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Solution of Linear Integral and Functional Equations With Special Right-Hand Side

Solution of Linear Integral and Functional Equations With Special Right-Hand Side

Here we describe some approaches to the construction of solutions of linear integral and functional equations with special right-hand side. These approaches are based on the application of auxiliary solutions that depend on a free parameter.

1. The General Scheme

Consider a linear equation, which we shall write in the following brief form:

$$\mathbf{L}[y] = f_g(x, \lambda), \tag{1}$$

where \mathbf{L} is a linear operator (integral, functional, differential, etc.) that acts with respect to the variable x and is independent of the parameter λ , and $f_g(x, \lambda)$ is a given function that depends on the variable x and the parameter λ .

Suppose that the solution of Eq. (1) is known:

$$y = y(x, \lambda). \tag{2}$$

Let \mathbf{M} be a linear operator (integral, functional, differential, etc.) that acts with respect to the parameter λ and is independent of the variable x . Consider the (usual) case in which \mathbf{M} commutes with \mathbf{L} . We apply the operator \mathbf{M} to Eq. (1) and find that the equation

$$\mathbf{L}[w] = f_M(x), \quad f_M(x) = \mathbf{M}[f_g(x, \lambda)], \tag{3}$$

has the solution

$$w = \mathbf{M}[y(x, \lambda)]. \tag{4}$$

By choosing the operator \mathbf{M} in a different way, we can obtain solutions for other right-hand sides of Eq. (1). The original function $f_g(x, \lambda)$ is called the *generating function* for the operator \mathbf{L} .

Examples of linear operators M :

$$M[y] = \frac{\partial^n}{\partial \lambda^n} y(x, \lambda) \quad \text{operator of differentiation with respect to parameter } \lambda,$$

$$M[y] = \int_a^b \varphi(\lambda) y(x, \lambda) d\lambda \quad \text{integration operator with respect to parameter } \lambda,$$

where $\varphi(\lambda)$ is an arbitrary function.

2. A Generating Function of Exponential Form

Consider a linear equation with exponential right-hand side

$$\mathbf{L}[y] = e^{\lambda x}. \tag{5}$$

Suppose that the solution is known and is given by formula (2). In Table 1 we present solutions of the equation $\mathbf{L}[y] = f(x)$ with various right-hand sides; these solutions are expressed via the solution of Eq. (5).

Remark 1. When applying the formulas indicated in the table, we need not know the left-hand side of the linear equation (5) (the equation can be integral, differential, etc.) provided that a particular solution of this equation for exponential right-hand side is known. It is only of importance that the left-hand side of the equation is independent of the parameter λ .

Remark 2. When applying formulas indicated in the table, the convergence of the integrals occurring in the resulting solution must be verified.

TABLE 1
Solutions of the equation $\mathbf{L}[y] = f(x)$ with generating function of the exponential form

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$e^{\lambda x}$	$y(x, \lambda)$	Original Equation
2	$A_1 e^{\lambda_1 x} + \dots + A_n e^{\lambda_n x}$	$A_1 y(x, \lambda_1) + \dots + A_n y(x, \lambda_n)$	Follows from linearity
3	$Ax + B$	$A \frac{\partial}{\partial \lambda} [y(x, \lambda)]_{\lambda=0} + B y(x, 0)$	Follows from linearity and the results of row No 4
4	$Ax^n, n = 0, 1, 2, \dots$	$A \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=0}$	Follows from the results of row No 6 for $\lambda = 0$
5	$\frac{A}{x+a}, a > 0$	$A \int_0^\infty e^{-a\lambda} y(x, -\lambda) d\lambda$	Integration with respect to the parameter λ
6	$Ax^n e^{\lambda x}, n = 0, 1, 2, \dots$	$A \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)]$	Differentiation with respect to the parameter λ
7	a^x	$y(x, \ln a)$	Follows from row No 1
8	$A \cosh(\lambda x)$	$\frac{1}{2} A [y(x, \lambda) + y(x, -\lambda)]$	Linearity and relations to the exponential
9	$A \sinh(\lambda x)$	$\frac{1}{2} A [y(x, \lambda) - y(x, -\lambda)]$	Linearity and relations to the exponential
10	$Ax^m \cosh(\lambda x), m = 1, 3, 5, \dots$	$\frac{1}{2} A \frac{\partial^m}{\partial \lambda^m} [y(x, \lambda) - y(x, -\lambda)]$	Differentiation with respect to λ and relation to the exponential
11	$Ax^m \cosh(\lambda x), m = 2, 4, 6, \dots$	$\frac{1}{2} A \frac{\partial^m}{\partial \lambda^m} [y(x, \lambda) + y(x, -\lambda)]$	Differentiation with respect to λ and relation to the exponential
12	$Ax^m \sinh(\lambda x), m = 1, 3, 5, \dots$	$\frac{1}{2} A \frac{\partial^m}{\partial \lambda^m} [y(x, \lambda) + y(x, -\lambda)]$	Differentiation with respect to λ and relation to the exponential
13	$Ax^m \sinh(\lambda x), m = 2, 4, 6, \dots$	$\frac{1}{2} A \frac{\partial^m}{\partial \lambda^m} [y(x, \lambda) - y(x, -\lambda)]$	Differentiation with respect to λ and relation to the exponential
14	$A \cos(\beta x)$	$A \operatorname{Re} [y(x, i\beta)]$	Selection of the real part for $\lambda = i\beta$
15	$A \sin(\beta x)$	$A \operatorname{Im} [y(x, i\beta)]$	Selection of the imaginary part for $\lambda = i\beta$
16	$Ax^n \cos(\beta x), n = 1, 2, 3, \dots$	$A \operatorname{Re} \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=i\beta}$	Differentiation with respect to λ and selection of the real part for $\lambda = i\beta$
17	$Ax^n \sin(\beta x), n = 1, 2, 3, \dots$	$A \operatorname{Im} \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=i\beta}$	Differentiation with respect to λ and selection of the imaginary part for $\lambda = i\beta$
18	$Ae^{\mu x} \cos(\beta x)$	$A \operatorname{Re} [y(x, \mu + i\beta)]$	Selection of the real part for $\lambda = \mu + i\beta$
19	$Ae^{\mu x} \sin(\beta x)$	$A \operatorname{Im} [y(x, \mu + i\beta)]$	Selection of the imaginary part for $\lambda = \mu + i\beta$
20	$Ax^n e^{\mu x} \cos(\beta x), n = 1, 2, 3, \dots$	$A \operatorname{Re} \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=\mu+i\beta}$	Differentiation with respect to λ and selection of the real part for $\lambda = \mu + i\beta$
21	$Ax^n e^{\mu x} \sin(\beta x), n = 1, 2, 3, \dots$	$A \operatorname{Im} \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=\mu+i\beta}$	Differentiation with respect to λ and selection of the imaginary part for $\lambda = \mu + i\beta$

Example 1. We seek a solution of the linear integral equation with exponential right-hand side

$$y(x) + \int_x^\infty K(x-t)y(t) dt = e^{\lambda x} \quad (6)$$

in the form

$$y(x, \lambda) = ke^{\lambda x} \quad (7)$$

by the method of indeterminate coefficients. Then we obtain

$$y(x, \lambda) = \frac{1}{B(\lambda)} e^{\lambda x}, \quad B(\lambda) = 1 + \int_0^\infty K(-z)e^{\lambda z} dz. \quad (8)$$

It follows from row 3 of Table 1 that the solution of the integral equation

$$y(x) + \int_x^\infty K(x-t)y(t) dt = Ax \quad (9)$$

can be obtained by differentiating the solution of (8) with respect to the parameter λ . Finally, we obtain

$$y(x) = \frac{A}{D} x - \frac{AC}{D^2},$$

$$D = 1 + \int_0^\infty K(-z) dz, \quad C = \int_0^\infty zK(-z) dz.$$

For such a solution to exist, it is necessary that the improper integrals of the functions $K(-z)$ and $zK(-z)$ exist. This holds if the function $K(-z)$ decreases more rapidly than z^{-2} as $z \rightarrow \infty$. Otherwise a solution can be nonexistent. It is of interest that for functions $K(-z)$ with power-law growth as $z \rightarrow \infty$ in the case $\lambda < 0$, the solution of Eq. (6) exists and is given by formula (8), whereas Eq. (9) does not have a solution. Therefore, we must be careful when using formulas from Table 1 and verify the convergence of the integrals occurring in the solution.

It follows from row 15 of Table 1 that the solution of the equation

$$y(x) + \int_x^\infty K(x-t)y(t) dt = A \sin(\lambda x) \quad (10)$$

is given by the formula

$$y(x) = \frac{A}{B_c^2 + B_s^2} [B_c \sin(\lambda x) - B_s \cos(\lambda x)],$$

$$B_c = 1 + \int_0^\infty K(-z) \cos(\lambda z) dz, \quad B_s = \int_0^\infty K(-z) \sin(\lambda z) dz.$$

Example 2. Consider the linear functional difference equation with exponential right-hand side

$$y(x+2) + ay(x+1) + by(x) = e^{\lambda x}.$$

We can find its solution in the form of (7) utilizing the method of indeterminate coefficients. After some calculations we find

$$y(x, \lambda) = \frac{1}{e^{2\lambda} + ae^\lambda + b} e^{\lambda x}.$$

It follows from row 3 of Table 1 that the solution of the difference equation

$$y(x+2) + ay(x+1) + by(x) = Ax$$

has the form

$$y(x) = \frac{x}{a+b+1} + \frac{a+2b}{(a+b+1)^2}.$$

Example 3. Consider the integral equation

$$Ay(x) + \int_{-\infty}^\infty Q(x+t)e^{\beta t} y(t) dt = e^{\lambda x}, \quad (11)$$

where $Q = Q(z)$ and $f(x)$ are arbitrary functions and A and β are arbitrary constants satisfying some constraints.

The solution of this equation in contrast to examples 1 and 2 cannot be found in the form of (7).

Denote the left hand side of (11) by $\mathbf{L}[y(x)]$ (see equation of (5)).

On substituting

$$y = e^{\lambda x}. \quad (12)$$

into the left-hand side of Eq. (33), after some algebraic manipulations we obtain

$$\mathbf{L}[e^{\lambda x}] = Ae^{\lambda x} + q(\lambda)e^{-(\lambda+\beta)x}, \quad \text{where } q(\lambda) = \int_{-\infty}^\infty Q(z)e^{(\lambda+\beta)z} dz. \quad (13)$$

Substituting λ for $-\lambda - \beta$ in Eq. (13), we obtain

$$\mathbf{L}[e^{-(\lambda+\beta)x}] = Ae^{-(\lambda+\beta)x} + q(-\lambda - \beta)e^{\lambda x}. \quad (14)$$

Let us multiply Eq. (13) by A and Eq. (14) by $-q(\lambda)$ and add the resulting relations. This yields

$$\mathbf{L}[Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}] = [A^2 - q(\lambda)q(-\lambda - \beta)]e^{\lambda x}. \quad (15)$$

On dividing Eq. (15) by the constant $A^2 - q(\lambda)q(-\lambda - \beta)$, we have

$$\mathbf{L}\left[\frac{Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}}{A^2 - q(\lambda)q(-\lambda - \beta)}\right] = e^{\lambda x}.$$

This yields that the solution of original equation is defined by the formula

$$y(x, \lambda) = \frac{Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}}{A^2 - q(\lambda)q(-\lambda - \beta)}. \quad (16)$$

3. Power-Law Generating Function

Consider the linear equation with power-law right-hand side

$$\mathbf{L}[y] = x^\lambda. \quad (17)$$

Suppose that the solution is known and is given by formula (2). In Table 2, solutions of the equation $\mathbf{L}[y] = f(x)$ with various right-hand sides are presented which can be expressed via the solution of Eq. (17).

TABLE 2
Solutions of the equation $\mathbf{L}[y] = f(x)$ with generating function of power-law form

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	x^λ	$y(x, \lambda)$	Original Equation
2	$\sum_{k=0}^n A_k x^k$	$\sum_{k=0}^n A_k y(x, k)$	Follows from linearity
3	$A \ln x + B$	$A \frac{\partial}{\partial \lambda} [y(x, \lambda)]_{\lambda=0} + B y(x, 0)$	Follows from linearity and from the results of row No 4
4	$A \ln^n x,$ $n = 0, 1, 2, \dots$	$A \left\{ \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)] \right\}_{\lambda=0}$	Follows from the results of row No 5 for $\lambda = 0$
5	$A x^\lambda \ln^n x,$ $n = 0, 1, 2, \dots$	$A \frac{\partial^n}{\partial \lambda^n} [y(x, \lambda)]$	Differentiation with respect to the parameter λ
6	$A \cos(\beta \ln x)$	$A \operatorname{Re} [y(x, i\beta)]$	Selection of the real part for $\lambda = i\beta$
7	$A \sin(\beta \ln x)$	$A \operatorname{Im} [y(x, i\beta)]$	Selection of the imaginary part for $\lambda = i\beta$
8	$A x^\mu \cos(\beta \ln x)$	$A \operatorname{Re} [y(x, \mu + i\beta)]$	Selection of the real part for $\lambda = \mu + i\beta$
9	$A x^\mu \sin(\beta \ln x)$	$A \operatorname{Im} [y(x, \mu + i\beta)]$	Selection of the imaginary part for $\lambda = \mu + i\beta$

Example 4. We seek a solution of the equation with power-law right-hand side

$$y(x) + \int_0^x \frac{1}{x} K\left(\frac{t}{x}\right) y(t) dt = x^\lambda$$

in the form $y(x, \lambda) = kx^\lambda$ by the method of indeterminate coefficients. We finally obtain

$$y(x, \lambda) = \frac{1}{1 + B(\lambda)} x^\lambda, \quad B(\lambda) = \int_0^1 K(t) t^\lambda dt.$$

It follows from row 3 of Table 2 that the solution of the equation with logarithmic right-hand side

$$y(x) + \int_0^x \frac{1}{x} K\left(\frac{t}{x}\right) y(t) dt = A \ln x$$

has the form

$$y(x) = \frac{A}{1 + I_0} \ln x - \frac{AI_1}{(1 + I_0)^2},$$

$$I_0 = \int_0^1 K(t) dt, \quad I_1 = \int_0^1 K(t) \ln t dt.$$

4. Generating Function Containing Sines and Cosines

Consider the linear equation

$$\mathbf{L}[y] = \sin(\lambda x). \quad (18)$$

We assume that the solution of this equation is known and is given by formula (2). In Table 3, solutions of the equation $\mathbf{L}[y] = f(x)$ with various right-hand sides are given, which are expressed via the solution of Eq. (18).

Consider the linear equation

$$\mathbf{L}[y] = \cos(\lambda x). \quad (19)$$

We assume that the solution of this equation is known and is given by formula (2). In Table 4, solutions of the equation $\mathbf{L}[y] = f(x)$ with various right-hand sides are given, which are expressed via the solution of Eq. (19).

TABLE 3
Solutions of the equation $\mathbf{L}[y] = f(x)$ with sine-shaped generating function

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$\sin(\lambda x)$	$y(x, \lambda)$	Original Equation
2	$\sum_{k=1}^n A_k \sin(\lambda_k x)$	$\sum_{k=1}^n A_k y(x, \lambda_k)$	Follows from linearity
3	$Ax^m,$ $m = 1, 3, 5, \dots$	$A(-1)^{\frac{m-1}{2}} \left[\frac{\partial^m}{\partial \lambda^m} y(x, \lambda) \right]_{\lambda=0}$	Follows from the results of row 5 for $\lambda = 0$
4	$Ax^m \sin(\lambda x),$ $m = 2, 4, 6, \dots$	$A(-1)^{\frac{m}{2}} \frac{\partial^m}{\partial \lambda^m} y(x, \lambda)$	Differentiation with respect to the parameter λ
5	$Ax^m \cos(\lambda x),$ $m = 1, 3, 5, \dots$	$A(-1)^{\frac{m-1}{2}} \frac{\partial^m}{\partial \lambda^m} y(x, \lambda)$	Differentiation with respect to the parameter λ
6	$\sinh(\beta x)$	$-iy(x, i\beta)$	Relation to the hyperbolic sine, $\lambda = i\beta$
7	$x^m \sinh(\beta x),$ $m = 2, 4, 6, \dots$	$i(-1)^{\frac{m+2}{2}} \left[\frac{\partial^m}{\partial \lambda^m} y(x, \lambda) \right]_{\lambda=i\beta}$	Differentiation with respect to λ and relation to the hyperbolic sine, $\lambda = i\beta$

TABLE 4
Solutions of the equation $\mathbf{L}[y] = f(x)$ with cosine-shaped generating function

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$\cos(\lambda x)$	$y(x, \lambda)$	Original Equation
2	$\sum_{k=1}^n A_k \cos(\lambda_k x)$	$\sum_{k=1}^n A_k y(x, \lambda_k)$	Follows from linearity
3	$Ax^m,$ $m = 0, 2, 4, \dots$	$A(-1)^{\frac{m}{2}} \left[\frac{\partial^m}{\partial \lambda^m} y(x, \lambda) \right]_{\lambda=0}$	Follows from the results of row 4 for $\lambda = 0$
4	$Ax^m \cos(\lambda x),$ $m = 2, 4, 6, \dots$	$A(-1)^{\frac{m}{2}} \frac{\partial^m}{\partial \lambda^m} y(x, \lambda)$	Differentiation with respect to the parameter λ
5	$Ax^m \sin(\lambda x),$ $m = 1, 3, 5, \dots$	$A(-1)^{\frac{m+1}{2}} \frac{\partial^m}{\partial \lambda^m} y(x, \lambda)$	Differentiation with respect to the parameter λ
6	$\cosh(\beta x)$	$y(x, i\beta)$	Relation to the hyperbolic cosine, $\lambda = i\beta$
7	$x^m \cosh(\beta x),$ $m = 2, 4, 6, \dots$	$(-1)^{\frac{m}{2}} \left[\frac{\partial^m}{\partial \lambda^m} y(x, \lambda) \right]_{\lambda=i\beta}$	Differentiation with respect to λ and relation to the hyperbolic cosine, $\lambda = i\beta$

Exercises

1. Find solutions to the integro-differential equation

$$\frac{dy}{dx} + \int_x^\infty e^{-k(x-t)^2} y(t) dt = f(x), \quad k > 0$$

for the following functions:

- (a) $f(x) = e^{\lambda x}$,
 (b) $f(x) = Ax$,
 (c) $f(x) = A \sin(kx)$.

2. Find solutions to the differential-difference equation

$$\frac{dy(x)}{dx} + ay(x+k) + by(x) = f(x)$$

for the following functions:

- (a) $f(x) = e^{\lambda x}$,
 (b) $f(x) = Ax$.

3. Find solutions to the integro-differential equation

$$x \frac{dy(x)}{dx} + \int_0^x \frac{1}{x} K\left(\frac{t}{x}\right) y(t) dt = f(x)$$

for the following functions:

- (a) $f(x) = x^\lambda$,
 (b) $f(x) = A \ln x$.

Reference

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