The International Conference MOGRAN 2000
Modern Group Analysis for the New Millennium
Ufa, RUSSIA, 27 September – 03 October, 2000

On group analysis of functional differential equations

Linchuk, L.V.

Russian State Pedagogical University,
Moyka 48, Saint-Petersburg, 191186, RUSSIA
e-mail: lidiya@osipenko.stu.neva.ru

Abstract. We suggest a group approach to research of functional-differential equations based on a search of symmetries of underdetermined differential equations by methods of classical and modern group analysis.

During the 20th century almost all types of differential equations were investigated by methods of group analysis. However the symmetry approach leaves aside practical significant and important class of functional differential equations (FDEs). For example, this class includes differential-delay equations, in particular, equations with lag. There exist a few equations, integrated in a closed form. Their number does not exceed several tens. Although some attempts to use a symmetry and to extend the methods of group analysis on the class of functional differential equations were undertaken.

A.N. Sharkovsky and G.P. Pelyukh [9, 10] considered FDEs with arguments related by transformations that generate a finite group or an infinite group with finit factor group, e.g., containing functions $y(x)$ and $y(-x)$ and their derivatives. Evidently, the re-notation $y(x) = y_1(x)$, $y(-x) = y_2(x)$ and replacement of the argument $x \to -x$ transform a given equation in a system of two ordinary equations with respect to $y_1(x)$, $y_2(x)$. However capabilities of this method are restricted not only by the requirement on the finiteness for the group of transformations of arguments (or its factor group), but also by the requirement on the absence of invariance of given equation with respect to this group.

Some preliminary researches on applications of methods of group analysis to FDEs are given in papers of V.P. Petukhov [11, 12], although any general approach was not found. The main problem arises: how to consider variables with a functional arbitrariness, and consequently, how to extend the operator to such variables. The ambiguity of the interpretation generates a set of approaches that as a rule, essentially depend on a type of functional relation between variables.

Therefore in our researches we replace a study of FDEs by a study of underdetermined equations [3, 5], supposing, for example that $y(\varphi(x)) = w(x)$. The reduction of FDE to a system consisting of an underdetermined differential equation (for example, with two unknown functions) and one or more functional equations allows to abstract from these relations and to concentrate on symmetries of the resulting generalized differential equation.

Underdetermined differential equations were considered, by I.M. Andersen [1], G.N. Yakovenko, V.I. Legenky [6], and V.I. Elkin [2] and othes. However, Andersen considered these equations as an example of the applications of the classical group analysis and the theory of Lie-Bäcklund groups. The others mathematicians treated only control problems and "superfluous" variables was components of a control vector and had no functional relation with basic unknown functions. In particular, V.I. Legenky pointed out that group analysis of underdetermined equations is ineffective since admissible operators have a functional arbitrariness (equations admit an infinite Lie algebra), induced not by real symmetries, but by the underdetermination of the equation. In fact, in this case the algebra turns out to be "empty", i.e. we did not succeed in simplification of the equation by the use of algebra.

However in the FDEs case, unknown variables are of the same type and are connected by some algebraic (not differential) relations. The inclusion of these additional relations leads to vanishing of the underdetermination. Therefore the functional arbitrariness which exists in general does not reduce efficiency of methods of the group analysis.

Using well known principle of factorization [3, 8], generalized differential equation can be reduced to a system of simpler equations embedded in each other. Let us consider the generalized differential equation of the nth order

$$F\big(x, y, y', \ldots, y^{(n)}, w, w', \ldots w^{(n)}\big) = 0. \qquad (1)$$

Theorem 1. The generalized differential equation (1) is factorized up to the system

$$\begin{cases} z_1 = G_1\big(x, y, y', \,\dots\, , y^{(k_1)}, w, w', \,\dots\, , w^{(s_1)}\big), \\ z_2 = G_2\big(x, y, y', \,\dots\, , y^{(k_2)}, w, w', \,\dots\, , w^{(s_2)}\big), \\ H\big(x, z_i, z_i', \,\dots\, , z_i^{(n-\sigma_i)}\big) = 0, \quad i = 1, 2, \end{cases}$$

where $\sigma_i = \max\{k_i, s_i\}$, $i = 1, 2$, if and only if it admits the (nonlocal) operator $X = \Phi\partial_y + \Psi\partial_w$ with coordinates satisfying the system of equations in total derivatives

$$\sum_{m=0}^{n-1} \left\{ D_x^m[\Phi] \frac{\partial z_i}{\partial y^{(m)}} + D_x^m[\Psi] \frac{\partial z_i}{\partial w^{(m)}} \right\} = 0, \; i = 1, 2,$$

and the functions z_1 and z_2 are distinct **lowest** invariants of this operator, differing from the universal invariant $I_0 = x$.

Remark 1. Note, that the invariants z_1 and z_2 are distinct if the only function Ψ for such $\Psi(x, z_1, z_2) = 0$ is the zero function.

Remark 2. The initial equation can be written also by one **lowest** invariant $z \neq f(x)$. In this case the resulting system consists of two equations.

It is known that the group approach is not effective for first order ordinary differential equations. However a special structure of generalized differential equations (the presence one more independent variable) enables to receive essential results using the same algorithm both for high order equations and for first order ones. In particular, the universal principle of factorization allows to reduce first order generalized differential equation up to either an ordinary differential equation of the first order or to a functional equation [4].

Example 1. The generalized first order differential equation

$$y' + G(x)w' + f(x)y + g(x)w + h(x) = 0,$$

where $g(x) = G'(x) + f(x)G(x)$, admits the infinitesimal operator

$$X = G(x)\partial_y - \partial_w$$

and factors up to the system

$$\begin{cases} z' + f(x)z + h(x) = 0, \\ z = y + G(x)w. \end{cases}$$

For any function $g(x)$ and $G(x) \not\equiv 0$ the equation admits the point operator

$$X = \exp\left[-\int f(x)dx\right]\partial_y + \\ + \exp\left[-\int \frac{g(x)}{G(x)}dx\right]\partial_w,$$

and if $w(x) = y(\tau(x))$, $g(x) = \tau'(x)G(x)f(\tau(x))$ then the given equation is reduced to the functional equation

$$\begin{cases} z(x) + G(x)z(\tau(x)) + h(x) = 0, \\ z(x) = y' + f(x)y. \end{cases}$$

The main concern of the paper is the study of second order generalized differential equations. Therefore further statements will be formulated for this class equations though all reasonings can be extended to any order equations.

Example 2. The second order generalized differential equation

$$y'' = Cw'^2 + (\psi_1 y + \psi_2)y' + (\chi_1 w + \chi_2)w' + \\ + \tfrac{1}{2}(\psi_1{}' + \psi_1\alpha - \psi_1\psi_2)y^2 + \\ + (\alpha' + \alpha^2 - \psi_2\alpha)y + h(x, w), \quad (2)$$

where $\psi_1, \psi_2, \chi_1, \chi_2, \alpha$ are enough smooth functions of x, $C \in \mathbb{R}$, admits the exponential nonlocal operator (ENO)

$$X = \exp\left[\int(\psi_1 y + \alpha)dx\right]\partial_y + \\ + \exp\left[-\int \frac{\chi_2 w' + h_{2w}}{2Cw' + \chi_2 w + \chi_3}dx\right]\partial_w,$$

if $2Cw' + \chi_1 w + \chi_2 \neq 0$. Except for the universal invariant x, this operator has two lowest invariants, namely two differential invariants of the first order. By using these invariant whereby the initial equation (2) is factorized up to the system

$$\begin{cases} z_1 = y' - \frac{1}{2}\psi_1 y^2 - \alpha y, \\ z_2 = Cw'^2 + \chi_1 w w' + \chi_2 w' + h, \\ z_1{}' = (\psi_2 - \alpha)z_1 + z_2, \end{cases} \quad (3)$$

in which the "external" equation is a generalized differential equation of the first order.

In some cases the existence of an additional functional differential relation between unknown functions can essentially simplify the system which the initial equation is reduced to. For example, the inheriting of the functional differential relation by invariants of the admissible operator is possible. Thus, the number of equations of the system resulting from the factorization can be considerably reduced.

Example 3. If we assume, that in the equation (3) the unknown functions y and w are connected by the functional relation $w(x) = y(\tau(x))$, $\tau' \neq 0$, and also $C = 0$, $\chi_1 = 0$, $\chi_2 = (\tau')^{-1}$ and $h = -\frac{1}{2}\psi_1(\tau(x))w^2 - \alpha(\tau(x))w$ then the same functional relation $z_2(x) = z_1(\tau(x))$ holds between the invariants. Therefore the system (3), resulting from the factorization of the equation

$$\begin{aligned} y'' = (\psi_1 y + \psi_2)y' + (\tau')^{-1}w' + \\ + \tfrac{1}{2}({\psi_1}' + \psi_1\alpha - \psi_1\psi_2)y^2 + \\ + (\alpha' + \alpha^2 - \psi_2\alpha)y - \tfrac{1}{2}\psi_1(\tau)w^2 - \alpha(\tau)w, \end{aligned}$$

will contain redundant conditions and it can be written as

$$\begin{cases} z = y' - \frac{1}{2}\psi_1 y^2 - \alpha y, \\ z' = (\psi_2 - \alpha)z + z(\tau). \end{cases}$$

In the example the "external" equation is FDE of the first order. Thus, the problem of finding the second order FDE was in essentially reduced to the sequential search of solutions of the first order FDE and the first order ordinary differential equation (Riccati equation).

Indeed, in the above example we reduced the order of the equation, using one invariant due to the specific form of the equation. However it may occur, that an admissible operator has one first order differential invariant and one universal invariant x on a manifold. Then, by Remark 2 to Theorem 1, the initial equation is factorized up to a system of two equations. This means that the "external" equation is a first order ordinary differential equation. Solved this equation we lower the order of the initial equation by one. We note that this type of factorization is independent of functional-differential relation between the unknown functions y and w.

Example 4. Let us consider the generalized differential equation

$$y'' = C{y'}^2 + (\psi_1 y - C\psi_2 w + \psi_3)y' + \psi_2 w' + H_1 w + H_0,$$

$C \in \mathbb{R}\backslash\{0\}$, ψ_1, ψ_2, ψ_3 are smooth functions of x, $\psi_2 \neq 0$, and H_1 and H_0 are follows

$$\begin{aligned} H_0 &= -\frac{1}{C^2{\alpha_1}^2}\Big\{\alpha_2 \exp(Cy) + \\ &\quad + \big[C(\psi_1 y + \psi_3) + \psi_1\big](C\alpha_3 + {\alpha_1}')\alpha_1 + \\ &\quad + \big[C({\psi_1}' y + {\psi_3}') + {\psi_1}'\big]{\alpha_1}^2 + C\big({\alpha_1}''\alpha_1 + \\ &\quad + C{\alpha_1}'\alpha_3 + C\alpha_1{\alpha_3}' + C^2{\alpha_3}^2\big)\Big\}, \\ H_1 &= \frac{\psi_2{\alpha_1}' + {\psi_2}'\alpha_1 + C\psi_2\alpha_3}{\alpha_1}, \end{aligned}$$

where $\alpha_i = \alpha_i(x), i = \overline{1,3}$, $\alpha_1 \neq 0$. By applying the classical algorithm of the solution of the direct problem, we find a family of admissible point operators in canonical form

$$\hat{X} = (\eta_1 - \xi y')\partial_y + (\eta_2 - \xi w')\partial_w,$$

and its coordinates are such that

$$\begin{aligned} \xi &= \alpha_1, \\ \eta_1 &= g\exp(Cy) + \alpha_3, \\ \eta_2 &= \frac{1}{\psi_2}\big[(g\exp(Cy) + \alpha_3)C\psi_2 w - N - \\ &\quad -(H_1 w + H_0)\alpha_1\big], \end{aligned}$$

where

$$\begin{aligned} N = \frac{1}{\alpha_1}\Big\{\big[(\psi_1 y + \psi_3)\alpha_1 g + {\alpha_1}' g - \\ - \alpha_1 g' + Cg\alpha_3\big]\exp(Cy) + (\psi_1 y + \psi_3)\alpha_1\alpha_3 + \\ + {\alpha_1}'\alpha_3 - \alpha_1{\alpha_3}' + C{\alpha_3}^2\Big\}, \end{aligned}$$

$g = g(x)$. For any function g (the case $g \equiv 0$ is not excluded) basis of invariants of this operator consists of two functions: the universal invariant x and one differential invariant. Therefore the investigated equation is factorized up to the system

$$\begin{cases} z = \dfrac{1}{C^2\alpha_1\exp(Cy)}\Big[C^2\alpha_1(y' - \psi_2 w) + \\ + \quad C\psi_1\alpha_1 y + (\psi_1 + C\psi_3)\alpha_1 + C(C\alpha_3 + {\alpha_1}'), \\ z' + \dfrac{{\alpha_1}' + C\alpha_3}{\alpha_1}z + \dfrac{\alpha_2}{C^2{\alpha_1}^2} = 0. \end{cases}$$

The second equation contains only one dependent variable z and is the first order linear differential equation (with respect to this variable) which is always solvable.

We note that the admissible Lie algebra L is infinite-dimensional and $L = L_\infty \bigoplus L_1$, where L_∞ and L_1 are determined by the operators

$$\begin{aligned} \hat{X}_1 = (\alpha_3 - \alpha_1 y')\partial_y + \Bigg\{\frac{1}{\psi_2\alpha_1}\Big[\alpha_1{\alpha_3}' - {\alpha_1}'\alpha_3 - \\ - C{\alpha_3}^2 + (C\psi_2 w - \psi_1 y - \psi_3)\alpha_1\alpha_3\Big] - \\ - \frac{(H_1 w + H_0)\alpha_1}{\psi_2} - \alpha_1 w'\Bigg\}\partial_w, \end{aligned}$$

and

$$\begin{aligned} \hat{X}_\infty = g\exp(Cy)\partial_y + \Bigg[(C\psi_2 w - \psi_1 y - \psi_3)\alpha_1 g - \\ - {\alpha_1}' g + \alpha_1 g' - Cg\alpha_3\Bigg]\frac{\exp(Cy)}{\psi_2\alpha_1}\partial_w. \end{aligned}$$

The factorization resulting by application of the operator $\hat{X}_1$ was given above. In contrast to $\hat{X}_1$ the basis of invariants of the second operator $\hat{X}_\infty$ consists of three invariants: two universal and one differential invariants. The system which the initial equation is reduced to takes the form

$$\begin{cases} z_1 = \dfrac{Cgy' + g'}{Cg\exp(Cy)}, \\ z_2 = \dfrac{1}{C^2\psi_2\alpha_1 g\exp(Cy)}\Big[\psi_1\alpha_1 g + C^2\alpha_3 g - \\ \quad - C\alpha_1 g' + C{\alpha_1}' g + \\ \quad + C\alpha_1 g(\psi_1 y - C\psi_2 w + \psi_3)\Big], \\ {z_1}' + \psi_2{z_2}' + \dfrac{C\psi_2\alpha_3 + \psi_2{\alpha_1}' + {\psi_2}'\alpha_1}{\alpha_1}z_1 + \\ \quad + \dfrac{C\alpha_3 + {\alpha_1}'}{\alpha_1}z_2 + \dfrac{\alpha_2}{C^2{\alpha_1}^2} = 0. \end{cases}$$

It must be emphasized that the substitution $z = z_1 + \psi_2 z_2$ reduces once again the initial equation to the ordinary differential equation of the first order resulting with use of the operator $\hat{X}_1$.

The examples considered above demonstrate, that a key role in a reduction of generalized differential equations is played not by admissible operators but by the structure of the set of its invariants [7]. As a rule, to find invariants of an admissible operator, we need to consider the invariance condition on a manifold determined by a generalized differential equation. Having several higher derivatives, we can express only one derivative from the equation. Therefore there are situations where the admissible operator has no differential invariants which order is less than the order of the equation) on the manifold, and consequently it is ineffective for factorization. Thus, we need to solve the problem of the number and the structure of invariants of an admissible operator depending on a type of a generalized differential equation. In addition, specification of the class of operators under condition of the availability of a factorization results to the condition for a type of the initial equation and allows to solve the inverse problem.

Theorem 2. In order that the canonical infinitesimal operator

$$\hat{X} = \left[\eta_1(x,y,w) - \xi(x,y,w)y'\right]\partial_y + \left[\eta_2(x,y,w) - \xi(x,y,w)w'\right]\partial_w, \quad (4)$$

admitted by the second order generalized differential equation

$$y'' = F(x,y,w,y'w',w''), \quad (5)$$

has differential invariants of the first order, it is necessary, that any of two conditions is fulfilled:

1) $\xi \neq 0$ and the equation is linear with respect to the higher derivative of w, i.e.

$$F = f_1 w'' + f_2, \quad F_{w''} \neq 0, \quad f_i = f_i(x,y,w,y',w'),\ i = 1,2, \quad (6)$$

or

$$F = g_1 w' + g_2, \quad F_{w'} \neq 0, \quad g_i = g_i(x,y,w,y'),\ i = 1,2. \quad (7)$$

2) $\xi = 0$.

Remark. We note that the condition (7) is sufficient for the class of equations

$$y'' = F(x,y,w,y',w'),$$

if the following relation holds:

$$\begin{aligned}&\eta_1 g_{1x} + (2\eta_1 - \xi y')y' g_{1y} + \eta_2 y' g_{1w} + \\ &+ \left[\eta_{1x} + (\eta_{1y} - \xi_x)y' - \xi_y {y'}^2\right] y' g_{1y'} - \eta_2 {g_1}^2 + \\ &+ (\eta_1 - \xi y')(g_{1y'} g_2 - g_1 g_{2y'} - g_{2w}) + \\ &+ \left(\eta_{1x} + \eta_{2w} y' + \xi_y {y'}^2\right) g_1 - \xi g_1 g_2 = 0. \quad (8)\end{aligned}$$

As mentioned above, both the existence of differential invariants and the dimension of the basis of invariants of the admissible operator are important because the structure of the system, which the initial equation is reduced to, depends on these special features.

In the case that the coordinate of the operator (4) $\xi = 0$, the basis of invariants consists of two universal invariants (one invariant is $J_0 = x$) and one differential invariant of the first order.

If the equation is of the form (7), and the condition (8) is fulfilled, the dimension of the basis of invariants of the operator (4) equals two: one invariant of the zero order $J_0 = x$ and one differential invariant of the first order. This case is of particular interest since the generalized differential equation (7) is reduced to a first order differential equation under factorization. If the structure of the given equation satisfies the condition (7) then the basis of invariants contains one universal invariant $J_0 = x$ and no more than two differential invariants of the first order.

Theorem 3. In order that the canonical infinitesimal operator (4), admitted by the equation (5), which coordinate $\xi \neq 0$, has two distinct first order differential invariants, it is necessary and sufficient that the right-hand side of the equation (5) is of the form (6), and the function f_1 and f_2 satisfy the relation

$$f_1 = \frac{\eta_1 - \xi y'}{\eta_2 - \xi w'},$$

$$f_1 f_{2y'} + f_{2w'} - 2D(f_1)\big|_{y''=f_1 w'' + f_2} = 0.$$

The considerable restrictions on the structure of invariants of point operators lead to the consideration of ENOs. In fact these operators has invariants of any type as opposed to point operators. Hence the extension of the considered class of operators enables to factorized more general FDEs classes.

We note that the algorithm for search of admissible ENO has more complex structure. The general form of ENO is

$$X = \exp\left(\int \zeta_1 dx\right)\partial_y + \exp\left(\int \zeta_2 dx\right)\partial_w, \quad (9)$$

where ζ_1 and ζ_2 can depend on x, y, w and their any order derivatives. For a simplicity of further reasonings we considered in more detail the case $\zeta_1 = \zeta_1(x, y, w, y', w')$, $\zeta_2 = \zeta_2(x, y, w, y', w')$ (see Example 2).

Let us set up the defining equation for the generalized differential equation of the second order

$$y'' = F(x, y, w, y'w', w'') \qquad (10)$$

and the operator (9). Using well known algorithm, we receive the relation

$$\begin{aligned} &\zeta_{1x} + y'\zeta_{1y} + w'\zeta_{1w} + F\zeta_{1y'} + w''\zeta_{1w'} + \\ &\quad + {\zeta_1}^2 - \zeta_1 F_{y'} - F_y - \big[(\zeta_{2x} + y'\zeta_{2y} + \\ &\quad + w'\zeta_{2w} + F\zeta_{2y'} + w''\zeta_{2w'} + {\zeta_2}^2)F_{w''} + \\ &\quad + F_{w'}\zeta_2 + F_w\big] \exp\left\{\int(\zeta_2 - \zeta_1)dx\right\} = 0. \qquad (11) \end{aligned}$$

Under splitting of the equation (11) we need to take into consideration the structure of the nonlocal factor $\exp\left\{\int(\zeta_2 - \zeta_1)dx\right\}$. If the integrand is a total derivative of a function then further reasonings are similar to construction of defining system for point operator. In the other case, if the integrand is not a total derivative, then at first we need to split the equation (11) by the nonlocal variable. Basis of invariants of admissible operator, which was found in the second case, can be chosen so that each invariant depends on x, y, y' or x, w, w'. The structure of invariants reflects on the type of factorization of the equation (10).

Theorem 4. The equation (10) is factorized up to the system

$$\begin{cases} u = J_1^1(x, y, y'), \\ v = J_1^2(x, w, w'), \\ G(x, u, v, u', v') = 0, \end{cases}$$

where $\dfrac{\partial J_1^1(x,y,y')}{\partial y'} \neq 0$, $\dfrac{\partial J_1^2(x,w,w')}{\partial w'} \neq 0$, if and only if it admits the operator (9), which the structural components ζ_1 and ζ_2 satisfy the system

$$\begin{cases} \zeta_{1x} + y'\zeta_{1y} + F\zeta_{1y'} + (\zeta_1 - F_{y'})\zeta_1 - F_y = 0, \\ F_{w''}\zeta_{2x} + F_{w''}w'\zeta_{2w} + F_{w''}w''\zeta_{2w'} + \\ \qquad\qquad + (F_{w'} + F_{w''}\zeta_2)\zeta_2 + F_w = 0. \end{cases}$$

In this case invariants turn out to be function inheriting the functional relation between y and w, consequently the initial equation is reduced to a system of first order FDE and first order ordinary differential equation (see Example 3).

Thus we suggested a new effective method of FDEs reduction allowing to extend the group approach to a new class of objects and increase a number of equations, which solutions can be written in a closed form.

Unfortunately the small volume of the article does not allow to include in the paper proves of the given statements.

References

[1] Anderson I.M., Kamran M., Olver P.J. Internal, External and Generalized Symmetries. – Preprint, 9/4/90, 51 pp.

[2] Elkin V.I. Reduction of nonlinear control system: Differential-geometric approach. – M.: Nauka. Fizmatlit, 1997, 317 pp. (in Russian).

[3] Zaitsev V.F. On modern group analysis of ordinary differential equations // Proceedings of II International Conference "Differential equations and applications". – Spb: SpbSTU, 1998, pp.137-151 (in Russian).

[4] Zaitsev V.F., Linchuk L.V. On some problems of modern group analysis of differential equations // Computer algebra in fundamental and applied research and education, proceedings of II Int. Sci. Conference. – Minsk: Belarusian St. Univ., 1999. – 76-81 pp.

[5] Zaitsev V.F., Linchuk L.V. On factorization of generalized differential equations // Proceedings of IX International Symposium "Methods of discrete singularity in problems of mathematical physics" – Oryol: OSU, 2000, pp. 222-226 (in Russian).

[6] Lekhenkyi V. The Integrability of some Underdetermined Systems // Proceedings of the 3-d International Conference "Symmetry in nonlinear mathematical physics" – Kyiv: Institute of Mathematics of NAS of Ukraine, 2000, V.30, part 1, pp.157-164.

[7] Linchuk L.V. On invariants of operators, admitted by second order generalized differential equation // Proceedings of IX International Symposium "Methods of discrete singularity in problems of mathematical physics" – Oryol: OSU, 2000, pp.275-278 (in Russian).

[8] Linchuk L.V. Local and nonlocal symmetries of functional-differential equations // Abstracts of III International Conference "Differential equations and applications". – SPb: SPbSTU, 2000, p.65.

[9] Maystrenko Yu.L., Sharkovsky A.N. On reduction of a number of argument transformations in functional and functional-differential equations // Qualitative methods of the theory of differential-delay equations. – Kiev: IM AS USSR, 1977, pp.57-70 (in Russian).

[10] Pelyukh G.P., Sharkovsky A.N. Introduction to the theory of functional equations. - Kiev: Naukova Dumka, 1974, 119 pp. (in Russian).

[11] Petukhov V.R. Theory of dynamical system. Educational manual. – Kalinin: KSU, 1981, 40 pp. (in Russian).

[12] Petukhov V.R. Group analysis of dynamical system with transformation of invariant groups. – M: ITEF, 1984, preprint N 34, 20 pp. (in Russian).