# SOME FUNCTIONAL EQUATIONS IN THE THEORY OF RELATIVITY.

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IN transcribing the process of light-signalling which leads to his kinematic equations of transformation, Einstein* obtains relations, connecting the space-time coordinates in two systems of reference, in the form of functional equations. He solves these by passing to partial differential equations, and restricts the solutions to linearity by a rather obscure appeal to the homogeneity of time and space. In view of the fundamental character that his theory has proved to possess it is natural to ask more closely what is implied in the term homogeneity, which occurs in various senses in physical science, and whether it is necessary to assume differentiability. These and some related questions bearing on the fundamentals of that theory suggested the following analysis, which aims at a more complete transcription by means of functional equations and their solution without appeal to differentiation.

The general setting of the problem is found in the supposition that for the set of space-time points there are various systems of reference, corresponding to observers and carriers of coordinate frames in uniform translatory motion with respect to each other; these systems being, under conditions to be described, all on a parity with each other in the sense to be defined as the meaning of relativity. It is understood that every space-time point is identifiable by each system in terms of a unique quadruple of coordinates, and conversely that each such quadruple that occurs at all identifies a unique space-time point. For a physical theory some limitation on the range of the coordinates might perhaps seem appropriate, and for the most part even if such a limitation of any natural kind were imposed the results would probably be unchanged if interpreted for corresponding regions. But in order to avoid the complication of boundary conditions that would accompany such a limitation it will be supposed that every

* Einstein, *Drude Annalen,* vol. 17, pp. 891–921 (1905).

point of euclidean space of three dimensions exists at each instant for every observer, that no other points ever exist, and that the space-time manifold is in a certain impartial sense continuous. For analytic purposes this is taken to mean that identification of the same space-time points by any two observers yields a unique one-to-one continuous correspondence of their quadruples of coordinate values, existing for the entire infinite real range. In other words the coordinates $(\tau, \xi, \eta, \zeta)$ in any system $\Sigma$ are single-valued continuous functions of the coordinates $(t, x, y, z)$ in any other system $S$, for all real values of the latter; these functions having then in some definite sense, though perhaps not algebraically, single-valued inverses. The problem is to determine the form of the functions by the imposition of conditions suitable to the notion of relativity as understood in the Einstein theory. For each system by itself the full kinematics of euclidean space and optical time is taken for granted.

Einstein's postulate of the constancy of the velocity of light has been much discussed, yet seems to invite still more detailed analysis than it has hitherto received. It clearly involves along with its physical content a considerable element of convention, with respect to orientation of axes and choice of units for instance. The essential physical feature seems to be that there is supposed to exist in every system by itself a cartesian scheme of measurement for space and time such that as interpreted in terms of that scheme space is optically isotropic; that is, light travels in straight lines with a velocity constant for each line and the same for all. This involves of course that much-discussed independence of the motion of the source of the light. For Einstein's plan of testing clocks by means of to-and-fro light signals, however, the rectilinearity of the light path is not directly made use of. It is in effect assumed that, starting at any instant, a light-signal can be sent from any point to any other, and that the distance is proportional to the time of transit. The factor of proportionality, the velocity of light, is assumed to be the same for all pairs of points, in any single system, and for convenience is taken as having the same numerical value $c$ in every system by means of a single admissible restriction on the units of length and time in each. For the present purpose it will be assumed that in every system the corresponding scheme of measurement is of the kind described. The nature of the conditions to be imposed is such that there remain arbitrary in each system the choice of origin, euclidean transformation of space-axes, and one further condition on the units.

That the system $\Sigma$ as observed by $S$ has a uniform translatory rigid motion means: that all points which in $\Sigma$ appear fixed have in $S$ a common velocity, of constant direction and of constant magnitude $v$, and conversely that any set of points having in $S$ this particular velocity appear in $\Sigma$ fixed. For consideration of a particular pair of systems let the direction of this velocity specify the direction of the $x$-axis in $S$. The light signals are to be thought of as passing between points fixed in $\Sigma$ and described in $S$. A signal is longitudinal when these points are at one instant and therefore always on a line in $S$ parallel to the $x$-axis; it is lateral when the points lie on a moving line perpendicular to the $x$-axis.

A forward longitudinal signal between points that in $S$ are at a distance $a$ apart may be described as starting at $(t, x, y, z)$, going to

$$\left(t + \frac{a}{c - v}, \ x + \frac{ca}{c - v}, \ y, \ z\right),$$

then to

$$\left(t + \frac{2ca}{c^2 - v^2}, \ x + \frac{2cva}{c^2 - v^2}, \ y, \ z\right);$$

or more conveniently, by the use of the relative abscissa $x' = x - vt$, as involving successively

$$[t, x', y, z], \qquad \left[t + \frac{a}{c - v}, \ x' + a, \ y, \ z\right],$$

and

$$\left[t + \frac{2ca}{c^2 - v^2}, \ x', \ y, \ z\right];$$

a backward longitudinal signal over an object of the same apparent length is like this except that $(a, c)$ are replaced by $(-a, -c)$. Since the first and third are spatially coincident in $\Sigma$ the intermediate $\tau$ must be the arithmetic mean of the others. Hence, if $\tau = F(x', t)$, where for brevity $(y, z, v)$ are omitted temporarily from the functional symbol because only parameters, the conditions are the Einstein functional equation and its mate for opposite signals:

$$
\begin{aligned}
&F(x', t) + F\left(x', t + \frac{2ca}{c^2 - v^2}\right) \\
(1) \qquad &= 2F\left(x' + a,\ t + \frac{a}{c - v}\right) = 2F\left(x' - a,\ t + \frac{a}{c + v}\right).
\end{aligned}
$$

The combined or right hand equality says in effect that $F(\alpha, \beta)$ is invariant under the transformation

$$\alpha' = \alpha + 2a, \qquad \beta' = \beta + \frac{2va}{c^2 - v^2},$$

with $a$ arbitrary, and is therefore a function of the invariant

$$\beta - \frac{v\alpha}{c^2 - v^2}$$

only, so that

$$F(x', t) = F\left(t - \frac{vx'}{c^2 - v^2}\right).$$

Substitution in the first equality in (1) reduces it to the form

$$F(u) + F(u + 2h) = 2F(u + h),$$

so that $F$, being continuous, is linear. Hence

$$\tau = \frac{\varphi(y, z, v)}{\kappa}\left(t - \frac{vx}{c^2}\right) + f(y, z, v), \tag{2}$$

where

$$\kappa^2 = 1 - \frac{v^2}{c^2}.$$

Here the first coefficient is written in a form later convenient, the $\varphi$ corresponding to Einstein's.

A lateral signal over an object of width $b$ in a plane of constant $z$ would involve

$$(t, x, y, z), \qquad \left(t + \frac{b}{c\kappa},\ x + \frac{vb}{c\kappa},\ y + b,\ z\right),$$

and

$$\left(t + \frac{2b}{c\kappa},\ x + \frac{2vb}{c\kappa},\ y,\ z\right).$$

The condition that here also the intermediate $\tau$ be the arithmetic mean reduces to

$$\left[t - \frac{vx}{c^2} + \frac{b}{c}\right][\varphi(y, z, v) - \varphi(y + b, z, v)] + [f(y, z, v) - f(y + b, z, v)] = 0,$$

showing that $\varphi$ and $f$ must be independent of $y$ and by symmetry of $z$ also.

If the lateral signal be considered first, and $\tau = F(x', y, t)$, then the corresponding condition is

$$(3)\qquad \begin{aligned} &F(x', y, t) + F\left(x', y, t + \frac{2b}{c\kappa}\right) \\ &\quad = 2F\left(x', y + b, t + \frac{b}{c\kappa}\right) = 2F\left(x', y - b, t + \frac{b}{c\kappa}\right), \end{aligned}$$

which shows that $\tau$ must be independent of $y$, and similarly of $z$, and must be linear in $t$, so that

$$(4)\qquad \tau = \psi(x', v)t + g(x', v).$$

Then the longitudinal signal gives the condition

$$\begin{aligned} &t[\psi(x', v) - \psi(x' + a, v)] \\ &\quad + \frac{a}{c - v}\left[\frac{c}{c + v}\psi(x', v) - \psi(x' + a, v)\right] \\ &\quad\quad + [g(x', v) - g(x' + a, v)] = 0 \end{aligned}$$

which shows that $\psi$ must be independent of $x'$ and that

$$(5)\qquad g(x' + a, v) - g(x', v) = -\frac{av}{c^2 - v^2}\psi(v),$$

so that $g(x', v) + vx'\psi(v)/(c^2 - v^2)$ must be a function of $v$ only. The two orders of proof thus lead to the result

$$(6)\qquad \tau = \frac{\varphi(v)}{\kappa}\left(t - \frac{vx}{c^2}\right) + f(v),$$

which evidently contains all the information that the signalling process can give about the time reckoning in $\Sigma$, since the undetermined functions $f$ and $\varphi$ correspond to arbitrary choice of origin and units. This equation shows that the time intervals in $\Sigma$ of one-way transit are $a\varphi(v)/\kappa c$ and $b\varphi(v)/c$ for the longitudinal and lateral signals respectively.

The condition that in each of the instances mentioned of to-and-fro signal the initial and final space points coincide in $\Sigma$ shows that $(\xi, \eta, \zeta)$ are functions of $(x', y, z, v)$ only, independent of $t$; which is in fact equivalent to the condition of rigid motion of $\Sigma$ with respect to $S$, when the time reckoning in $\Sigma$ is given by (6). The relation between the two systems of space coordinates will appear from consideration of a one-way signal of arbitrary direction and time interval. For convenience let $x'/\kappa = x''$; then in general

$$\Delta x = \kappa \Delta x'' + v\Delta t, \qquad \Delta\tau = \varphi(v)(\kappa\Delta t - v\Delta x''/c^2).$$

From these it follows that the distance $r$, which must be the same as $c\Delta t$, travelled in the time interval $\Delta t$ by the signal in $(x, y, z)$ space, is related to the apparent distance $r''$ in the space of the auxiliary coordinates $(x'', y, z)$ by

$$r^2 = c^2(\Delta t)^2 = r''^2 - v^2(\Delta x'')^2/c^2 + v^2(\Delta t)^2 + 2\kappa v\Delta x''\Delta t,$$

which reduces to

$$r''^2 = (\kappa c\Delta t - v\Delta x'')^2 = c^2(\Delta\tau)^2/\varphi^2.$$

This shows that the distance in $(\xi, \eta, \zeta)$ space is always the same as in $(\varphi x'', \varphi y, \varphi z)$ space, so that these two triples must be linear orthogonal transforms of each other; the known proof of this requires no assumption of differentiability. Combined with (6) this gives essentially Einstein's result, except that there is here no initial supposition as to the orientation of the axes in $\Sigma$.

If, however, the special supposition be made that $\xi$ is a function of $(x, t)$ only, therefore of $(x', t)$ only, then the longitudinal signal gives

$$G\left(x', t + \frac{2ca}{c^2 - v^2}\right) = G(x', t), \tag{7}$$

making $G$ independent of $t$; also

$$G(x' + a) = G(x') + a\varphi(v)/\kappa, \tag{8}$$

so that, $G$ being continuous, $\xi = \varphi(v)x'/\kappa + g(v)$. Then the remaining conditions show that the pair $(\eta, \zeta)$ is an orthogonal transform of $(\varphi y, \varphi z)$. The general result is conveniently viewed from Minkowski's point of view, as giving $(ic\tau, \xi, \eta, \zeta)$ from $(ict, x, y, z)$ by a rigid rotation about an arbitrary origin, followed by a magnification with factor $\varphi$, whose arbitrariness indicates the freedom remaining in the choice of units.

Thus far the proof refers to a particular pair of systems only, and the undetermined functions $\varphi$, $f$, $g$, and the like, instead of being written as functions of $v$, are thus more suitably to be regarded as merely functions of the pair of systems whose relation is sought. For there is nothing in the general assumptions to imply that their values need be the same for two $\Sigma$'s moving with respect to $S$ with the same velocity, even if the directions were also the same, in which case they would be relatively to each other at rest. But in the case of a set of systems, each standing to each in the relation restricted by the general postulates, certain conditions are imposed on these functions by the requirement of transitivity, or the group property as it here becomes. For example, with $n$ systems there would be $n(n-1)/2$ factors $\varphi$ and their inverses, but only $(n-1)$ arbitrary ratios of units. If then instead of $\varphi(v)$ there be written $\varphi(\Sigma, S)$, which for this purpose might be more suggestively $\varphi(\Sigma/S)$, these conditions have the form

$$\varphi(U, T)\varphi(T, S) = \varphi(U, S). \tag{9}$$

The algebraic form of the implications of this equation would depend on the set of systems considered. If the artificiality be discarded that would allow $\varphi$ to be different for two $\Sigma$'s at rest with respect to each other or moving with respect to $S$ with the same speed in different directions, the condition reduces to

$$\varphi(v)\varphi(v') = \varphi(v''), \tag{10}$$

where $v''$ is the resultant of $v$ and $v'$ in Einstein's sense. If any rigid translation with respect to any $S$ gives a possible $\Sigma$, then, by alteration of the direction of $v'$, $v''$ can vary while $v$ and $v'$ are constant, so that $\varphi$ must be constant and therefore unity, as Einstein makes it. Poincaré* had already indicated this as a condition for a group in the Lorentz transformation.† If, however, only parallel motions were admitted, and $v$ taken as with sign, the condition would be

$$\varphi(v)\varphi(v') = \varphi\left(\frac{v+v'}{1+vv'/c^2}\right). \tag{11}$$

The continuous solution of this, readily obtained through the transformation $v/c = \tanh \alpha$, is

* Poincaré, *Comptes Rendus*, vol. 140, pp. 1504–1508 (1905).

† Lorentz, *Proc. Amsterdam A.*, vol. 6, pp. 809–831 (1904); and Theory of Electrons, p. 197.

$$\varphi(v) = \left(\frac{c+v}{c-v}\right)^n, \tag{12}$$

where $n$ is an arbitrary constant. Einstein's condition that reversal of $v$ leave $\varphi$ unaltered makes $n = 0$; as to physical interpretation it may be noted that this is the only case where the contraction factor would have an expansion in powers of $v$ free from a linear term.

Einstein's original proof referred only to signals transmitted through a vacuum. The case of a refracting medium of arbitrary index $\mu$, at rest with respect to $\Sigma$, was considered by Laub,* who imposed the requirement that the comparison of space and time reckoning be independent of the value of $\mu$. He obtained thereby the various velocities of propagation with respect to $S$, leading to the Fresnel-Fizeau convection coefficients. These results were then shown by Laue† to be instances of the Einstein law of composition of velocities. The following examples illustrate the modification of the functional equations to suit this case.

The medium is supposed to appear isotropic in $\Sigma$, with velocity of propagation $c/\mu$; but in $S$ light will have different velocities in different directions, except when $\mu$ is unity. Considering only the longitudinal signal let the forward and backward velocities in $S$ be $c'$, $c''$. Then (1) is replaced by

$$\begin{aligned} F(x', t) + F\left(x', t + a\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right]\right) & \\ = F\left(x' + a, t + \frac{a}{c'-v}\right) & \qquad (13) \\ = F\left(x' - a, t + \frac{a}{c''+v}\right), & \end{aligned}$$

making $F(\alpha, \beta)$ invariant under the transformation

$$\alpha' = \alpha + 2a, \qquad \beta' = \beta + a\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right],$$

so that

$$F(x', t) = F\left(t - \frac{x'}{2}\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right]\right).$$

Linearity appearing as before, the result is

* Laub, *Ann. d. Phys.*, vol. 23, pp. 738–744 (1907).
† Laue, *Ann. d. Phys.*, vol. 23, pp. 989–990 (1907).

$$(14)\quad \tau = \frac{\varphi(v, \mu)}{\kappa}\left(t - \frac{x'}{2}\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right]\right) + f(v, \mu).$$

Equation (7) is replaced by

$$(15)\qquad G(x', t) = G\left(x',\ t + a\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right]\right),$$

the equation (8) by

$$(16)\qquad G(x' + a) = G(x') + \frac{c\varphi a}{2\mu\kappa}\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right],$$

whence

$$(17)\qquad \xi = \frac{c\varphi x'}{2\mu}\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right] + g(x, \mu).$$

The corresponding modifications for lateral and other signals are readily made.

It appears therefore that Einstein's assumption of differentiability is unnecessary, and that the needed features of "homogeneity" are already implicit in the optical and kinematic postulates as here interpreted.

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# FORMULAS FOR CONSTRUCTING ABRIDGED MORTALITY TABLES FOR DECENNIAL AGES.

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MR. GEORGE KING presented in the British Registrar-General's Report for 1914 a method of constructing abridged mortality tables which consist merely of the values appearing in an ordinary mortality table but corresponding only to each quinquennial age. The computation of the values corresponding to the other ages is eliminated by the method and an enormous saving of labor is thus gained.

Such tables promise to prove very useful in investigations in human vitality because many problems which have heretofore involved too much computation to permit much investigation now become relatively easy. For this reason it seems