Solution of Functional Equations by Argument Elimination Method

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Here we describe a class of functional equations that can be solved by eliminating an argument in an auxiliary, more general functional equation. Examples of solving specific functional equations are given.

1. Equation classes in question. Method description

We will study functional equations of the form

$$w(x,t)=f(x,t,a)\,w\big(g(x,t,a),h(x,t,a)\big), \tag{1}$$

where x and t are independent variables, $w=w(x,t)$ is the unknown functions, $f=f(x,t,a)$, $g=g(x,t,a)$, and $h=h(x,t,a)$ are prescribed functions, and a is a free parameter that can take any values (from a certain interval).

Let us consider, instead of equation (1), a more general, auxiliary functional equation

$$w(x,t)=f(x,t,\xi)\,w\big(g(x,t,\xi),h(x,t,\xi)\big), \tag{2}$$

where $\xi=\xi(x,t)$ is an arbitrary function.

Basic idea: if one succeeds in obtaining an exact solution to equation (2), it will also be a solution to the original functional equation (1), corresponding to the special case $\xi=a$ in (2).

Since the function $\xi=\xi(x,t)$ is arbitrary, we take to satisfy the condition

$$h(x,t,\xi)=b, \tag{3}$$

where b is some constant (it can usually be set $b=1$ or $b=0$). On solving (3) for ξ and on substituting the resulting expression $\xi=\xi(x,t)$ into (2), we obtain

$$w(x,t)=f\big(x,t,\xi(x,t)\big)\,\Phi\big(g(x,t,\xi(x,t))\big), \tag{4}$$

where the notation $\Phi(g)\equiv w(g,b)$ is used.

Expression (4) provides a basis for exact solution of the original functional equation. One should substitute (4) into (1) and see what functions $\Phi(g)$ provide solutions for (1); constrains may arise on the for of the determining functions f, g, and h.

Remark 1. Condition (3) implies the "elimination" of one argument (since it is "replaced" by a constant) on the right-hand side of equation (2).

Remark 2. Instead of (3), a similar condition $g(x,t,\xi)=b$ can be used to select the function $\xi=\xi(x,t)$.

2. Examples of solving functional equations

Example 1. Self-similar solutions, which are common in mathematical physics [1–3], may be defined as solutions invariant under a scaling transformation, i.e., a solution satisfying the functional equation

$$w(x,t) = a^k w(a^m x, a^n t), \tag{5}$$

where k, m, n are some given constants and a is an arbitrary positive number.

Equation (5) is a special case of equation (1) where $f(x,t,a) = a^k$, $g(x,t,a) = a^m x$, and $h(x,t,a) = a^n t$.

Following the procedure outlined in Section 1, let us consider the auxiliary equation

$$w(x,t) = \xi^k w(\xi^m x, \xi^n t), \tag{6}$$

where the function ξ will be defined, according to (3), by the condition

$$\xi^n t = 1 \qquad (b = 1) \tag{7}$$

It follows that $\xi = t^{-1/n}$. Substituting this into (6), we have

$$w(x,t) = t^{-k/n} \Phi(t^{-m/n} x), \tag{8}$$

where the notation $\Phi(g) = w(g,1)$ is used.

It is easy to verify by straightforward substitution that expression (8) is a solution to the original functional equation (5) for arbitrary function Φ.

Remark 3. One could use any one nonzero constant b on the right-hand side of (7) instead of 1; the result would be the same up to redefining the arbitrary function Φ.

Example 2. Consider the functional equation

$$w(x,t) = a^k w(a^m x, t + \beta \ln a), \tag{9}$$

with k, m, β being some given constants and a an arbitrary positive number.

Equation (9) is a special case of equation (1) where $f(x,t,a) = a^k$, $g(x,t,a) = a^m x$, and $h(x,t,a) = t + \beta \ln a$.

Following the procedure outlined above, let us consider the more general, auxiliary equation

$$w(x,t) = \xi^k w(\xi^m x, t + \beta \ln \xi). \tag{10}$$

The function ξ will be defined by

$$t + \beta \ln \xi = 0 \qquad (b = 0).$$

We have $\xi = \exp(-t/\beta)$. On substituting this expression into (10), we obtain

$$w(x,t) = e^{-kt/\beta} \Phi(x e^{-mt/\beta}), \tag{11}$$

where $\Phi(g) = w(g,0)$. It can be verified by straightforward substitution that expression (11) is a solution to the functional equation (9) for arbitrary function Φ.

Remark 4. A solution of the form (11) is called a limiting self-similar solution [3].

Example 3. Likewise, it can be shown that the functional equation

$$w(x,t) = a^k w(x + \beta \ln a, t + \gamma \ln a),$$

has a solution

$$w(x,t) = e^{-kt/\gamma} \Phi\left(x - \frac{\beta}{\gamma} t\right),$$

where $\Phi(z)$ is an arbitrary function.

Example 4. Now consider the functional equation

$$w(x,t) = a^k w\bigl(x + (1-a)t,\ a^n t\bigr), \tag{12}$$

where a is any positive number and n is some constant.

Equation (12) is a special case of equation (1) with $f(x,t,a) = a^k$, $g(x,t,a) = x + (1-a)t$, and $h(x,t,a) = a^n t$.

Following the procedure outlined above, let us consider the auxiliary equation

$$w(x,t) = \xi^k w\bigl(x + (1-\xi)t,\ \xi^n t\bigr). \tag{13}$$

where the function ξ will be defined, according to (3), by condition (7). We have $\xi = t^{-1/n}$. On substituting this expression into (13), we find that

$$w(x,t) = t^{-k/n}\Phi(z), \qquad z = x + t - t^{(n-1)/n}. \tag{14}$$

Let us substitute (14) into the original equation (12) and divide the resulting relation by $t^{-k/n}$ to obtain

$$\Phi\bigl(x + t - t^{(n-1)/n}\bigr) = \Phi\bigl(x + (1 - a + a^n)t - a^{n-1}t^{(n-1)/n}\bigr). \tag{15}$$

Since this equality must hold for any $a > 0$, there are two cases:

$$\begin{aligned} &n \text{ is any}, \quad && \Phi = C = \text{const} \quad && \text{(case 1)};\\ &n = 1, && \Phi \text{ is any} && \text{(case 2)}. \end{aligned} \tag{16}$$

In the second case ($n = 1$), the functional equation (12) has a solution

$$w(x,t) = t^{-k}\Psi(x + t), \tag{17}$$

where $\Psi(z)$ is an arbitrary function such that $\Psi(z) = \Phi(z - 1)$.

Remark 5. The solutions to the above functional equations, considered in Examples 1–4, obtained by the method of argument elimination coincide with those obtained by the method of differentiation with respect to a parameter [4].

Remark 6. Intermediate results of solving by the two methods may be different. For example, with the method of parameter differentiation, one obtains a solution [4],

$$w(x,t) = t^{-k/n}\Phi(nx + t), \tag{18}$$

that differs from formula (14) in Example 4. However, both intermidiate solutions, (14) and (18), result in the same final solutions (16)–(17).

Remark 7. It is important that the method of argument elimination is much simpler than that of parameter differentiation, since the former only involves solution of algebraic (transcendental) equations of the form (3) for ξ whereas the latter requires derivation and solution of first-order partial differential equations.

References

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