x, t) \frac{\partial w}{\partial x} + h(x, t)w$	373
3.8.7	Equations of the Form $\frac{\partial w}{\partial t} = f(t) \frac{\partial^2 w}{\partial x^2} + g(x, t) \frac{\partial w}{\partial x} + h(x, t)w$	382
3.8.8	Equations of the Form $\frac{\partial w}{\partial t} = f(x, t) \frac{\partial^2 w}{\partial x^2} + g(x, t) \frac{\partial w}{\partial x} + h(x, t)w$	385
3.8.9	Equations of the Form $s(x) \frac{\partial w}{\partial t} = \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w + \Phi(x, t)$	388
3.9	Equations of Special Form	393
3.9.1	Equations of the Diffusion (Thermal) Boundary Layer	393
3.9.2	One-Dimensional Schrödinger Equation $i\hbar \frac{\partial w}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 w}{\partial x^2} + U(x)w$...	396
4	Second-Order Parabolic Equations with Two Space Variables	401
4.1	Heat Equation $\frac{\partial w}{\partial t} = a\Delta_2 w$	401
4.1.1	Boundary Value Problems in Cartesian Coordinates	401
4.1.2	Problems in Polar Coordinates	416
4.1.3	Axisymmetric Problems	423
4.2	Heat Equation with a Source $\frac{\partial w}{\partial t} = a\Delta_2 w + \Phi(x, y, t)$	434
4.2.1	Problems in Cartesian Coordinates	434
4.2.2	Problems in Polar Coordinates	442
4.2.3	Axisymmetric Problems	445
4.3	Other Equations	455
4.3.1	Equations Containing Arbitrary Parameters	455
4.3.2	Equations Containing Arbitrary Functions	457
5	Second-Order Parabolic Equations with Three or More Space Variables	463
5.1	Heat Equation $\frac{\partial w}{\partial t} = a\Delta_3 w$	463
5.1.1	Problems in Cartesian Coordinates	463
5.1.2	Problems in Cylindrical Coordinates	487
5.1.3	Problems in Spherical Coordinates	517
5.2	Heat Equation with Source $\frac{\partial w}{\partial t} = a\Delta_3 w + \Phi(x, y, z, t)$	522
5.2.1	Problems in Cartesian Coordinates	522
5.2.2	Problems in Cylindrical Coordinates	528
5.2.3	Problems in Spherical Coordinates	534
5.3	Other Equations with Three Space Variables	537
5.3.1	Equations Containing Arbitrary Parameters	537
5.3.2	Equations Containing Arbitrary Functions	539
5.3.3	Equations of the Form $\rho(x, y, z) \frac{\partial w}{\partial t} = \text{div}[a(x, y, z)\nabla w] - q(x, y, z)w + \Phi(x, y, z, t)$	542
5.4	Equations with n Space Variables	545
5.4.1	Equations of the Form $\frac{\partial w}{\partial t} = a\Delta_n w + \Phi(x_1, \dots, x_n, t)$	545
5.4.2	Other Equations Containing Arbitrary Parameters	548
5.4.3	Equations Containing Arbitrary Functions	549
6	Second-Order Hyperbolic Equations with One Space Variable	557
6.1	Constant Coefficient Equations	557
6.1.1	Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$	557
6.1.2	Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x, t)$	563
6.1.3	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw + \Phi(x, t)$	567

6.1.4	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - b \frac{\partial w}{\partial x} + \Phi(x, t)$	571
6.1.5	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + cw + \Phi(x, t)$	574
6.2	Wave Equations with Axial or Central Symmetry	577
6.2.1	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)$	577
6.2.2	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$	580
6.2.3	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right)$	581
6.2.4	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$	585
6.2.5	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - bw + \Phi(r, t)$	586
6.2.6	Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) - bw + \Phi(r, t)$	590
6.3	Equations Containing Power Functions and Arbitrary Parameters	593
6.3.1	Equations of the Form $\frac{\partial^2 w}{\partial t^2} = (ax + b) \frac{\partial^2 w}{\partial x^2} + c \frac{\partial w}{\partial x} + kw + \Phi(x, t)$	593
6.3.2	Equations of the Form $\frac{\partial^2 w}{\partial t^2} = (ax^2 + b) \frac{\partial^2 w}{\partial x^2} + cx \frac{\partial w}{\partial x} + kw + \Phi(x, t)$	598
6.3.3	Other Equations	600
6.4	Equations Containing the First Time Derivative	607
6.4.1	Equations of the Form $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + cw + \Phi(x, t)$	607
6.4.2	Equations of the Form $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = f(x) \frac{\partial^2 w}{\partial x^2} + g(x) \frac{\partial w}{\partial x} + h(x)w + \Phi(x, t)$	616
6.4.3	Other Equations	621
6.5	Equations Containing Arbitrary Functions	623
6.5.1	Equations of the Form $s(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w + \Phi(x, t)$	623
6.5.2	Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a(t) \frac{\partial w}{\partial t} =$ $b(t) \left\{ \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w \right\} + \Phi(x, t)$	626
6.5.3	Other Equations	628
7	Second-Order Hyperbolic Equations with Two Space Variables	633
7.1	Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w$	633
7.1.1	Problems in Cartesian Coordinates	633
7.1.2	Problems in Polar Coordinates	639
7.1.3	Axisymmetric Problems	645
7.2	Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w + \Phi(x, y, t)$	651
7.2.1	Problems in Cartesian Coordinates	651
7.2.2	Problems in Polar Coordinates	653
7.2.3	Axisymmetric Problems	656
7.3	Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w - bw + \Phi(x, y, t)$	659
7.3.1	Problems in Cartesian Coordinates	659
7.3.2	Problems in Polar Coordinates	664
7.3.3	Axisymmetric Problems	670
7.4	Telegraph Equation $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \Delta_2 w - bw + \Phi(x, y, t)$	676
7.4.1	Problems in Cartesian Coordinates	676
7.4.2	Problems in Polar Coordinates	681
7.4.3	Axisymmetric Problems	688
7.5	Other Equations with Two Space Variables	693

8 Second-Order Hyperbolic Equations with Three or More Space Variables	695
8.1 Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w$	695
8.1.1 Problems in Cartesian Coordinates	695
8.1.2 Problems in Cylindrical Coordinates	701
8.1.3 Problems in Spherical Coordinates	712
8.2 Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w + \Phi(x, y, z, t)$	717
8.2.1 Problems in Cartesian Coordinates	717
8.2.2 Problems in Cylindrical Coordinates	718
8.2.3 Problems in Spherical Coordinates	719
8.3 Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w - bw + \Phi(x, y, z, t)$	720
8.3.1 Problems in Cartesian Coordinates	720
8.3.2 Problems in Cylindrical Coordinates	726
8.3.3 Problems in Spherical Coordinates	738
8.4 Telegraph Equation $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \Delta_3 w - bw + \Phi(x, y, z, t)$	743
8.4.1 Problems in Cartesian Coordinates	743
8.4.2 Problems in Cylindrical Coordinates	748
8.4.3 Problems in Spherical Coordinates	760
8.5 Other Equations with Three Space Variables	765
8.5.1 Equations Containing Arbitrary Parameters	765
8.5.2 Equation of the Form $\rho(x, y, z) \frac{\partial^2 w}{\partial t^2} =$ $\text{div}[a(x, y, z) \nabla w] - q(x, y, z)w + \Phi(x, y, z, t)$	765
8.6 Equations with n Space Variables	768
8.6.1 Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w$	768
8.6.2 Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w + \Phi(x_1, \dots, x_n, t)$...	770
8.6.3 Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w - bw + \Phi(x_1, \dots, x_n, t)$	773
8.6.4 Equations Containing the First Time Derivative	776
9 Second-Order Elliptic Equations with Two Space Variables	781
9.1 Laplace Equation $\Delta_2 w = 0$	781
9.1.1 Problems in Cartesian Coordinate System	781
9.1.2 Problems in Polar Coordinate System	787
9.1.3 Other Coordinate Systems. Conformal Mappings Method	792
9.2 Poisson Equation $\Delta_2 w = -\Phi(\mathbf{x})$	794
9.2.1 Preliminary Remarks. Solution Structure	794
9.2.2 Problems in Cartesian Coordinate System	796
9.2.3 Problems in Polar Coordinate System	803
9.2.4 Arbitrary Shape Domain. Conformal Mappings Method	807
9.3 Helmholtz Equation $\Delta_2 w + \lambda w = -\Phi(\mathbf{x})$	809
9.3.1 General Remarks, Results, and Formulas	809
9.3.2 Problems in Cartesian Coordinate System	813
9.3.3 Problems in Polar Coordinate System	824
9.3.4 Other Orthogonal Coordinate Systems. Elliptic Domain	830
9.4 Other Equations	832
9.4.1 Stationary Schrödinger Equation $\Delta_2 w = f(x, y)w$	832
9.4.2 Convective Heat and Mass Transfer Equations	835
9.4.3 Equations of Heat and Mass Transfer in Anisotropic Media	843
9.4.4 Other Equations Arising in Applications	851
9.4.5 Equations of the Form $a(x) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x) \frac{\partial w}{\partial x} + c(x)w = -\Phi(x, y)$..	855

10 Second-Order Elliptic Equations with Three or More Space Variables	859
10.1 Laplace Equation $\Delta_3 w = 0$	859
10.1.1 Problems in Cartesian Coordinates	859
10.1.2 Problems in Cylindrical Coordinates	862
10.1.3 Problems in Spherical Coordinates	863
10.1.4 Other Orthogonal Curvilinear Systems of Coordinates	866
10.2 Poisson Equation $\Delta_3 w + \Phi(\mathbf{x}) = 0$	866
10.2.1 Preliminary Remarks. Solution Structure	866
10.2.2 Problems in Cartesian Coordinates	871
10.2.3 Problems in Cylindrical Coordinates	883
10.2.4 Problems in Spherical Coordinates	888
10.3 Helmholtz Equation $\Delta_3 w + \lambda w = -\Phi(\mathbf{x})$	892
10.3.1 Homogeneous Helmholtz Equation. Eigenvalue problems	892
10.3.2 Nonhomogeneous Helmholtz Equation. General Remarks, Results, and Formulas	893
10.3.3 Problems in Cartesian Coordinates	900
10.3.4 Problems in Cylindrical Coordinates	915
10.3.5 Problems in Spherical Coordinates	924
10.3.6 Other Orthogonal Curvilinear Coordinates	929
10.4 Other Equations with Three Space Variables	931
10.4.1 Equations Containing Arbitrary Functions	931
10.4.2 Equations of the Form $\operatorname{div}[a(x, y, z)\nabla w] - q(x, y, z)w = -\Phi(x, y, z)$	934
10.5 Equations with n Space Variables	936
10.5.1 Laplace Equation $\Delta_n w = 0$	936
10.5.2 Other Equations	937
11 Higher-Order Partial Differential Equations	941
11.1 Third-Order Partial Differential Equations	941
11.1.1 One-Dimensional Equations Containing the First Derivative in t	941
11.1.2 One-Dimensional Equations Containing the Second Derivative in t	944
11.1.3 One-Dimensional Equations Containing a Mixed Derivative and the First Derivative in t	945
11.1.4 One-Dimensional Equations Containing a Mixed Derivative and the Second Derivative in t	951
11.1.5 Two- and Three-Dimensional Equations	954
11.2 Fourth-Order One-Dimensional Nonstationary Equations	957
11.2.1 Equation of the Form $\frac{\partial w}{\partial t} + a^2 \frac{\partial^4 w}{\partial x^4} = \Phi(x, t)$	957
11.2.2 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0$	960
11.2.3 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = \Phi(x, t)$	962
11.2.4 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} + kw = \Phi(x, t)$	965
11.2.5 Other Equations without Mixed Derivatives	968
11.2.6 Equations Containing Second Derivative in x and Mixed Derivatives	971
11.2.7 Equations Containing Fourth Derivative in x and Mixed Derivatives	980
11.3 Two-Dimensional Nonstationary Fourth-Order Equations	986
11.3.1 Equation of the Form $\frac{\partial w}{\partial t} + a^2 \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) = \Phi(x, y, t)$	986
11.3.2 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w = 0$	988
11.3.3 Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w + kw = \Phi(x, y, t)$	991

11.3.4	Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + kw = \Phi(x, y, t)$	993
11.3.5	Other Two-Dimensional Nonstationary Fourth-Order Equations	994
11.4	Three- and n -Dimensional Nonstationary Fourth-Order Equations	997
11.4.1	Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w = 0$	997
11.4.2	Equations Containing Mixed Derivatives	999
11.5	Fourth-Order Stationary Equations	1003
11.5.1	Biharmonic Equation $\Delta \Delta w = 0$	1003
11.5.2	Equation of the Form $\Delta \Delta w = \Phi$	1009
11.5.3	Equation of the Form $\Delta \Delta w - \lambda w = \Phi(x, y)$	1012
11.5.4	Equation of the Form $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} = \Phi(x, y)$	1014
11.5.5	Equation of the Form $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + kw = \Phi(x, y)$	1016
11.5.6	Stokes Equation (Axisymmetric Flows of Viscous Fluids)	1017
11.6	Higher-Order Linear Equations with Constant Coefficients	1020
11.6.1	Fundamental Solutions. Cauchy Problem	1020
11.6.2	Elliptic Operators and Elliptic Equations	1022
11.6.3	Hyperbolic Operators and Hyperbolic Equations	1025
11.6.4	Regular Equations. Number of Initial Conditions in the Cauchy Problem	1025
11.6.5	Some Equations with Two Independent Variables Containing the First Derivative in t	1029
11.6.6	Some Equations with Two Independent Variables Containing the Second Derivative in t	1035
11.6.7	Other Equations with Two Independent Variables	1039
11.6.8	Equations with Three and More Independent Variables	1041
11.7	Higher-Order Linear Equations with Variable Coefficients	1045
11.7.1	Equations Containing the First Time Derivative	1045
11.7.2	Equations Containing the Second Time Derivative	1050
11.7.3	Nonstationary Problems with Many Space Variables	1052
11.7.4	Some Special Equations with Variable Coefficients	1054
12	Systems of Linear Partial Differential Equations	1059
12.1	Preliminary Remarks. Some Notation and Helpful Relations	1059
12.2	Systems of Two First-Order Equations	1059
12.3	Systems of Two Second-Order Equations	1063
12.3.1	Systems of Parabolic Equations	1063
12.3.2	Systems of Hyperbolic or Elliptic Equations	1064
12.4	Systems of Two Higher-Order Equations	1064
12.5	Simplest Systems Containing Vector Functions and Operators div and curl	1066
12.5.1	Equation $\text{curl } \mathbf{u} = \mathbf{A}(\mathbf{x})$	1066
12.5.2	Simple Systems of Equations Containing Operators div and curl	1067
12.5.3	Two Representations of Vector Functions	1069
12.6	Elasticity Equations	1071
12.6.1	Elasticity Equations in Various Coordinate Systems	1071
12.6.2	Various Forms of Decompositions of Homogeneous Elasticity Equations with $\mathbf{f} = \mathbf{0}$	1073
12.6.3	Various Forms of Decompositions for Nonhomogeneous Elasticity Equations	1075
12.6.4	Cauchy Problem and Its Solution. Fundamental Solution Matrix	1076

12.7	Stokes Equations for Viscous Incompressible Fluids	1077
12.7.1	Stokes Equations in Various Coordinate Systems	1077
12.7.2	Various Forms of Decompositions for the Stokes Equations with $\mathbf{f} = \mathbf{0}$	1081
12.7.3	Various Forms of Decompositions for the Stokes Equations with $\mathbf{f} \neq \mathbf{0}$	1082
12.7.4	General Solution of the Steady-State Homogeneous Stokes Equations	1084
12.7.5	Solution of the Steady-State Nonhomogeneous Stokes Equations	1085
12.7.6	Solution of the Cauchy Problem	1085
12.8	Oseen Equations for Viscous Incompressible Fluids	1086
12.8.1	Vector Form of Oseen Equations. Some Remarks	1086
12.8.2	Various Forms of Decompositions for the Oseen Equations with $\mathbf{f} = \mathbf{0}$	1087
12.8.3	Various Forms of Decompositions for the Oseen Equations with $\mathbf{f} \neq \mathbf{0}$	1088
12.8.4	Oseen Equations with Variable Coefficients	1088
12.9	Maxwell Equations for Viscoelastic Incompressible Fluids	1089
12.9.1	Vector Form of the Maxwell Equations	1089
12.9.2	Various Forms of Decompositions for the Maxwell Equations with $\mathbf{f} = \mathbf{0}$	1089
12.9.3	Various Forms of Decompositions for the Maxwell Equations with $\mathbf{f} \neq \mathbf{0}$	1090
12.10	Equations of Viscoelastic Incompressible Fluids (General Model)	1091
12.10.1	Linearized Equations of Viscoelastic Incompressible Fluids. Some Models of Viscoelastic Fluids	1091
12.10.2	Various Forms of Decompositions for Equations of Viscoelastic Incompressible Fluids with $\mathbf{f} = \mathbf{0}$	1092
12.10.3	Various Forms of Decompositions for Equations of Viscoelastic Incompressible Fluids with $\mathbf{f} \neq \mathbf{0}$	1093
12.11	Linearized Equations for Inviscid Compressible Barotropic Fluids	1094
12.11.1	Vector Form of Equations without Mass Forces. Some Remarks	1094
12.11.2	Decompositions of Equations for Inviscid Compressible Barotropic Fluid	1095
12.11.3	Vector Form of Equations with Mass Forces	1095
12.12	Stokes Equations for Viscous Compressible Barotropic Fluids	1096
12.12.1	Linearized Equations of Viscous Compressible Barotropic Fluids	1096
12.12.2	Decompositions of Equations of Viscous Compressible Barotropic Fluid with $\mathbf{f} = \mathbf{0}$	1097
12.12.3	Decompositions of Equations of a Viscous Compressible Barotropic Fluid with $\mathbf{f} \neq \mathbf{0}$	1098
12.12.4	Reduction to One Vector Equation and Its Decompositions	1098
12.12.5	Independent Equations for \mathbf{u} and p	1099
12.13	Oseen Equations for Viscous Compressible Barotropic Fluids	1100
12.13.1	Vector Form of Equations. Some Remarks	1100
12.13.2	Decompositions of Equations with $\mathbf{f} = \mathbf{0}$	1100
12.13.3	Decomposition of Equations with $\mathbf{f} \neq \mathbf{0}$	1101
12.14	Equations of Thermoelasticity	1101
12.14.1	Vector Form of Thermoelasticity Equations	1101
12.14.2	Decompositions of Thermoelasticity Equations with $\mathbf{f} = \mathbf{0}$	1101
12.14.3	Decompositions of Thermoelasticity Equations with $\mathbf{f} \neq \mathbf{0}$	1102

12.15	Nondissipative Thermoelasticity Equations (the Green–Naghdi Model)	1103
12.15.1	Vector Form of the Nondissipative Thermoelasticity Equations	1103
12.15.2	Decompositions of the Nondissipative Thermoelasticity Equations with $\mathbf{f} = \mathbf{0}$	1104
12.15.3	Decompositions of Thermoelasticity Equations with $\mathbf{f} \neq \mathbf{0}$	1105
12.16	Viscoelasticity Equations	1106
12.16.1	Vector Form of Viscoelasticity Equations	1106
12.16.2	Decompositions of Viscoelasticity Equations with $\mathbf{f} = \mathbf{0}$	1106
12.16.3	Various Forms of Decompositions for Viscoelasticity Equations with $\mathbf{f} \neq \mathbf{0}$	1108
12.17	Maxwell Equations (Electromagnetic Field Equations)	1108
12.17.1	Maxwell Equations in a Medium and Constitutive Relations	1108
12.17.2	Some Transformations and Solutions of the Maxwell Equations	1109
12.18	Vector Equations of General Form	1110
12.18.1	Vector Equations Containing Operators div and ∇	1110
12.18.2	Decompositions of the Homogeneous Vector Equation	1111
12.18.3	Decompositions of the Nonhomogeneous Vector Equation	1112
12.18.4	Vector Equations Containing More General Operators	1113
12.19	General Systems Involving Vector and Scalar Equations: Part I	1114
12.19.1	Systems Containing Operators div and ∇	1114
12.19.2	Decompositions of Systems with Homogeneous Vector Equation	1115
12.19.3	Decompositions of Systems with Nonhomogeneous Vector Equation	1115
12.19.4	Equations for \mathbf{u} and p . Reduction to One Vector Equation	1116
12.19.5	Systems Containing More General Operators	1117
12.20	General Systems Involving Vector and Scalar Equations: Part II	1118
12.20.1	Class of Systems Considered	1118
12.20.2	Asymmetric Decomposition	1118
12.20.3	Symmetric Decomposition	1119
Part II Analytical Methods		1121
13 Methods for First-Order Linear PDEs		1123
13.1	Linear PDEs with Two Independent Variables	1123
13.1.1	Special First-Order Linear PDEs with Two Independent Variables	1123
13.1.2	General First-Order Linear PDE with Two Independent Variables	1126
13.2	First-Order Linear PDEs with Three or More Independent Variables	1129
13.2.1	Characteristic System. General Solution	1129
13.2.2	Cauchy Problems	1131
14 Second-Order Linear PDEs: Classification, Problems, Particular Solutions		1133
14.1	Classification of Second-Order Linear Partial Differential Equations	1133
14.1.1	Equations with Two Independent Variables	1133
14.1.2	Equations with Many Independent Variables	1138
14.2	Basic Problems of Mathematical Physics	1140
14.2.1	Initial and Boundary Conditions. Cauchy Problem. Boundary Value Problems	1140
14.2.2	First, Second, Third, and Mixed Boundary Value Problems	1142

14.3	Properties and Particular Solutions of Linear Equations	1144
14.3.1	Homogeneous Linear Equations. Basic Properties of Particular Solutions	1144
14.3.2	Separable Solutions. Solutions in the Form of Infinite Series	1147
14.3.3	Nonhomogeneous Linear Equations and Their Properties	1150
14.3.4	General Solutions of Some Hyperbolic Equations	1150
15	Separation of Variables and Integral Transform Methods	1153
15.1	Separation of Variables (Fourier Method)	1153
15.1.1	Description of Separation of Variables. General Stage of Solution . . .	1153
15.1.2	Problems for Parabolic Equations: Final Stage of Solution	1157
15.1.3	Problems for Hyperbolic Equations: Final Stage of Solution	1159
15.1.4	Solution of Boundary Value Problems for Elliptic Equations	1160
15.1.5	Solution of Boundary Value Problems for Higher-Order Equations . . .	1163
15.2	Integral Transform Method	1165
15.2.1	Laplace Transform and Its Application in Mathematical Physics	1165
15.2.2	Fourier Transform and Its Application in Mathematical Physics	1170
15.2.3	Fourier Sine and Cosine Transforms	1173
15.2.4	Mellin, Hankel, and Other Integral Transforms	1177
16	Cauchy Problem. Fundamental Solutions	1181
16.1	Dirac Delta Function. Fundamental Solutions	1181
16.1.1	Dirac Delta Function and Its Properties	1181
16.1.2	Fundamental Solutions. Constructing Particular Solutions	1182
16.2	Representation of the Solution of the Cauchy Problem via the Fundamental Solution	1185
16.2.1	Cauchy Problem for Ordinary Differential Equations	1185
16.2.2	Cauchy Problem for Parabolic Equations	1187
16.2.3	Cauchy Problem for Hyperbolic Equations	1190
16.2.4	Higher-Order Linear PDEs. Generalized Cauchy Problem	1193
17	Boundary Value Problems. Green's Function	1199
17.1	Boundary Value Problems for Parabolic Equations with One Space Variable. Green's Function	1199
17.1.1	Representation of Solutions via the Green's Function	1199
17.1.2	Problems for Equation $s(x) \frac{\partial w}{\partial t} = \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w + \Phi(x, t)$. . .	1202
17.2	Boundary Value Problems for Hyperbolic Equations with One Space Variable. Green's Function. Goursat Problem	1205
17.2.1	Representation of Solutions via the Green's Function	1205
17.2.2	Problems for Equation $s(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w + \Phi(x, t)$. . .	1207
17.2.3	Problems for Equation $\frac{\partial^2 w}{\partial t^2} + a(t) \frac{\partial w}{\partial t} =$ $b(t) \{ \frac{\partial}{\partial x} [p(x) \frac{\partial w}{\partial x}] - q(x)w \} + \Phi(x, t)$	1208
17.2.4	Generalized Cauchy Problem with Initial Conditions Set along a Curve. Riemann Function	1210
17.2.5	Goursat Problem (a Problem with Initial Data on Characteristics)	1212
17.3	Boundary Value Problems for Elliptic Equations with Two Space Variables . .	1214
17.3.1	Problems and the Green's Functions for Equation $a(x) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x) \frac{\partial w}{\partial x} + c(x)w = -\Phi(x, y)$	1214
17.3.2	Representation of Solutions of Boundary Value Problems via Green's Functions	1216

17.4	Boundary Value Problems with Many Space Variables. Green's Function	1218
17.4.1	Problems for Parabolic Equations	1218
17.4.2	Problems for Hyperbolic Equations	1220
17.4.3	Problems for Elliptic Equations	1221
17.4.4	Comparison of the Solution Structures for Boundary Value Problems for Equations of Various Types	1222
17.5	Construction of the Green's Functions. General Formulas and Relations	1223
17.5.1	Green's Functions of Boundary Value Problems for Equations of Various Types in Bounded Domains	1223
17.5.2	Green's Functions Admitting Incomplete Separation of Variables	1224
17.5.3	Construction of Green's Functions via Fundamental Solutions	1227
18	Duhamel's Principles. Some Transformations	1233
18.1	Duhamel's Principles in Nonstationary Problems	1233
18.1.1	Problems for Homogeneous Linear Equations	1233
18.1.2	Problems for Nonhomogeneous Linear Equations	1235
18.2	Transformations Simplifying Initial and Boundary Conditions	1237
18.2.1	Transformations That Lead to Homogeneous Boundary Conditions . .	1237
18.2.2	Transformations That Lead to Homogeneous Initial and Boundary Conditions	1238
19	Systems of Linear Coupled PDEs. Decomposition Methods	1239
19.1	Asymmetric and Symmetric Decompositions	1239
19.1.1	Asymmetric Decomposition. Order of Decomposition	1239
19.1.2	Symmetric Decomposition. Invariant Transformations	1242
19.2	First-Order Decompositions. Examples	1244
19.2.1	Systems of Linear PDEs without Mass Forces ($\mathbf{f} = \mathbf{0}$)	1244
19.2.2	Systems of Linear PDEs with Mass Forces	1249
19.3	Higher-Order Decompositions	1251
19.3.1	Decomposition of Systems Consisting of One Vector Equation	1251
19.3.2	Decomposition of Systems Consisting of a Vector Equation and a Scalar Equation (the First Approach)	1252
19.3.3	Decomposition of Systems Consisting of a Vector Equation and a Scalar Equation (the Second Approach)	1253
20	Some Asymptotic Results and Formulas	1255
20.1	Regular Perturbation Theory Formulas for the Eigenvalues	1256
20.1.1	Statement of the Problem	1256
20.1.2	Formulas for the Coefficients of the Expansion	1256
20.2	Singular Perturbation Theory	1257
20.2.1	Cauchy Problem for the Schrödinger Equation	1257
20.2.2	Stationary Phase Method	1261
20.2.3	Fourier Transform with a Parameter	1264
21	Elements of Theory of Generalized Functions	1265
21.1	Generalized Functions of One Variable	1265
21.1.1	Important Terminological Remark	1265
21.1.2	Test Function Space	1265
21.1.3	Distribution Space. Dirac Delta Function	1266
21.1.4	Derivatives of Distributions. Some Formulas	1267
21.1.5	Operations on Distributions and Some Transformations	1269

21.1.6	Tempered Distributions and Fourier Transform	1270
21.1.7	Generalized Solutions of Linear Ordinary Differential Equations	1271
21.2	Generalized Functions of Several Variables	1271
21.2.1	Some Definitions. Partial Derivatives. Direct Product. Linear Transformations	1271
21.2.2	Dirac Delta Function. Generalized Solutions of Linear PDEs	1273

Part III Symbolic and Numerical Solutions with Maple, Mathematica, and MATLAB® 1275

22 Linear Partial Differential Equations with Maple 1277

22.1	Introduction	1277
22.1.1	Preliminary Remarks	1277
22.1.2	Brief Introduction to Maple	1279
22.1.3	Maple Language	1280
22.2	Analytical Solutions and Their Visualizations	1282
22.2.1	Constructing Analytical Solutions in Terms of Predefined Functions	1282
22.2.2	Constructing General Solutions via the Method of Characteristics	1289
22.2.3	Constructing General Solutions via Transformations to Canonical Forms	1291
22.2.4	Constructing Analytical Solutions of Cauchy Problems	1293
22.2.5	Constructing Analytical Solutions of Boundary Value Problems	1297
22.2.6	Constructing Analytical Solutions of Initial-Boundary Value Problems	1298
22.2.7	Constructing Analytical Solutions of Systems of Linear PDEs	1299
22.3	Analytical Solutions of Mathematical Problems	1301
22.3.1	Constructing Separable Solutions	1301
22.3.2	Constructing Analytical Solutions via Integral Transform Methods	1305
22.3.3	Constructing Analytical Solutions in Terms of Green's Functions	1306
22.4	Numerical Solutions and Their Visualizations	1309
22.4.1	Constructing Numerical Solutions in Terms of Predefined Functions	1310
22.4.2	Numerical Methods Embedded in Maple	1312
22.4.3	Numerical Solutions of Initial-Boundary Value Problems	1318
22.4.4	Numerical Solutions of Boundary Value Problems	1322
22.4.5	Numerical Solutions of Cauchy Problems	1323
22.4.6	Numerical Solutions of Systems of Linear PDEs	1325

23 Linear Partial Differential Equations with Mathematica 1327

23.1	Introduction	1327
23.1.1	Some Notational Conventions	1327
23.1.2	Brief Introduction to Mathematica	1327
23.1.3	Mathematica Language	1329
23.1.4	Dynamic Computation and Visualization in Mathematica Notebook	1332
23.2	Analytical Solutions and Their Visualizations	1333
23.2.1	Constructing Analytical Solutions in Terms of Predefined Functions	1333
23.2.2	Constructing General Solutions via the Method of Characteristics	1335
23.2.3	Constructing General Solutions via Conversion to Canonical Forms	1338
23.2.4	Constructing Analytical Solutions of Cauchy Problems	1340
23.2.5	Constructing Analytical Solutions of Boundary Value Problems	1342
23.2.6	Constructing Analytical Solutions of Initial-Boundary Value Problems	1344
23.2.7	Constructing Analytical Solutions of Systems of Linear PDEs	1345

23.3	Analytical Solutions of Mathematical Problems	1347
23.3.1	Constructing Separable Solutions	1347
23.3.2	Constructing Analytical Solutions via Integral Transform Methods	1350
23.3.3	Constructing Analytical Solutions in Terms of Green's Functions	1352
23.4	Numerical Solutions and Their Visualizations	1356
23.4.1	Constructing Numerical Solutions in Terms of Predefined Functions	1356
23.4.2	Numerical Methods Embedded in Mathematica	1358
23.4.3	Numerical Solutions of Initial-Boundary Value Problems	1359
23.4.4	Numerical Solutions of Boundary Value Problems	1363
23.4.5	Numerical Solutions of Cauchy Problems	1364
23.4.6	Numerical Solutions of Systems of Linear PDEs	1365
24	Linear Partial Differential Equations with MATLAB®	1367
24.1	Introduction	1367
24.1.1	Preliminary Remarks	1367
24.1.2	Brief Introduction to MATLAB	1368
24.1.3	MATLAB Language	1371
24.2	Numerical Solutions of Linear PDEs	1374
24.2.1	Constructing Numerical Solutions via Predefined Functions	1375
24.2.2	Numerical Methods Embedded in MATLAB	1383
24.2.3	Numerical Solutions of Cauchy Problems	1383
24.2.4	Numerical Solutions of Initial-Boundary Value Problems	1385
24.2.5	Numerical Solutions of Boundary Value Problems	1388
24.3	Constructing Finite-Difference Approximations	1392
24.3.1	Explicit Finite Difference Solutions	1392
24.3.2	Implicit Finite Difference Solutions	1395
24.4	Numerical Solutions of Systems of Linear PDEs	1396
24.4.1	Linear Systems of 1D PDEs	1396
24.4.2	Linear Systems of 2D PDEs	1399
	Part IV Tables and Supplements	1403
25	Elementary Functions and Their Properties	1405
25.1	Power, Exponential, and Logarithmic Functions	1405
25.1.1	Properties of the Power Function	1405
25.1.2	Properties of the Exponential Function	1405
25.1.3	Properties of the Logarithmic Function	1406
25.2	Trigonometric Functions	1407
25.2.1	Simplest Relations	1407
25.2.2	Reduction Formulas	1408
25.2.3	Relations between Trigonometric Functions of Single Argument	1408
25.2.4	Addition and Subtraction of Trigonometric Functions	1408
25.2.5	Products of Trigonometric Functions	1409
25.2.6	Powers of Trigonometric Functions	1409
25.2.7	Addition Formulas	1409
25.2.8	Trigonometric Functions of Multiple Arguments	1410
25.2.9	Trigonometric Functions of Half Argument	1410
25.2.10	Differentiation Formulas	1410
25.2.11	Integration Formulas	1410
25.2.12	Expansion in Power Series	1411

25.2.13	Representation in the Form of Infinite Products	1411
25.2.14	Euler and de Moivre Formulas. Relationship with Hyperbolic Functions	1411
25.3	Inverse Trigonometric Functions	1411
25.3.1	Definitions of Inverse Trigonometric Functions	1411
25.3.2	Simplest Formulas	1412
25.3.3	Some Properties	1412
25.3.4	Relations between Inverse Trigonometric Functions	1413
25.3.5	Addition and Subtraction of Inverse Trigonometric Functions	1413
25.3.6	Differentiation Formulas	1414
25.3.7	Integration Formulas	1414
25.3.8	Expansion in Power Series	1414
25.4	Hyperbolic Functions	1414
25.4.1	Definitions of Hyperbolic Functions	1414
25.4.2	Simplest Relations	1415
25.4.3	Relations between Hyperbolic Functions of Single Argument ($x \geq 0$)	1415
25.4.4	Addition and Subtraction of Hyperbolic Functions	1415
25.4.5	Products of Hyperbolic Functions	1415
25.4.6	Powers of Hyperbolic Functions	1416
25.4.7	Addition Formulas	1416
25.4.8	Hyperbolic Functions of Multiple Argument	1416
25.4.9	Hyperbolic Functions of Half Argument	1417
25.4.10	Differentiation Formulas	1417
25.4.11	Integration Formulas	1417
25.4.12	Expansion in Power Series	1417
25.4.13	Representation in the Form of Infinite Products	1417
25.4.14	Relationship with Trigonometric Functions	1418
25.5	Inverse Hyperbolic Functions	1418
25.5.1	Definitions of Inverse Hyperbolic Functions	1418
25.5.2	Simplest Relations	1418
25.5.3	Relations between Inverse Hyperbolic Functions	1418
25.5.4	Addition and Subtraction of Inverse Hyperbolic Functions	1418
25.5.5	Differentiation Formulas	1419
25.5.6	Integration Formulas	1419
25.5.7	Expansion in Power Series	1419
26	Finite Sums and Infinite Series	1421
26.1	Finite Numerical Sums	1421
26.1.1	Progressions	1421
26.1.2	Sums of Powers of Natural Numbers Having the Form $\sum k^m$	1421
26.1.3	Alternating Sums of Powers of Natural Numbers, $\sum (-1)^k k^m$	1422
26.1.4	Other Sums Containing Integers	1422
26.1.5	Sums Containing Binomial Coefficients	1422
26.1.6	Other Numerical Sums	1423
26.2	Finite Functional Sums	1424
26.2.1	Sums Involving Hyperbolic Functions	1424
26.2.2	Sums Involving Trigonometric Functions	1425
26.3	Infinite Numerical Series	1426
26.3.1	Progressions	1426
26.3.2	Other Numerical Series	1426

26.4	Infinite Functional Series	1428
26.4.1	Power Series	1428
26.4.2	Trigonometric Series in One Variable Involving Sine	1429
26.4.3	Trigonometric Series in One Variable Involving Cosine	1431
26.4.4	Trigonometric Series in Two Variables	1433
27	Indefinite and Definite Integrals	1435
27.1	Indefinite Integrals	1435
27.1.1	Integrals Involving Rational Functions	1435
27.1.2	Integrals Involving Irrational Functions	1439
27.1.3	Integrals Involving Exponential Functions	1442
27.1.4	Integrals Involving Hyperbolic Functions	1443
27.1.5	Integrals Involving Logarithmic Functions	1446
27.1.6	Integrals Involving Trigonometric Functions	1447
27.1.7	Integrals Involving Inverse Trigonometric Functions	1451
27.2	Definite Integrals	1452
27.2.1	Integrals Involving Power-Law Functions	1452
27.2.2	Integrals Involving Exponential Functions	1455
27.2.3	Integrals Involving Hyperbolic Functions	1456
27.2.4	Integrals Involving Logarithmic Functions	1457
27.2.5	Integrals Involving Trigonometric Functions	1457
27.2.6	Integrals Involving Bessel Functions	1460
28	Integral Transforms	1463
28.1	Tables of Laplace Transforms	1463
28.1.1	General Formulas	1463
28.1.2	Expressions with Power-Law Functions	1465
28.1.3	Expressions with Exponential Functions	1465
28.1.4	Expressions with Hyperbolic Functions	1466
28.1.5	Expressions with Logarithmic Functions	1467
28.1.6	Expressions with Trigonometric Functions	1467
28.1.7	Expressions with Special Functions	1469
28.2	Tables of Inverse Laplace Transforms	1470
28.2.1	General Formulas	1470
28.2.2	Expressions with Rational Functions	1472
28.2.3	Expressions with Square Roots	1476
28.2.4	Expressions with Arbitrary Powers	1477
28.2.5	Expressions with Exponential Functions	1478
28.2.6	Expressions with Hyperbolic Functions	1479
28.2.7	Expressions with Logarithmic Functions	1480
28.2.8	Expressions with Trigonometric Functions	1481
28.2.9	Expressions with Special Functions	1481
28.3	Tables of Fourier Cosine Transforms	1482
28.3.1	General Formulas	1482
28.3.2	Expressions with Power-Law Functions	1482
28.3.3	Expressions with Exponential Functions	1483
28.3.4	Expressions with Hyperbolic Functions	1484
28.3.5	Expressions with Logarithmic Functions	1485
28.3.6	Expressions with Trigonometric Functions	1485
28.3.7	Expressions with Special Functions	1486

28.4	Tables of Fourier Sine Transforms	1488
28.4.1	General Formulas	1488
28.4.2	Expressions with Power-Law Functions	1488
28.4.3	Expressions with Exponential Functions	1489
28.4.4	Expressions with Hyperbolic Functions	1489
28.4.5	Expressions with Logarithmic Functions	1490
28.4.6	Expressions with Trigonometric Functions	1490
28.4.7	Expressions with Special Functions	1492
29	Curvilinear Coordinates, Vectors, Operators, and Differential Relations	1495
29.1	Arbitrary Curvilinear Coordinate Systems	1495
29.1.1	General Nonorthogonal Curvilinear Coordinates	1495
29.1.2	General Orthogonal Curvilinear Coordinates	1497
29.2	Cartesian, Cylindrical, and Spherical Coordinate Systems	1498
29.2.1	Cartesian Coordinates	1498
29.2.2	Cylindrical Coordinates	1499
29.2.3	Spherical Coordinates	1500
29.3	Other Special Orthogonal Coordinates	1502
29.3.1	Coordinates of a Prolate Ellipsoid of Revolution	1502
29.3.2	Coordinates of an Oblate Ellipsoid of Revolution	1503
29.3.3	Coordinates of an Elliptic Cylinder	1504
29.3.4	Conical Coordinates	1505
29.3.5	Parabolic Cylinder Coordinates	1506
29.3.6	Parabolic Coordinates	1506
29.3.7	Bicylindrical Coordinates	1507
29.3.8	Bipolar Coordinates (in Space)	1507
29.3.9	Toroidal Coordinates	1508
30	Special Functions and Their Properties	1509
30.1	Some Coefficients, Symbols, and Numbers	1509
30.1.1	Binomial Coefficients	1509
30.1.2	Pochhammer Symbol	1510
30.1.3	Bernoulli Numbers	1510
30.1.4	Euler Numbers	1511
30.2	Error Functions, Exponential and Logarithmic Integrals	1512
30.2.1	Error Function and Complementary Error Function	1512
30.2.2	Exponential Integral	1512
30.2.3	Logarithmic Integral	1513
30.3	Sine Integral and Cosine Integral, Fresnel Integrals	1514
30.3.1	Sine Integral	1514
30.3.2	Cosine Integral	1515
30.3.3	Fresnel Integrals	1515
30.4	Gamma Function, Psi Function, and Beta Function	1516
30.4.1	Gamma Function	1516
30.4.2	Psi Function (Digamma Function)	1517
30.4.3	Beta Function	1518
30.5	Incomplete Gamma and Beta Functions	1519
30.5.1	Incomplete Gamma Function	1519
30.5.2	Incomplete Beta Function	1520

30.6	Bessel Functions (Cylindrical Functions)	1520
30.6.1	Definitions and Basic Formulas	1520
30.6.2	Integral Representations and Asymptotic Expansions	1522
30.6.3	Zeros and Orthogonality Properties of Bessel Functions	1524
30.6.4	Hankel Functions (Bessel Functions of the Third Kind)	1525
30.7	Modified Bessel Functions	1526
30.7.1	Definitions. Basic Formulas	1526
30.7.2	Integral Representations and Asymptotic Expansions	1528
30.8	Airy Functions	1529
30.8.1	Definition and Basic Formulas	1529
30.8.2	Power Series and Asymptotic Expansions	1529
30.9	Degenerate Hypergeometric Functions (Kummer Functions)	1530
30.9.1	Definitions and Basic Formulas	1530
30.9.2	Integral Representations and Asymptotic Expansions	1533
30.9.3	Whittaker Functions	1534
30.10	Hypergeometric Functions	1534
30.10.1	Various Representations of the Hypergeometric Function	1534
30.10.2	Basic Properties	1536
30.11	Legendre Polynomials, Legendre Functions, and Associated Legendre Functions	1536
30.11.1	Legendre Polynomials and Legendre Functions	1536
30.11.2	Associated Legendre Functions with Integer Indices and Real Argument	1539
30.11.3	Associated Legendre Functions. General Case	1539
30.12	Parabolic Cylinder Functions	1542
30.12.1	Definitions. Basic Formulas	1542
30.12.2	Integral Representations, Asymptotic Expansions, and Linear Relations	1543
30.13	Elliptic Integrals	1544
30.13.1	Complete Elliptic Integrals	1544
30.13.2	Incomplete Elliptic Integrals (Elliptic Integrals)	1545
30.14	Elliptic Functions	1547
30.14.1	Jacobi Elliptic Functions	1547
30.14.2	Weierstrass Elliptic Function	1551
30.15	Jacobi Theta Functions	1553
30.15.1	Series Representation of the Jacobi Theta Functions. Simplest Properties	1553
30.15.2	Various Relations and Formulas. Connection with Jacobi Elliptic Functions	1554
30.16	Mathieu Functions and Modified Mathieu Functions	1555
30.16.1	Mathieu Functions	1555
30.16.2	Modified Mathieu Functions	1557
30.17	Orthogonal Polynomials	1557
30.17.1	Laguerre Polynomials and Generalized Laguerre Polynomials	1558
30.17.2	Chebyshev Polynomials and Functions	1559
30.17.3	Hermite Polynomials	1561
30.17.4	Jacobi Polynomials and Gegenbauer Polynomials	1563

30.18 Nonorthogonal Polynomials	1564
30.18.1 Bernoulli Polynomials	1564
30.18.2 Euler Polynomials	1565
References	1569
Index	1587