Solution of Functional Equations and Functional-Differential Equations by the Differentiation Method

© 2004 **A. D. Polyanin and A. I. Zhurov**

21 December 2004

Here we describe various classes of functional equations and functional-differential equations that can be solved by differentiating with respect to a parameter or independent variables. Some of the functional and functional-differential equations in question arise in constructing exact solutions to nonlinear partial differential equations with the method of Lie groups [1–3] and the methods of generalized and functional separation of variables [3–5].

1. Reduction of functional equations to a partial differential equation by differentiating with respect to a parameter

1.1. Classes of functional equations in question. Method description

We will consider functional equations of the form

$$w(x,y) = \theta(x,y,a)w\bigl(\varphi(x,y,a),\psi(x,y,a)\bigr), \tag{1}$$

where x and y are independent variables, $w = w(x,y)$ is the unknown function, $\theta = \theta(x,y,a)$, $\varphi = \varphi(x,y,a)$, and $\psi = \psi(x,y,a)$ are prescribed functions, and a is a free parameter that can take any values (from a certain interval). We assume that the relations

$$\theta(x,y,a_0) = 1, \quad \varphi(x,y,a_0) = x, \quad \psi(x,y,a_0) = y \tag{2}$$

hold true for a specific value of the parameter, $a = a_0$, which means that the functional equation (1) is satisfied identically at $a = a_0$.

Let us expand (1) into a power series in the parameter a at the point a_0 taking relations (2) into account and let us divide the resulting expression by $a - a_0$. Then let us take the limit as $a \to a_0$ to obtain a first-order linear partial differential equation for w:

$$\varphi_a^\circ(x,y)\frac{\partial w}{\partial x} + \psi_a^\circ(x,y)\frac{\partial w}{\partial y} + \theta_a^\circ(x,y)w = 0, \tag{3}$$

where the following notation is used:

$$\varphi_a^\circ(x,y) = \frac{\partial\varphi}{\partial a}\Big|_{a=a_0}, \quad \psi_a^\circ(x,y) = \frac{\partial\psi}{\partial a}\Big|_{a=a_0}, \quad \theta_a^\circ(x,y) = \frac{\partial\theta}{\partial a}\Big|_{a=a_0}.$$

In order to solve equation (3), one should consider the corresponding characteristic system of equations

$$\frac{dx}{\varphi_a^\circ(x,y)} = \frac{dy}{\psi_a^\circ(x,y)} = -\frac{dw}{\theta_a^\circ(x,y)w}. \tag{4}$$

Suppose

$$u_1(x,y) = C_1, \quad u_2(x,y,w) = C_2 \tag{5}$$

are independent integrals of the characteristic system (4). Then the general solution of equation (3) is expressed as

$$u_2(x, y, w) = F\big(u_1(x, y)\big), \tag{6}$$

where $F(z)$ is an arbitrary function.

Equation (6) should be solved for w and the resulting expression should then be substituted into the original equation (1) for verification, since redundant solutions could arise. Situations are possible where the solution to the partial differential equation (3) does not satisfy the functional equation (1) at all; see example 3 below.

Remark 1. Equation (3) can be obtained from (1) by differentiating with respect to a followed by setting $a = a_0$.

Remark 2. It is convenient to take the second integral in (5) to be linear in w, so that $u_2(x, y, w) = \xi(x, y)w$, and then rewrite formula (6) to solve for w.

Remark 3. Although the solutions obtained by the differentiation method must be smooth, one can easily verify, given an explicit form of the solution, that they are also suitable as continuous solutions.

Remark 4. Some functional equations of the form (1) are treated in the book [6].

1.2. Examples of the solution of functional equations by the differentiation with respect to a parameter

Example 1. Self-similar solutions, which are common in mathematical physics [1–3], can be defined as solutions invariant under scaling transformations, i.e., those satisfying the functional equation

$$w(x, t) = a^k w(a^m x, a^n t), \tag{7}$$

where k, m, n are some given constants and a is a positive number.

Equation (7) is satisfied identically at $a = 1$. Differentiating (7) with respect to a and then setting $a = 1$, one arrives at the first-order partial differential equation

$$mx\frac{\partial w}{\partial x} + nt\frac{\partial w}{\partial t} + kw = 0. \tag{8}$$

First integrals of the corresponding characteristic system of ordinary differential equations,

$$\frac{dx}{mx} = \frac{dt}{nt} = -\frac{dw}{kw},$$

are expressed as

$$xt^{-m/n} = C_1, \qquad t^{k/n}w = C_2,$$

provided $n \neq 0$. Therefore the general solution of the partial differential equation (8) has the form

$$w(x, t) = t^{-k/n}F(z), \qquad z = xt^{-m/n}, \tag{9}$$

where $F(z)$ is an arbitrary function. One can verify by straightforward substitution that expression (9) is a solution to the functional equation in question (7).

Example 2. Consider the functional equation

$$w(x, t) = a^k w(a^m x, t + b\ln a), \tag{10}$$

where k, m, b are some given constants and a is an arbitrary positive number.

Equation (10) is satisfied identically at $a = 1$. Differentiating (10) with respect to a and then setting $a = 1$, one arrives at the first-order partial differential equation

$$mx\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial t} + kw = 0. \tag{11}$$

The corresponding characteristic system of ordinary differential equations,

$$\frac{dx}{mx} = \frac{dt}{b} = -\frac{dw}{kw}$$

admits the first integrals

$$x \exp(-mt/b) = C_1, \qquad w \exp(kt/b) = C_2.$$

Therefore the general solution of the partial differential equation (11) has the form

$$w(x,t) = \exp(-kt/b)F(z), \qquad z = x \exp(-mt/b), \tag{12}$$

where $F(z)$ is an arbitrary function. By straightforward substitution, one can verify that expression (12) is a solution to the functional equation (10).

Remark 5. A solution of the form (12) is called a limit self-similar solution [3].

Example 3. Now consider the functional equation

$$w(x,t) = a^k w\big(x + (1-a)t,\ a^n t\big), \tag{13}$$

where a is any positive number and n is some constant.

Equation (13) is satisfied identically at $a = 1$. Differentiating (13) with respect to a and then setting $a = 1$, one arrives at the first-order partial differential equation

$$-t\frac{\partial w}{\partial x} + nt\frac{\partial w}{\partial t} + kw = 0. \tag{14}$$

The corresponding characteristic system

$$-\frac{dx}{t} = \frac{dt}{nt} = -\frac{dw}{kw}$$

has the first integrals

$$t + nx = C_1, \qquad wt^{k/n} = C_2.$$

Therefore the general solution of the partial differential equation (14) has the form

$$w(x,t) = t^{-k/n}F(nx+t), \tag{15}$$

where $F(z)$ is an arbitrary function.

Let us substitute expression (15) into the original equation (13). On cancelling out $t^{-k/n}$, one obtains

$$F(nx+t) = F(nx+\sigma t), \qquad \sigma = (1-a)n + a^n. \tag{16}$$

If $F(z) \neq \text{const}$, it follows that $\sigma = 1$, or

$$(1-a)n + a^n = 1. \tag{17}$$

Since (16) must hold for any $a > 0$, relation (17) must also hold for any $a > 0$. This is only possible for a single value of n,

$$n = 1. \tag{18}$$

In this case, the solution to equation (13) is given by (see (15) with $n = 1$)

$$w(x,t) = t^{-k}F(x+t),$$

where $F(z)$ is an arbitrary function.

If $n \neq 1$, equation (13) admits only a degenerate solution, $w(x,t) = Ct^{-k/n}$, where C is an arbitrary constant; this solution corresponds to $F = \text{const}$ in (16).

➠ *Exercises for Section 1.*

1. Solve the functional equation $w(x,t) = w(x + a\lambda, t + ak)$, where k and λ are some constants and a is an arbitrary constant.

2. Solve the functional equation $w(x,t) = a^k w(x + b\ln a, t + \ln a)$, where b and k are some constants and a is any positive number.

3. What conditions must the functions $\varphi(a)$, $\psi_1(a)$, $\psi_2(a)$ satisfy in order that the functional equation

$$w(x,y) = \varphi(a)w\big(\psi_1(a)x, \psi_2(a)y\big)$$

admit solutions for all positive values of a?

4. For which values of the constants k and n does the functional equation $w(x,t) = a^k w(a^n x, t + a - 1)$ admit solutions for all positive a?

2. Reduction of functional equations to ordinary differential equations by differentiating with respect to independent variables

2.1. Preliminary remarks

1°. There are a number of cases where some arguments may be eliminated from the functional equation under consideration by differentiating with respect to its independent variables. The equation is thus reduced to an ordinary differential equation. The resulting solution should then be substituted into the original equation in order to "remove" redundant constants of integration that may arise due to differentiation.

2°. In some cases, the differentiation with respect to independent variables should be combined with the multiplication (division) of the equation and its consequences by appropriate functions. Sometimes it may be useful to take the logarithm of the equation or/and its differential consequences.

3°. Sometimes the differentiation of the functional equation in question with respect to independent variables allows the elimination of some arguments and the reduction of the equation to a simpler functional equation whose solution is known.

4°. In some cases, the differentiation of the functional equation in question with respect to independent variables allows the elimination of some arguments and the reduction of the equation to a simpler functional-differential equation (see Subsection 3.3).

2.2. Examples of the solution of functional equations by the differentiation with respect to independent variables

Example 4. Consider Pexider's equation [6]

$$f(x) + g(y) = h(x + y). \tag{19}$$

Here, $f(x)$, $g(y)$, and $h(z)$ are unknown functions.

Differentiating the functional equation (19) with respect to x and y, one arrives at the ordinary differential equation $h''_{zz}(z) = 0$ (where $z = x + y$), whose solution is a linear function,

$$h(z) = az + b. \tag{20}$$

Substituting this expression into (19) yields

$$f(x) + g(y) = ax + ay + b.$$

On separating the variables, one finds the functions f and g,

$$f(x) = ax + b + c, \qquad g(y) = ay - c. \tag{21}$$

Thus, the solution to Pexider's equation is given by formulas (20) and (21), where a, b, and c are arbitrary constants.

Example 5. Consider the nonlinear equation

$$f(x + y) = f(x) + f(y) + af(x)f(y), \qquad a \neq 0. \tag{22}$$

It arises in probability theory in the case $a = -1$.

Differentiating the equation with respect to x and y, one obtains

$$f''_{zz}(z) = af'_x(x)f'_y(y), \tag{23}$$

where $z = x + y$. Let us take the logarithm of (23) and differentiate the resulting equation with respect to x and y to obtain the ordinary differential equation

$$[\ln f''_{zz}(z)]''_{zz} = 0 \tag{24}$$

Integrating (24) twice with respect to z yields

$$f''_{zz}(z) = C_1 \exp(C_2 z), \tag{25}$$

where C_1 and C_2 are arbitrary constants. On substituting (25) into (23), one arrives at the equation

$$C_1 \exp[C_2(x+y)] = a f'_x(x) f'_y(y),$$

which admits separation of variables. Integrating yields

$$f(x) = A \exp(C_2 x) + B, \qquad A = \pm\frac{1}{C_2}\sqrt{\frac{C_1}{a}}. \tag{26}$$

On substituting (26) into the original equation (22), one finds that $A = -B = 1/a$ and $C_2 = \beta$ is any number. Thus, one obtains the desired solution

$$f(x) = \frac{1}{a}\left(e^{\beta x} - 1\right).$$

Exercises for Section 2.

1. Solve Cauchy's exponential equation $f(x+y) = f(x)f(y)$. Hint: first take the logarithm of the equation.

2. Solve Cauchy's power equation $f(xy) = f(x)f(y)$.

3. Solve Lobachevsky's functional equation $f(x+y)f(x-y) = f^2(x)$.

4. Solve the functional equation $f(x+y)g(x-ay) = h(x)$, where a is a given number.

5. Solve the functional equation $f(x+y) = a^{xy} g(x)h(y)$, where a is a given positive number. Hint: first take the logarithm of the equation.

6. Solve the generalized d'Alembert equation $f_1(y+x) + f_2(y-x) = g_(x)g_2(y)$. Hint: first pass to the new variables $\xi = x+y$, $\eta = y-x$.

3. Solution of Some Functional-Differential Equations by Differentiation

3.1. Classes of functional-differential equations in question. Method description

Consider functional-differential equations of the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_k(x)g_k(y) = 0, \tag{27}$$

where the functionals $f_i(x)$ and $g_j(x)$ are prescribed and have the form, respectively,

$$\begin{aligned} f_j(x) &\equiv F_j\left(x, \varphi_1, \varphi'_1, \varphi''_1, \ldots, \varphi_n, \varphi'_n, \varphi''_n\right), \\ g_j(y) &\equiv G_j\left(y, \psi_1, \psi'_1, \psi''_1, \ldots, \psi_m, \psi'_m, \psi''_m\right). \end{aligned} \tag{28}$$

The functions $\varphi_i = \varphi_i(x)$ and $\psi_j = \psi_j(y)$, dependent on different arguments, are to be found. Here, for simplicity, an equation involving second derivatives is considered; in the general case, the right-hand sides of relations (28) will contain higher-order derivatives of $\varphi_i = \varphi_i(x)$ and $\psi_j = \psi_j(y)$.

Below we describe a procedure for constructing solutions to functional-differential equations (27)–(28) by the differentiation method. It involves three successive stages.

1°. Assume that $g_k \not\equiv 0$. We divide equation (27) by g_k and differentiate with respect to y. This results in a similar equation but with fewer terms:

$$\begin{gathered} \widetilde{f}_1(x)\widetilde{g}_1(y) + \widetilde{f}_2(x)\widetilde{g}_2(y) + \cdots + \widetilde{f}_{k-1}(x)\widetilde{g}_{k-1}(y) = 0, \\ \widetilde{f}_j(x) = f_j(x), \quad \widetilde{g}_j(y) = [g_j(y)/g_k(y)]'_y. \end{gathered}$$

We continue the above procedure until we obtain a separable two-term equation

$$\widehat{f}_1(x)\widehat{g}_1(y) + \widehat{f}_2(x)\widehat{g}_2(y) = 0. \qquad (29)$$

Three cases must be considered.

Nondegenerate case: $|\widehat{f}_1(x)|+|\widehat{f}_2(x)|\not\equiv 0$ and $|\widehat{g}_1(y)|+|\widehat{g}_2(y)|\not\equiv 0$. Then equation (29) is equivalent to the ordinary differential equations

$$\widehat{f}_1(x) + C\widehat{f}_2(x) = 0, \quad C\widehat{g}_1(y) - \widehat{g}_2(y) = 0,$$

where C is an arbitrary constant. The equations $\widehat{f}_2 = 0$ and $\widehat{g}_1 = 0$ correspond to the limit case $C = \infty$.

Two degenerate cases:

$$\widehat{f}_1(x) \equiv 0, \quad \widehat{f}_2(x) \equiv 0 \quad \Longrightarrow \quad \widehat{g}_{1,2}(y) \text{ are any;}$$
$$\widehat{g}_1(y) \equiv 0, \quad \widehat{g}_2(y) \equiv 0 \quad \Longrightarrow \quad \widehat{f}_{1,2}(x) \text{ are any.}$$

2°. The solutions of the two-term equation (29) should be substituted into the original functional-differential equation (27) to "remove" redundant constants of integration [these arise because equation (29) is obtained from (27) by differentiation].

3°. The case $g_k \equiv 0$ should be treated separately (since we divided the equation by g_k at the first stage). Likewise, we have to study all other cases where the functionals by which the intermediate functional-differential equations were divided vanish.

Remark 6. Functional-differential equations of the form (27) play an important role in the method of generalized separation of variables for the nonlinear PDEs [3–5].

3.2. Example of the solution of a functional-differential equation by differentiation

Below we demonstrate with a specific example how functional-differential equations of the form (27) arise and how they can be solved.

Example 6. The two-dimensional stationary equations of motion of a viscous incompressible fluid are reduced to a single fourth-order nonlinear equation for the stream function

$$\frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w) - \frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w, \qquad \Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \qquad (30)$$

We seek exact separable solutions of equation (30) in the form

$$w = f(x) + g(y). \qquad (31)$$

Substituting (31) into (30) yields a functional-differential equation of the form (27):

$$g'_y f'''_{xxx} - f'_x g'''_{yyy} = \nu f''''_{xxxx} + \nu g''''_{yyyy}. \qquad (32)$$

Differentiating (32) with respect to x and y, we obtain

$$g''_{yy} f''''_{xxxx} - f''_{xx} g''''_{yyyy} = 0. \qquad (33)$$

If $f''_{xx} \not\equiv 0$ and $g''_{yy} \not\equiv 0$, we separate the variables in (33) to obtain the ordinary differential equations

$$f''''_{xxxx} = Cf''_{xx}, \qquad (34)$$
$$g''''_{yyyy} = Cg''_{yy}, \qquad (35)$$

which have different solutions depending on the value of the integration constant C.

Integrating the constant-coefficient linear ordinary differential equations (34)–(35) followed by substituting the resulting solutions into the functional-differential equation (32), we finally arrive at three different solutions for the case $C > 0$:

$$
\begin{aligned}
&f(x) = C_1 e^{-\lambda y} + C_2 y + C_3, \quad && g(y) = \nu\lambda x;\\
&f(x) = C_1 e^{-\lambda x} + \nu\lambda x, && g(y) = C_2 e^{-\lambda y} - \nu\lambda y + C_3;\\
&f(x) = C_1 e^{-\lambda x} - \nu\lambda x, && g(y) = C_2 e^{\lambda y} - \nu\lambda y + C_3,
\end{aligned}
$$

where C_1, C_2, C_3, and λ are arbitrary constants (for details and other solutions to the equation in question, see [3, 4]).

3.3. Solution of some functional equations with a composite argument

1°. The functional equation

$$S(t) + R_1(x)Q_1(y) + \cdots + R_n(x)Q_n(y) = 0, \qquad \text{where} \quad y = \varphi(x) + \psi(t), \tag{36}$$

can be reduced, by differentiation with respect to x, to a functional-differential equation in two variables x and y of the form (27).

2°. Consider a functional equation of the form

$$S_1(t)R_1(x) + \cdots + S_m(t)R_m(x) + h_1(x)Q_1(y) + \cdots + h_n(x)Q_n(y) = 0, \qquad \text{where} \quad y = \varphi(x) + \psi(t). \tag{37}$$

Assume that $R_m(x) \not\equiv 0$. We divide equation (37) by $R_m(x)$ and differentiate with respect to x. This results in an equation,

$$S_1(t)\bar{R}_1(x) + \cdots + S_{m-1}(t)\bar{R}_{m-1}(x) + \sum_{i=1}^{2n} F_i(x)G_i(z) = 0,$$

with fewer functions $S_i(t)$. Proceeding likewise, all functions $S_i(t)$ can eventually be eliminated resulting in a functional-differential equation in two variables of the form (27).

Remark 7. Functional equations of the form (36) and (37) play an important role in the method of functional separation of variables for the nonlinear PDEs [3–5]. Solutions to a number of specific functional equations of this form and their application to nonlinear PDEs can be found in [3].

Exercises for Section 3.

1. Find generalized separable solutions of the nonlinear first-order partial differential equation $w_x = aw_y^2 + f(x)$. Hint: look for solutions in the form $w = \varphi(x)\theta(y) + \psi(x)$.

2. Find generalized separable solutions of the nonlinear heat equation $w_t = a(ww_x)_x + b$. Hint: look for solutions in the form $w = f(t)\theta(x) + g(t)$.

3. Find generalized separable solutions of the nonhomogeneous Monge–Ampère equation $w_{xy}^2 - w_{xx}w_{yy} = f(x)y^k$. Hint: look for solutions in the form $w = \varphi(x)\theta(y) + \psi(x)$.

4. Solve the functional equation

$$f(t) + g(x) + h(x)Q(y) + R(y) = 0, \qquad \text{where} \quad y = x + t,$$

where the functions $f(t)$, $g(x)$, $h(x)$, $Q(y)$, and $R(y)$ are assumed unknown.

5. Solve the functional equation

$$f(t) + g(x)Q(y) + h(x)R(y) = 0, \qquad \text{where} \quad y = x + t.$$

References

1. **Ovsiannikov, L. V.,** *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
2. **Olver, P. J.,** *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.
3. **Polyanin, A. D. and Zaitsev, V. F.,** *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC Press, Boca Raton, 2004.
4. **Polyanin, A. D.,** *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Supplement B), Chapman & Hall/CRC Press, Boca Raton, 2002.
5. **Polyanin, A. D. and Zhurov, A. I.,** The generalized and functional separation of variables in mathematical physics and mechanics, *Doklady Mathematics*, Vol. 65, No. 1, pp. 129–134, 2002.
6. **Aczél, J. and Dhombres, J.,** *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, 1989.