Method of Model Solutions in the Theory of Linear Integral Equations

1. Preliminary Remarks. Test and Model Solutions

Consider a linear equation, which we briefly write out in the form

$$\mathbf{L}\,[y(x)] = f(x), \tag{1}$$

where $\mathbf{L}$ is a linear (integral) operator, $y(x)$ is an unknown function, and $f(x)$ is a known function.

We first define arbitrarily a test solution

$$y_0 = y_0(x, \lambda), \tag{2}$$

which depends on an auxiliary parameter λ (it is assumed that the operator $\mathbf{L}$ is independent of λ and $y_0 \not\equiv \text{const}$). By means of Eq. (1) we define the right-hand side that corresponds to the test solution (2):

$$f_0(x, \lambda) = \mathbf{L}\,[y_0(x, \lambda)].$$

Let us multiply Eq. (1), for $y = y_0$ and $f = f_0$, by some function $\varphi(\lambda)$ and integrate the resulting relation with respect to λ over an interval $[a, b]$. We finally obtain

$$\mathbf{L}\,[y_\varphi(x)] = f_\varphi(x), \tag{3}$$

where

$$y_\varphi(x) = \int_a^b y_0(x, \lambda)\varphi(\lambda)\,d\lambda, \qquad f_\varphi(x) = \int_a^b f_0(x, \lambda)\varphi(\lambda)\,d\lambda. \tag{4}$$

Here and in what follows we suppose all integrals to be convergent.

It follows from formulas (3) and (4) that, for the right-hand side $f = f_\varphi(x)$, the function $y = y_\varphi(x)$ is a solution of the original equation (1). Since the choice of the function $\varphi(\lambda)$ (as well as of the integration interval) is arbitrary, the function $f_\varphi(x)$ can be arbitrary in principle. Here the main problem is how to choose a function $\varphi(\lambda)$ to obtain a given function $f_\varphi(x)$. This problem can be solved if we can find a test solution such that the right-hand side of Eq. (1) is the kernel of a known inverse integral transform (we denote such a test solution by $Y(x, \lambda)$ and call it a *model solution*).

2. Description of the Method

Indeed, let $\mathfrak{P}$ be an invertible integral transform that takes each function $f(x)$ to the corresponding transform $F(\lambda)$ by the rule

$$F(\lambda) = \mathfrak{P}\{f(x)\}. \tag{5}$$

Assume that the inverse transform $\mathfrak{P}^{-1}$ has the kernel $\psi(x, \lambda)$ and acts as follows:

$$\mathfrak{P}^{-1}\{F(\lambda)\} = f(x), \qquad \mathfrak{P}^{-1}\{F(\lambda)\} \equiv \int_a^b F(\lambda)\psi(x, \lambda)\,d\lambda. \tag{6}$$

The limits of integration a and b and the integration path in (6) may well lie in the complex plane.

Suppose that we succeeded in finding a model solution $Y(x, \lambda)$ of the auxiliary problem for Eq. (1) whose right-hand side is the kernel of the inverse transform $\mathfrak{P}^{-1}$:

$$\mathbf{L}\,[Y(x, \lambda)] = \psi(x, \lambda). \tag{7}$$

Let us multiply Eq. (7) by $F(\lambda)$ and integrate with respect to λ within the same limits that stand in the inverse transform (6). Taking into account the fact that the operator $\mathbf{L}$ is independent of λ and applying the relation $\mathfrak{P}^{-1}\{F(\lambda)\} = f(x)$, we obtain

$$\mathbf{L}\left[\int_a^b Y(x,\lambda)F(\lambda)\,d\lambda\right] = f(x).$$

Therefore, the solution of Eq. (1) for an arbitrary function $f(x)$ on the right-hand side is expressed via a solution of the simpler auxiliary equation (7) by the formula

$$y(x) = \int_a^b Y(x,\lambda)F(\lambda)\,d\lambda, \tag{8}$$

where $F(\lambda)$ is the transform (5) of the function $f(x)$.

For the right-hand side of the auxiliary equation (7) we can take, for instance, exponential, power-law, and trigonometric function, which are the kernels of the Laplace, Mellin, and sine and cosine Fourier transforms (up to a constant factor). Sometimes it is rather easy to find a model solution by means of the method of indeterminate coefficients (by prescribing its structure). Afterwards, to construct a solution of the equation with arbitrary right-hand side, we can apply formulas written out below in Sections 3–5.

3. Model Solution in the Case of an Exponential Right-Hand Side

Assume that we have found a model solution $Y = Y(x,\lambda)$ that corresponds to the exponential right-hand side:

$$\mathbf{L}\,[Y(x,\lambda)] = e^{\lambda x}. \tag{9}$$

Consider two cases:

1°. *Equations on the semiaxis,* $0 \le x < \infty$. Let $\widetilde{f}(p)$ be the Laplace transform of the function $f(x)$:

$$\widetilde{f}(p) = \mathfrak{L}\{f(x)\}, \qquad \mathfrak{L}\{f(x)\} \equiv \int_0^\infty f(x)e^{-px}\,dx. \tag{10}$$

The solution of Eq. (1) for an arbitrary right-hand side $f(x)$ can be expressed via the solution of the simpler auxiliary equation with exponential right-hand side (9) for $\lambda = p$ by the formula

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(x,p)\widetilde{f}(p)\,dp. \tag{11}$$

2°. *Equations on the entire axis,* $-\infty < x < \infty$. Let $\widetilde{f}(u)$ the Fourier transform of the function $f(x)$:

$$\widetilde{f}(u) = \mathfrak{F}\{f(x)\}, \qquad \mathfrak{F}\{f(x)\} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iux}\,dx. \tag{12}$$

The solution of Eq. (1) for an arbitrary right-hand side $f(x)$ can be expressed via the solution of the simpler auxiliary equation with exponential right-hand side (9) for $\lambda = iu$ by the formula

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(x,\,iu)\widetilde{f}(u)\,du. \tag{13}$$

In the calculation of the integrals on the right-hand sides in (11) and (13), methods of the theory of functions of a complex variable are applied, including the Jordan lemma and the Cauchy residue theorem.

Remark 1. The structure of a model solution $Y(x,\lambda)$ can differ from that of the kernel of the Laplace or Fourier inversion formula.

Remark 2. When applying the method under consideration, the left-hand side of Eq. (1) need not be known (the equation can be integral, differential, functional, etc.) if a particular solution of this equation is known for the exponential right-hand side. Here only the most general information is important, namely, that the equation is linear, and its left-hand side is independent of the parameter λ.

Remark 3. The above method can be used in the solution of linear integral (differential, integro-differential, and functional) equations with composed argument of the unknown function.

Example 1. Consider the following Volterra equation of the second kind with difference kernel:

$$y(x) + \int_x^\infty K(x-t)y(t)\,dt = f(x). \tag{14}$$

This equation cannot be solved by direct application of the Laplace transform because the convolution theorem cannot be used here.

In accordance with the method of model solutions, we consider the auxiliary equation with exponential right-hand side

$$Y(x,p) + \int_x^\infty K(x-t)Y(t,p)\,dt = e^{px} \qquad (\lambda = p). \tag{15}$$

We seek a solution of the linear integral equation with exponential right-hand side (15) in the form $Y(x,p) = ke^{pz}$ by the method of indeterminate coefficients. Then we obtain

$$Y(x,p) = \frac{1}{1+\widetilde{K}(-p)}e^{px}, \qquad \widetilde{K}(-p) = \int_0^\infty K(-z)e^{pz}\,dz. \tag{16}$$

This, by means of formula (11), yields a solution of Eq. (12) for an arbitrary right-hand side,

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\widetilde{f}(p)}{1+\widetilde{K}(-p)}e^{px}\,dp, \tag{17}$$

where $\widetilde{f}(p)$ is the Laplace transform (10) of the function $f(x)$.

Example 2. Consider the integral equation

$$Ay(x) + \int_{-\infty}^\infty Q(x+t)e^{\beta t}y(t)\,dt = f(x), \tag{18}$$

where $Q = Q(z)$ and $f(x)$ are arbitrary functions and A and β are arbitrary constants satisfying some constraints.

In order to find the model solution that corresponds to the equation with the exponential right-hand side

$$AY(x,p) + \int_{-\infty}^\infty Q(x+t)e^{\beta t}Y(t,p)\,dt = e^{px} \qquad (\lambda = p).$$

let us proceed as follows.

For clarity, instead of the original equation (18) we write

$$\mathbf{L}\,[y(x)] = f(x). \tag{19}$$

For a test solution, we take the exponential function

$$y_0 = e^{px}. \tag{20}$$

On substituting (20) into the left-hand side of Eq. (19), after some algebraic manipulations we obtain

$$\mathbf{L}\,[e^{px}] = Ae^{px} + q(p)e^{-(p+\beta)x}, \qquad \text{where} \quad q(p) = \int_{-\infty}^\infty Q(z)e^{(p+\beta)z}\,dz. \tag{21}$$

The right-hand side of (21) can be regarded as a functional equation for the kernel e^{px} of the inverse Laplace transform. To solve it, we replace p by $-p-\beta$ in Eq. (19). We finally obtain

$$\mathbf{L}\,[e^{-(p+\beta)x}] = Ae^{-(p+\beta)x} + q(-p-\beta)e^{px}. \tag{22}$$

Let us multiply Eq. (21) by A and Eq. (22) by $-q(p)$ and add the resulting relations. This yields

$$\mathbf{L}\,[Ae^{px} - q(p)e^{-(p+\beta)x}] = [A^2 - q(p)q(-p-\beta)]e^{px}. \tag{23}$$

On dividing Eq. (23) by the constant $A^2 - q(p)q(-p-\beta)$, we obtain the original model solution

$$Y(x,p) = \frac{Ae^{px} - q(p)e^{-(p+\beta)x}}{A^2 - q(p)q(-p-\beta)}, \qquad \mathbf{L}\,[Y(x,p)] = e^{px}. \tag{24}$$

Since here $-\infty < x < \infty$, one must set $p = iu$ and use the formulas from Section 3. Then the solution of Eq. (18) for an arbitrary function $f(x)$ can be represented in the form

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty Y(x,iu)\widetilde{f}(u)\,du, \qquad \widetilde{f}(u) = \int_{-\infty}^\infty f(x)e^{-iux}\,dx. \tag{25}$$

4. Model Solution in the Case of a Power-Law Right-Hand Side

Suppose that we have succeeded in finding a model solution $Y = Y(x,s)$ that corresponds to a power-law right-hand side of the equation:

$$\mathbf{L}\,[Y(x,s)] = x^{-s}, \qquad \lambda = -s. \tag{26}$$

Let $\hat{f}(s)$ be the Mellin transform of the function $f(x)$:

$$\hat{f}(s) = \mathfrak{M}\{f(x)\}, \qquad \mathfrak{M}\{f(x)\} \equiv \int_0^\infty f(x)x^{s-1}\,dx. \tag{27}$$

The solution of Eq. (1) for an arbitrary right-hand side $f(x)$ can be expressed via the solution of the simpler auxiliary equation with power-law right-hand side (26) by the formula

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(x,s)\hat{f}(s)\,ds. \tag{28}$$

In the calculation of the corresponding integrals on the right-hand side of formula (28), one can use tables of inverse Mellin transforms, as well as methods of the theory of functions of a complex variable, including the Jordan lemma and the Cauchy residue theorem.

Example 3. Consider the equation

$$y(x) + \int_0^x \frac{1}{x}K\Bigl(\frac{t}{x}\Bigr)y(t)\,dt = f(x). \tag{29}$$

In accordance with the method of model solutions, we consider the following auxiliary equation with power-law right-hand side:

$$y(x) + \int_0^x \frac{1}{x}K\Bigl(\frac{t}{x}\Bigr)y(t)\,dt = x^{-s}. \tag{30}$$

Its solution has the form

$$Y(x,s) = \frac{1}{1+B(s)}x^{-s}, \qquad B(s) = \int_0^1 K(t)t^{-s}\,dt. \tag{31}$$

This, by means of formula (28), yields the solution of Eq. (29) for an arbitrary right-hand side:

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\hat{f}(s)}{1+B(s)}x^{-s}\,ds, \tag{32}$$

where $\hat{f}(s)$ is the Mellin transform (27) of the function $f(x)$.

5. Model Solution in the Case of a Sine-Shaped Right-Hand Side

Suppose that we have succeeded in finding a model solution $Y = Y(x,u)$ that corresponds to the sine on the right-hand side:

$$\mathbf{L}\,[Y(x,u)] = \sin(ux), \qquad \lambda = u. \tag{33}$$

Let $\check{f}_{\rm s}(u)$ be the asymmetric sine Fourier transform of the function $f(x)$:

$$\check{f}_{\rm s}(u) = \mathcal{F}_{\rm s}\{f(x)\}, \qquad \mathcal{F}_{\rm s}\{f(x)\} \equiv \int_0^\infty f(x)\sin(ux)\,dx.$$

The solution of Eq. (1) for an arbitrary right-hand side $f(x)$ can be expressed via the solution of the simpler auxiliary equation with sine-shape right-hand side (33) by the formula

$$y(x) = \frac{2}{\pi}\int_0^\infty Y(x,u)\check{f}_{\rm s}(u)\,du.$$

Reference

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