Notices of the South African
Mathematical Society
Volume 31, No. 1, April 2000, 1 - 8

A remark on Sincov's functional equation

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Abstract

This is a historical note on Sincov's functional equation

$$\varphi(x, y) + \varphi(y, z) = \varphi(x, z).$$

Sincov gave in 1903 an elegant proof of its general real solution, which has the form

$$\varphi(x, y) = \psi(x) - \psi(y),$$

where ψ is an arbitrary function in one variable. Others, like Moritz Cantor and Gottlob Frege treated this equation before Sincov.

AMS Subject Classification: 01A55, 39-03, 39B12

Key words and phrases: history of mathematics, functional equations in several variables

1. Introduction

The functional equation

$$\varphi(x, y) + \varphi(y, z) = \varphi(x, z) \tag{1}$$

is called *Sincov equation* (see e.g. [2] and [3]), after Dmitrii Matveevich Sincov (November 21, 1867 – January 28, 1946) or Sintsov (Д. М. Синцовъ, german transscription Sintzow). It seemed that he was the first who gave (in two papers [21], [22] in 1903) elementary simple proofs of its general real solutions.

But before, it was Moritz Cantor (August 8, 1829 – April 9, 1920) who proposed these equations (as we will see there are two equations). In his journal "Zeitschrift für Mathematik und Physik", whose editor he was jointly with O. Schlömilch he published in 1896 a note [5]. Cantor quotes these equations as examples of equations in three variables which can be solved by the method of differential calculus due to Niels Henrik Abel (August 5, 1802 – April 6, 1829).

Let us quote Sincov from [22]:

"Herr M. Cantor hat in der Zeitschrift für Math. und Phys. **41,** 161-163 die Beispiele zweier Funktionalgleichungen mit drei Veränderlichen angegeben:

$$(1) \quad \varphi(x, y) + \varphi(y, z) = \varphi(x, z), \qquad (2) \quad \varphi(x, y) \cdot \varphi(y, z) = \varphi(x, z).$$

Seine Lösungen sind sehr einfach; doch differenziert der Verf. dabei die unbekannte Funktion und setzt also ihre Differenzierbarkeit voraus. Ich will daher eine andere Lösung angeben, welche eine derartige Voraussetzung vermeidet."[1]

It happened to me that I found in the mathematical work of the famous logician Gottlob Frege (November 8, 1848 – July 26, 1925), i.e. in his habilitation thesis [8] *"Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen"*, Jena 1874, also the Sincov equation (1). Frege discovered this equation in connection with dynamical systems to solve the so-called "translation equation".

In this note I will write about the early history of the Sincov equation.

2. Moritz Cantor

Before we come to Sincov's proof let us see what M. Cantor is doing in the mentioned paper [5], published as "Kleine Mittheilung". Cantor gives firstly a remark on Cauchy's functional equations (treated in the 5-th chapter of his "Analyse algébrique", 1821), and how Abel in his treatise "Ueber die Functionen, welche der Gleichung $\varphi(x) + \varphi(y) = \psi[x.f(y) + y.f(x)]$ genugthun" in the Crelle journal II, 386-394, solved these type of equations by differential calculus. He refered also to one of the functional equations treated by Abel: $\varphi(x)+\varphi(y) = \psi[x \cdot f(y)+y \cdot f(y)]$. He refers to Abel's method solving functional equations in two variables by (possibly repeated partial) differentiation of the unknown function in two variables. The completed works of Niels Henrik Abel firstly appeared in 1839, a new edition was edited in 1881 by L. Sylow and S. Lie [1]; so Cantor had easely access to Abel's work. Cantor in [5]: *"Seit dieser Zeit sind die Cauchy'schen Beispiele bald ohne, bald mit Differentialrechnung behandelt mehrfach in Lehrbücher und Übungsbücher übergegangen. Beispiele ähnlicher Aufgaben mit mehr als zwei voneinander unabhängigen Variablen sind uns dagegen in der Literatur nicht begegnet."*[2] Cantor gives no hint about the origin, nor any application, of the considered equations (1) and (2).

In the following we give a shortened version of Cantors lengthy arguments [5]: In solving equation (1), Cantor concludes $\varphi(x, x) = 0$ by putting $z = y = x$ and afterwards by putting $z = x$

$$\varphi(y, x) = -\varphi(x, y). \tag{3}$$

Then he takes the partial derive of equation (1)

$$\frac{\partial\varphi(y, z)}{\partial z} = \frac{\partial\varphi(x, z)}{\partial z},$$

[1]Mr. Cantor gave in Z. f. Math. Phys. **41,** 161-163, two examples of functional equations in three varables:

(1) $\varphi(x, y) + \varphi(y, z) = \varphi(x, z)$, (2) $\varphi(x, y) \cdot \varphi(y, z) = \varphi(x, z)$.

Its solutions ar very simple. But the author differentiates the unknown function, and therefore he supposes their differentiability. I will give another solution avoiding such conditions.

[2]Since that time the Cauchy's examples entered into textbooks, treated without or with differential calculus. But it never happened that we saw similar examples in more than to variables in the bibliography.

which shows that each of these differential quotients does not contain either x nor y, hence φ must have the form

$$\varphi(x,z) = \psi(x) + \chi(z). \tag{4}$$

By (3) we get

$$\varphi(z,x) = -\varphi(x,z) \quad \text{or} \quad \psi(x) + \chi(x) = -[\psi(z) + \chi(z)],$$

hence we have $\chi(x) = -\psi(x)$, $\chi(z) = -\psi(z)$. Thus

$$\varphi(x,z) = \psi(x) - \psi(z)$$

is the unique solution of (1) (with arbitrary, differentiable ψ).

With a similar argument Cantor shows that the general solution of the multiplicatively written equation (2) is of the form

$$\varphi(x,z) = \frac{\psi(x)}{\psi(z)}$$

with an arbitrary nonzero differentiable function ψ.

Moritz Benedikt Cantor was a wellknown mathematician. He was, together with O. Schlömilch the editor of the *Zeitschrift für Mathematik und Physik* and his four volume work on history of mathematics [6], published 1899 – 1908, was a standard reference on this topic for a long time. Nevertheless, the above cited note [5] is badly written. Cantor pursues stupidly the method of differentiation and ignores the question of nondifferentiable solutions. In the case of the multiplicative equation (2) Cantor calculates with the fraction $\frac{\varphi(x,z)}{\varphi(y,z)}$ disregarding, as usual at that time, the fact that the denominator may be zero.

3. Sincov's proof

The proof of Sincov in [22] is much simpler and elegant. Sincov argues as follows: "Die Gleichung (1) schreiben wir in der Form:

$$\varphi(x,y) = \varphi(x,z) - \varphi(y,z).$$

Die linke Seite ist von z unabhängig; z muß also auch in der rechten Seite herausfallen. Wir können daher ohne die Allgemeinheit zu beeinträchtigen, z irgendeinen bestimmten Wert beilegen, z.B. $z = a$, was gewiß voraussetzt, $\varphi(x,a)$ sei nicht durchgehend unendlich. Indem wir noch $\varphi(x,a) = \theta(x)$, gleich einer willkürlichen Funktion von x, setzen, gelangen wir zur Lösung des Herrn M. Cantor:

$$\varphi(x,y) = \theta(x) - \theta(y).$$

. . . "[3]

[3] Rewriting (1) we get $\varphi(x,y) = \varphi(x,z) - \varphi(y,z)$. The left hand side is independent of z, hence also the right hand side does not depend on z. So we may put $z = a$ and take $\theta(x) = \varphi(x,a)$ as an arbitrary function of x, thus we get the solution of M. Cantor: $\varphi(x,y) = \theta(x) - \theta(y)$.

This is the short and concise argument of Sincov.

The multiplicative equation (2) is solved in [22] by similar arguments and, alternatively, by taking the logarithms. Sincov too disregards the possible fact that a solution $\varphi(x, y)$ may vanish at a point. To handle this case, one can see easily, that any solution of (2) must be either constant zero, otherwise it is nonzero at each point (x, y). As a side remark one can mention that also Cauchy, in his *Cours d'analyse de L'École Polytechnique*, Vol. 1, Analyse algébrique V, Paris 1821, disregards the constant zero solution of the functional equation $f(x+y) = f(x) \cdot f(y)$ of the exponential function.

Dmitrii Matveevich Sincov was an esteemed mathematician from the Ukraine, who mainly was concerned with differential equations and differential geometry. For more about his life see [17]. J. Aczél informed me that he preferred to introduce the terminology "Sincov equation" instead of "Cantor equation" to avoid confusion with Georg Cantor.

4. Gottlob Frege

Also the famous logician Gottlob Frege dealt with functional equations in his habilitation thesis [8] which appeared 1874, as a 27 page booklet with the title *"Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen"* (Methods of calculations, based on an extension of the notion of magnitudes). It is available as a reprint from a collection of Frege's work ([9], pp. 50-84) and also in English translation ([10], p. 56-92).

Frege investigates there the translation equation in explicit form and he uses the Sincov equation and also differential equations and the infinitesimal generator to determine solutions of the translation equation. For more details on this see [11] or [12].

Frege introduces as "an extension of the notion of magnitudes" a generalized n-th iterate, also for nonintegers n, of a given function f. This is the nowadays socalled *"continuous iterate"* of the function f. If a function ϕ is the n-th iterate of f, then Frege denotes the (not necessary integer) number n as the "magnitude" of ϕ.

On [9], p. 53-54, ([10], p. 59-60), Frege writes:

"One can raise the following questions:

What is the function, whose magnitude stands in a given ratio to that of the magnitude of a given function?

Given two functions, are they in the same Größengebiet and, if so, what is the ratio of their magnitudes?"

The first question means: given a real n, what is the n-th iterate of the given function? The second question raises up the problem to find a geodesic through the two given functions.

"The answer to these questions is closely connected to the knowledge of the general form of a function which is the n-fold of a given one. More precisely, one has to have a function of n and x which, for $n = 1$ turns into the given function of x, and for which, generally, the functional equation [Frege writes indeed *Functionalgleichung* (D.G.)]

$$f(n_0, f(n_1, x)) = f(n_0 + n_1, x) \tag{1}$$

holds. [This is the socalled *translation equation* (D.G.).]

If one denotes the value of this function by X, then one can also say: one has to have an equation connecting the magnitude n, the value X and the arguments x of a function. We will call such an equation a "Größengleichung" (magnitude equation). If n is given by X and x via

$$n = \psi(X, x),$$

then the function ψ must have the property that by elimination of x_0 from $n_0 = \psi(X, x_0)$ and $n_1 = \psi(x_0, x_1)$ we get the equation

$$n_0 + n_1 = \psi(X, x_1)$$

or

$$\psi(X, x_0) + \psi(x_0, x_1) = \psi(X, x_1). \tag{2}$$

. . . "

Here we recover the Sincov equation. And Frege gives also the solution of this equation.

Frege continues ([9], p. 55, [10], p. 61-62):

"If we set $x_1 = X$ then (2) becomes

$$\psi(x_1, x_0) + \psi(x_0, x_1) = 0.$$

Subtracting this from (2) we get

$$\psi(X, x_0) - \psi(x_1, x_0) = \psi(X, x_1).$$

If we consider x_0 as a constant, we can write the Größengleichung $n = \psi(X, x)$ in the form

$$n = \vartheta(X) - \vartheta(x),$$

where $\vartheta(x) = \psi(x, x_0)$."

Here we find the general solution of Sincov's equation

$$\psi(X, x) = \vartheta(X) - \vartheta(x).$$

Frege uses this representation to calculate solutions of the translation equation (note that $X = f(n, x)$) in the form

$$f(n, x) = \vartheta^{-1}\left(n + \vartheta(x)\right),$$

a first step to determine the general solution of the translation equation (see [2], [15]).

Unfortunately this work of Frege was completely forgotten and had no influence in further research on functional equations and iteration theory.

5. Further work on Sincov's equation

Of course, the Sincov equation is also applicable to functions with more general domains and codomains. The following result is easy to prove.

Theorem: *Let $\varphi : S \times S \to X$ be a function where S is an arbitrary set.*
a.) If X is an arbitrary (not necessarily abelian) group. Then φ is a solution of

$$\varphi(x,y)\cdot\varphi(y,z) = \varphi(x,z) \text{ for all } x,y,z \in S \tag{1}$$

if and only if there exist a function $\psi : S \to X$, such that

$$\varphi(x,y) = \psi(x)\cdot\psi(y)^{-1}. \tag{2}$$

b.) If X is an arbitrary (not necessarily abelian) ring with unit element 1, where the only idempotent elements (i.e. $x \in X$ with $x^2 = x$) are 0 and 1. Then φ is a solution of (1) if and only if either $\varphi \equiv 0$ or $\varphi(x,y)$ is invertible for all $x,y \in S$ and there exists a function $\psi : S \to X$, such that $\psi(x)$ is invertible for all $x \in S$ and (2) holds.

In [2], p. 223 and also in [3], p. 377 are some of the application of the Sincov equations and a list of many references where the Sincov equation play some role.

Here we will quote some further topics, where also the Sincov equation is involved:

- *The principal fundamental solution* $Y(x_0,x)$ of the linear differential equation system for n unknown functions

  $$y'(x) = y(x)\cdot A(x), \tag{3}$$

  where the coefficients of the $n \times n$ matrix $A(x)$ are continuous, satisfies the multiplicative Sincov equation; i.e. $Y(x_0,x)$ is defined as the unique matrix solution of (3) which takes at x_0 the identity matrix as initial value. Here we have $Y(x_0,x_1)\cdot Y(x_1,x) = Y(x_0,x)$, see e.g. Lappo-Danilevsky [14], Vol. I, p. 232. Similar arguments apply also for the principal fundamental solution of a linear difference equation of order n.
- *For the practical rule of interest compounding* Jens Schwaiger in [20] applied the general solution of the multiplicative Sincov equation to determine the general solution of a functional equation which is satisfied by the capital function $C(K,t,s)$, which denotes the value of the capital K if it was invested in the time interval beween t and s (a problem of economics).
- *The Sincov equation on restricted domain* were treated by B. Ebanks and others, see e.g. [7] in connection with information measures.
- *In generalizing the translation equation* (Pexiderization of the translation equation) Z. Moszner [16] introduced a generalized Sincov equation for three unknown function $G_1,\ G_2,\ G_3$: $G_3(x,z) = G_2(x,y)\cdot G_1(y,z)$.

I am quite sure that there are a lot of mathematical papers which contain the Sincov equation in implicit or explicit form. Also in some recent papers I discovered the Sincov equation. Here I may only quote as examples [4], [18] and [19].

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