Solution of Linear Integral and Functional Equations With Special Right-Hand Side

Here we describe some approaches to the construction of solutions of linear integral and functional equations with special right-hand side. These approaches are based on the application of auxiliary solutions that depend on a free parameter.

1. The General Scheme

Consider a linear equation, which we shall write in the following brief form:

$$\mathbf{L}\,[y] = f_{\rm g}(x,\lambda), \tag{1}$$

where $\mathbf{L}$ is a linear operator (integral, functional, differential, etc.) that acts with respect to the variable x and is independent of the parameter λ, and $f_{\rm g}(x,\lambda)$ is a given function that depends on the variable x and the parameter λ.

Suppose that the solution of Eq. (1) is known:

$$y = y(x,\lambda). \tag{2}$$

Let $\mathbf{M}$ be a linear operator (integral, functional, differential, etc.) that acts with respect to the parameter λ and is independent of the variable x. Consider the (usual) case in which $\mathbf{M}$ commutes with $\mathbf{L}$. We apply the operator $\mathbf{M}$ to Eq. (1) and find that the equation

$$\mathbf{L}\,[w] = f_M(x), \qquad f_M(x) = \mathbf{M}\left[f_{\rm g}(x,\lambda)\right], \tag{3}$$

has the solution

$$w = \mathbf{M}\left[y(x,\lambda)\right]. \tag{4}$$

By choosing the operator $\mathbf{M}$ in a different way, we can obtain solutions for other right-hand sides of Eq. (1). The original function $f_{\rm g}(x,\lambda)$ is called the *generating function* for the operator $\mathbf{L}$.

Examples of linear operators M:

$$M[y] = \frac{\partial^n}{\partial\lambda^n} y(x,\lambda) \quad \text{operator of differentiation with respect to parameter } \lambda,$$

$$M[y] = \int_a^b \varphi(\lambda) y(x,\lambda)\,d\lambda \quad \text{integration operator with respect to parameter } \lambda,$$

where $\varphi(\lambda)$ is an arbitrary function.

2. A Generating Function of Exponential Form

Consider a linear equation with exponential right-hand side

$$\mathbf{L}\,[y] = e^{\lambda x}. \tag{5}$$

Suppose that the solution is known and is given by formula (2). In Table 1 we present solutions of the equation $\mathbf{L}\,[y] = f(x)$ with various right-hand sides; these solutions are expressed via the solution of Eq. (5).

Remark 1. When applying the formulas indicated in the table, we need not know the left-hand side of the linear equation (5) (the equation can be integral, differential, etc.) provided that a particular solution of this equation for exponential right-hand side is known. It is only of importance that the left-hand side of the equation is independent of the parameter λ.

Remark 2. When applying formulas indicated in the table, the convergence of the integrals occurring in the resulting solution must be verified.

TABLE 1
Solutions of the equation $\mathbf{L}\,[y]=f(x)$ with generating function of the exponential form

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$e^{\lambda x}$	$y(x,\lambda)$	Original Equation
2	$A_1e^{\lambda_1 x}+\dots+A_ne^{\lambda_n x}$	$A_1y(x,\lambda_1)+\dots+A_ny(x,\lambda_n)$	Follows from linearity
3	$Ax+B$	$A\dfrac{\partial}{\partial\lambda}\Big[y(x,\lambda)\Big]_{\lambda=0}+By(x,0)$	Follows from linearity and the results of row No 4
4	Ax^n, $n=0,\,1,\,2,\,\dots$	$A\left\{\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=0}$	Follows from the results of row No 6 for $\lambda=0$
5	$\dfrac{A}{x+a},\;\; a>0$	$A\displaystyle\int_0^\infty e^{-a\lambda}y(x,-\lambda)\,d\lambda$	Integration with respect to the parameter λ
6	$Ax^ne^{\lambda x}$, $n=0,\,1,\,2,\,\dots$	$A\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]$	Differentiation with respect to the parameter λ
7	a^x	$y(x,\ln a)$	Follows from row No 1
8	$A\cosh(\lambda x)$	$\frac{1}{2}A\big[y(x,\,\lambda)+y(x,\,-\lambda)\big]$	Linearity and relations to the exponential
9	$A\sinh(\lambda x)$	$\frac{1}{2}A\big[y(x,\,\lambda)-y(x,\,-\lambda)\big]$	Linearity and relations to the exponential
10	$Ax^m\cosh(\lambda x)$, $m=1,\,3,\,5,\,\dots$	$\frac{1}{2}A\dfrac{\partial^m}{\partial\lambda^m}\big[y(x,\,\lambda)-y(x,\,-\lambda)\big]$	Differentiation with respect to λ and relation to the exponential
11	$Ax^m\cosh(\lambda x)$, $m=2,\,4,\,6,\,\dots$	$\frac{1}{2}A\dfrac{\partial^m}{\partial\lambda^m}\big[y(x,\,\lambda)+y(x,\,-\lambda)\big]$	Differentiation with respect to λ and relation to the exponential
12	$Ax^m\sinh(\lambda x)$, $m=1,\,3,\,5,\,\dots$	$\frac{1}{2}A\dfrac{\partial^m}{\partial\lambda^m}\big[y(x,\,\lambda)+y(x,\,-\lambda)\big]$	Differentiation with respect to λ and relation to the exponential
13	$Ax^m\sinh(\lambda x)$, $m=2,\,4,\,6,\,\dots$	$\frac{1}{2}A\dfrac{\partial^m}{\partial\lambda^m}\big[y(x,\,\lambda)-y(x,\,-\lambda)\big]$	Differentiation with respect to λ and relation to the exponential
14	$A\cos(\beta x)$	$A\,\mathrm{Re}\big[y(x,\,i\beta)\big]$	Selection of the real part for $\lambda=i\beta$
15	$A\sin(\beta x)$	$A\,\mathrm{Im}\big[y(x,\,i\beta)\big]$	Selection of the imaginary part for $\lambda=i\beta$
16	$Ax^n\cos(\beta x)$, $n=1,\,2,\,3,\,\dots$	$A\,\mathrm{Re}\left\{\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=i\beta}$	Differentiation with respect to λ and selection of the real part for $\lambda=i\beta$
17	$Ax^n\sin(\beta x)$, $n=1,\,2,\,3,\,\dots$	$A\,\mathrm{Im}\left\{\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=i\beta}$	Differentiation with respect to λ and selection of the imaginary part for $\lambda=i\beta$
18	$Ae^{\mu x}\cos(\beta x)$	$A\,\mathrm{Re}\big[y(x,\,\mu+i\beta)\big]$	Selection of the real part for $\lambda=\mu+i\beta$
19	$Ae^{\mu x}\sin(\beta x)$	$A\,\mathrm{Im}\big[y(x,\,\mu+i\beta)\big]$	Selection of the imaginary part for $\lambda=\mu+i\beta$
20	$Ax^ne^{\mu x}\cos(\beta x)$, $n=1,\,2,\,3,\,\dots$	$A\,\mathrm{Re}\left\{\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=\mu+i\beta}$	Differentiation with respect to λ and selection of the real part for $\lambda=\mu+i\beta$
21	$Ax^ne^{\mu x}\sin(\beta x)$, $n=1,\,2,\,3,\,\dots$	$A\,\mathrm{Im}\left\{\dfrac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=\mu+i\beta}$	Differentiation with respect to λ and selection of the imaginary part for $\lambda=\mu+i\beta$

Example 1. We seek a solution of the linear integral equation with exponential right-hand side

$$y(x) + \int_x^\infty K(x-t)y(t)\,dt = e^{\lambda x} \tag{6}$$

in the form

$$y(x,\lambda) = ke^{\lambda x} \tag{7}$$

by the method of indeterminate coefficients. Then we obtain

$$y(x,\lambda) = \frac{1}{B(\lambda)}e^{\lambda x}, \qquad B(\lambda) = 1 + \int_0^\infty K(-z)e^{\lambda z}\,dz. \tag{8}$$

It follows from row 3 of Table 1 that the solution of the integral equation

$$y(x) + \int_x^\infty K(x-t)y(t)\,dt = Ax \tag{9}$$

can be obtained by differentiating the solution of (8) with respect to the parameter λ. Finally, we obtain

$$y(x) = \frac{A}{D}x - \frac{AC}{D^2},$$
$$D = 1 + \int_0^\infty K(-z)\,dz, \quad C = \int_0^\infty zK(-z)\,dz.$$

For such a solution to exist, it is necessary that the improper integrals of the functions $K(-z)$ and $zK(-z)$ exist. This holds if the function $K(-z)$ decreases more rapidly than z^{-2} as $z \to \infty$. Otherwise a solution can be nonexistent. It is of interest that for functions $K(-z)$ with power-law growth as $z \to \infty$ in the case $\lambda < 0$, the solution of Eq. (6) exists and is given by formula (8), whereas Eq. (9) does not have a solution. Therefore, we must be careful when using formulas from Table 1 and verify the convergence of the integrals occurring in the solution.

It follows from row 15 of Table 1 that the solution of the equation

$$y(x) + \int_x^\infty K(x-t)y(t)\,dt = A\sin(\lambda x) \tag{10}$$

is given by the formula

$$y(x) = \frac{A}{B_\mathrm{c}^2 + B_\mathrm{s}^2}\left[B_\mathrm{c}\sin(\lambda x) - B_\mathrm{s}\cos(\lambda x)\right],$$
$$B_\mathrm{c} = 1 + \int_0^\infty K(-z)\cos(\lambda z)\,dz, \quad B_\mathrm{s} = \int_0^\infty K(-z)\sin(\lambda z)\,dz.$$

Example 2. Consider the linear functional difference equation with exponential right-hand side

$$y(x+2) + ay(x+1) + by(x) = e^{\lambda x}.$$

We can find its solution in the form of (7) utilizing the method of indeterminate coefficients. After some calculations we find

$$y(x,\lambda) = \frac{1}{e^{2\lambda} + ae^{\lambda} + b}e^{\lambda x}.$$

It follows from row 3 of Table 1 that the solution of the difference equation

$$y(x+2) + ay(x+1) + by(x) = Ax$$

has the form

$$y(x) = \frac{x}{a+b+1} + \frac{a+2b}{(a+b+1)^2}.$$

Example 3. Consider the integral equation

$$Ay(x) + \int_{-\infty}^\infty Q(x+t)e^{\beta t}y(t)\,dt = e^{\lambda x}, \tag{11}$$

where $Q = Q(z)$ and $f(x)$ are arbitrary functions and A and β are arbitrary constants satisfying some constraints.

The solution of this equation in contrast to examples 1 and 2 cannot be found in the form of(7).

Denote the left hand side of (11) by $\mathbf{L}\,[y(x)]$ (see equation of (5)).

On substituting

$$y = e^{\lambda x}. \tag{12}$$

into the left-hand side of Eq. (33), after some algebraic manipulations we obtain

$$\mathbf{L}\,[e^{\lambda x}] = Ae^{\lambda x} + q(\lambda)e^{-(\lambda+\beta)x}, \qquad \text{where} \quad q(\lambda) = \int_{-\infty}^\infty Q(z)e^{(\lambda+\beta)z}\,dz. \tag{13}$$

Substituting λ for $-\lambda-\beta$ in Eq. (13), we obtain

$$\mathbf{L}\,[e^{-(\lambda+\beta)x}] = Ae^{-(\lambda+\beta)x} + q(-\lambda-\beta)e^{\lambda x}. \tag{14}$$

Let us multiply Eq. (13) by A and Eq. (14) by $-q(\lambda)$ and add the resulting relations. This yields

$$\mathbf{L}\,[Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}] = [A^2 - q(\lambda)q(-\lambda-\beta)]e^{\lambda x}. \tag{15}$$

On dividing Eq. (15) by the constant $A^2 - q(\lambda)q(-\lambda-\beta)$, we have

$$\mathbf{L}\left[\frac{Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}}{A^2 - q(\lambda)q(-\lambda-\beta)}\right] = e^{\lambda x}.$$

This yields that the solution of original equation is defined by the formula

$$y(x,\lambda) = \frac{Ae^{\lambda x} - q(\lambda)e^{-(\lambda+\beta)x}}{A^2 - q(\lambda)q(-\lambda-\beta)}. \tag{16}$$

3. Power-Law Generating Function

Consider the linear equation with power-law right-hand side

$$\mathbf{L}\,[y] = x^{\lambda}. \tag{17}$$

Suppose that the solution is known and is given by formula (2). In Table 2, solutions of the equation $\mathbf{L}\,[y] = f(x)$ with various right-hand sides are presented which can be expressed via the solution of Eq. (17).

TABLE 2

Solutions of the equation $\mathbf{L}\,[y] = f(x)$ with generating function of power-law form

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	x^{λ}	$y(x,\lambda)$	Original Equation
2	$\sum\limits_{k=0}^{n} A_k x^k$	$\sum\limits_{k=0}^{n} A_k y(x,k)$	Follows from linearity
3	$A\ln x + B$	$A\frac{\partial}{\partial\lambda}\Big[y(x,\lambda)\Big]_{\lambda=0} + By(x,0)$	Follows from linearity and from the results of row No 4
4	$A\ln^n x$, $n = 0,\,1,\,2,\,\ldots$	$A\left\{\frac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]\right\}_{\lambda=0}$	Follows from the results of row No 5 for $\lambda = 0$
5	$Ax^{\lambda}\ln^n x$, $n = 0,\,1,\,2,\,\ldots$	$A\frac{\partial^n}{\partial\lambda^n}\Big[y(x,\lambda)\Big]$	Differentiation with respect to the parameter λ
6	$A\cos(\beta\ln x)$	$A\,\mathrm{Re}\big[y(x,\,i\beta)\big]$	Selection of the real part for $\lambda = i\beta$
7	$A\sin(\beta\ln x)$	$A\,\mathrm{Im}\big[y(x,\,i\beta)\big]$	Selection of the imaginary part for $\lambda = i\beta$
8	$Ax^{\mu}\cos(\beta\ln x)$	$A\,\mathrm{Re}\big[y(x,\,\mu+i\beta)\big]$	Selection of the real part for $\lambda = \mu+i\beta$
9	$Ax^{\mu}\sin(\beta\ln x)$	$A\,\mathrm{Im}\big[y(x,\,\mu+i\beta)\big]$	Selection of the imaginary part for $\lambda = \mu+i\beta$

Example 4. We seek a solution of the equation with power-law right-hand side

$$y(x) + \int_0^x \frac{1}{x}K\left(\frac{t}{x}\right)y(t)\,dt = x^{\lambda}$$

in the form $y(x,\lambda) = kx^{\lambda}$ by the method of indeterminate coefficients. We finally obtain

$$y(x,\lambda) = \frac{1}{1+B(\lambda)}x^{\lambda}, \qquad B(\lambda) = \int_0^1 K(t)t^{\lambda}\,dt.$$

It follows from row 3 of Table 2 that the solution of the equation with logarithmic right-hand side

$$y(x) + \int_0^x \frac{1}{x} K\Big(\frac{t}{x}\Big) y(t)\,dt = A \ln x$$

has the form

$$y(x) = \frac{A}{1+I_0}\ln x - \frac{AI_1}{(1+I_0)^2},$$
$$I_0 = \int_0^1 K(t)\,dt, \quad I_1 = \int_0^1 K(t)\ln t\,dt.$$

4. Generating Function Containing Sines and Cosines

Consider the linear equation

$$\mathbf{L}\,[y] = \sin(\lambda x). \tag{18}$$

We assume that the solution of this equation is known and is given by formula (2). In Table 3, solutions of the equation $\mathbf{L}\,[y] = f(x)$ with various right-hand sides are given, which are expressed via the solution of Eq. (18).

Consider the linear equation

$$\mathbf{L}\,[y] = \cos(\lambda x). \tag{19}$$

We assume that the solution of this equation is known and is given by formula (2). In Table 4, solutions of the equation $\mathbf{L}\,[y] = f(x)$ with various right-hand sides are given, which are expressed via the solution of Eq. (19).

TABLE 3
Solutions of the equation $\mathbf{L}\,[y] = f(x)$ with sine-shaped generating function

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$\sin(\lambda x)$	$y(x,\lambda)$	Original Equation
2	$\sum_{k=1}^{n} A_k \sin(\lambda_k x)$	$\sum_{k=1}^{n} A_k y(x,\lambda_k)$	Follows from linearity
3	Ax^m, $m = 1, 3, 5, \ldots$	$A(-1)^{\frac{m-1}{2}}\left[\frac{\partial^m}{\partial\lambda^m} y(x,\lambda)\right]_{\lambda=0}$	Follows from the results of row 5 for $\lambda = 0$
4	$Ax^m \sin(\lambda x)$, $m = 2, 4, 6, \ldots$	$A(-1)^{\frac{m}{2}}\frac{\partial^m}{\partial\lambda^m} y(x,\lambda)$	Differentiation with respect to the parameter λ
5	$Ax^m \cos(\lambda x)$, $m = 1, 3, 5, \ldots$	$A(-1)^{\frac{m-1}{2}}\frac{\partial^m}{\partial\lambda^m} y(x,\lambda)$	Differentiation with respect to the parameter λ
6	$\sinh(\beta x)$	$-iy(x,i\beta)$	Relation to the hyperbolic sine, $\lambda = i\beta$
7	$x^m \sinh(\beta x)$, $m = 2, 4, 6, \ldots$	$i(-1)^{\frac{m+2}{2}}\left[\frac{\partial^m}{\partial\lambda^m} y(x,\lambda)\right]_{\lambda=i\beta}$	Differentiation with respect to λ and relation to the hyperbolic sine, $\lambda = i\beta$

TABLE 4
Solutions of the equation $\mathbf{L}[y] = f(x)$ with cosine-shaped generating function

No	Right-Hand Side $f(x)$	Solution y	Solution Method
1	$\cos(\lambda x)$	$y(x,\lambda)$	Original Equation
2	$\sum\limits_{k=1}^{n} A_k \cos(\lambda_k x)$	$\sum\limits_{k=1}^{n} A_k y(x,\lambda_k)$	Follows from linearity
3	Ax^m, $m = 0,\ 2,\ 4,\ \ldots$	$A(-1)^{\frac{m}{2}} \left[\dfrac{\partial^m}{\partial\lambda^m} y(x,\ \lambda)\right]_{\lambda=0}$	Follows from the results of row 4 for $\lambda = 0$
4	$Ax^m \cos(\lambda x)$, $m = 2,\ 4,\ 6,\ \ldots$	$A(-1)^{\frac{m}{2}} \dfrac{\partial^m}{\partial\lambda^m} y(x,\ \lambda)$	Differentiation with respect to the parameter λ
5	$Ax^m \sin(\lambda x)$, $m = 1,\ 3,\ 5,\ \ldots$	$A(-1)^{\frac{m+1}{2}} \dfrac{\partial^m}{\partial\lambda^m} y(x,\ \lambda)$	Differentiation with respect to the parameter λ
6	$\cosh(\beta x)$	$y(x, i\beta)$	Relation to the hyperbolic cosine, $\lambda = i\beta$
7	$x^m \cosh(\beta x)$, $m = 2,\ 4,\ 6,\ \ldots$	$(-1)^{\frac{m}{2}} \left[\dfrac{\partial^m}{\partial\lambda^m} y(x,\ \lambda)\right]_{\lambda=i\beta}$	Differentiation with respect to λ and relation to the hyperbolic cosine, $\lambda = i\beta$

Exersises

1. Find solutions to the integro-differential equation

$$\frac{dy}{dx} + \int_x^\infty e^{-k(x-t)^2} y(t)\,dt = f(x), \qquad k > 0$$

for the following functions:

$$(a)\quad f(x) = e^{\lambda x},$$
$$(b)\quad f(x) = Ax,$$
$$(c)\quad f(x) = A\sin(kx).$$

2. Find solutions to the differential-difference equation

$$\frac{dy(x)}{dx} + ay(x+k) + by(x) = f(x)$$

for the following functions:

$$(a)\quad f(x) = e^{\lambda x},$$
$$(b)\quad f(x) = Ax.$$

3. Find solutions to the integro-differential equation

$$x\frac{dy(x)}{dx} + \int_0^x \frac{1}{x} K\left(\frac{t}{x}\right) y(t)\,dt = f(x)$$

for the following functions:

$$(a)\quad f(x) = x^\lambda,$$
$$(b)\quad f(x) = A\ln x.$$

Reference

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