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A Heat Transfer with a Source: the Complete Set of Invariant Difference Schemes

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Abstract

In this letter we present the set of invariant difference equations and meshes which preserve the Lie group symmetries of the equation $u_t = (K(u)u_x)_x + Q(u)$. All special cases of $K(u)$ and $Q(u)$ that extend the symmetry group admitted by the differential equation are considered. This paper completes the paper [*J. Phys. A: Math. Gen.* **30**, Nr. 23 (1997), 8139–8155], where a few invariant models for heat transfer equations were presented.

1 Introduction

Symmetries are fundamental features of the differential equations of mathematical physics. It yield a number of useful properties such as integrability of ODEs, symmetry reduction of PDEs, existence of various types invariant solutions, conservation laws for the invariant variational problems etc. Therefore, preserving symmetries in discrete schemes, we retain qualitative properties of the underlying differential equations.

The purpose of this paper is to develop the entire set of invariant difference schemes for the heat transfer equation

$$u_t = (K(u)u_x)_x + Q(u), \tag{1.1}$$

for all special cases of the coefficients $K(u)$, $Q(u)$ which extend the symmetry group admitted by equation (1.1). This paper is based on the Lie group classification [4] (see also [1]) of the equation (1.1) with arbitrary $K(u)$ and $Q(u)$. This classification contains the result of L V Ovsyannikov [17] for equation (1.1) with $Q \equiv 0$ as well as symmetries of the linear case $K \equiv 1$, $Q \equiv 0$, which were known by S Lie.

A few examples of the invariant difference schemes and meshes were considered in [2]. In the present paper we complete the paper [2] going through all cases of $K(u)$ and $Q(u)$ identified in the group classification [4], we construct difference equations and meshes (lattices) which admit the same Lie groups of point transformations as their continuous limits.

Lie group analysis of difference equations is a very active field of research where many contributions were done, and various approaches were applied by several authors (see [18]). In our approach which we are following in this paper we pose the question: How does one discretize a differential equation while preserving all of its Lie point symmetries? Thus a differential equation and its Lie group symmetry are *a priory* given but not a difference model. One then looks for a difference scheme, i.e. a difference equation and a mesh, that have the same symmetry group and the same Lie algebra. The basic steps in this direction were done [5, 6, 7, 8, 9, 10, 11], which were summarized in a recent book [12]. The main idea is that the invariant difference equations and meshes can be constructed with the help of the entire set of difference invariants of the corresponding Lie group. In the next section we explain how to construct difference models that conserve the whole group of point transformations admitted by the differential equations.

The article is organized as follows. Section 2 provides a brief overview of the invariant discretization procedure. In Sections 3, 4, 5 and 6 we consider the cases of an arbitrary heat transfer coefficient $K(u)$, the exponential heat transfer coefficient e^u, the power heat transfer coefficient u^σ and the special case of power heat transfer coefficient: $u^{-4/3}$ correspondingly.

Section 7 is devoted to the linear heat conductivity with a source. In particular, this section covers detailed study of the invariant difference scheme for the linear heat equation without a source ($Q = 0$) including such aspects as superposition principle, reduction of the invariant scheme on the optimal system of subalgebras and the way to transform the moving mesh scheme into a stationary one. Notice that in the paper [19] there were considered some difference approximations of the linear heat transfer equation, which preserve its different symmetries on different meshes. In [19] a difference equation and a mesh are *a priori* given, then it was shown that for some kind of the mesh there were preserved some symmetries of the linear heat equation and another meshes preserve other parts of the symmetries. Thus, there are no difference schemes which conserve the *entire set of symmetries* in one difference model. In Section 7 we will develop the difference mesh and difference equation, which conserve the complete set of original symmetries in the one and the same difference scheme.

The same approach we will apply for some tens of other nonlinear models of heat equation (1.1). Summarizing conclusions end up the consideration of the entire set of invariant schemes for the equation (1.1).

2 Symmetry preserving discretization procedure

1. Let us briefly describe this method, which was called *the method of finite-difference invariants* [6].

Let the differential equation

$$F(t, x, u, u_t, u_x, \dots) = 0 \tag{2.1}$$

admit a known symmetry group G_n, whose Lie algebra is spanned by the operators $X_1, \dots, X_n$ of the form

$$X_i = \xi_i^t \frac{\partial}{\partial t} + \xi_i^x \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial u}, \qquad i = 1, \dots, n, \tag{2.2}$$

where the coefficients ξ^x, ξ^t and η are functions of t, x, u, since we consider Lie point symmetries.

Then we would like to propose a discrete model

$$F(z) = 0,$$
$$\Omega(z, h) = 0, \tag{2.3}$$

where the first equation is the approximation of the initial differential equation and the second one defines a difference mesh. Both of these equations *a priori* are not given and we have to establish the invariant mesh, on which we should approximate the original heat equation. We will show, that in all special cases of equation (1.1) one can construct the system (2.3), starting from the entire set of finite-difference invariants of the corresponding Lie group.

We denote by z in (2.3) a finite number of difference variables, which are used in the considered *difference stencil*, i.e. a finite number of mesh points, which are needed for the approximation of the differential equation (2.1). The equations (2.3) can be explicitly connected with each other (if invariant mesh depends on solution) or not. In the last case we can choose the invariant mesh firstly (for example, a fixed mesh) and then construct the invariant approximation of the original equation. If a mesh depends on the solution, all specifications of the mesh made in advance lead to restrictions on the symmetries which may be admitted by the considered discrete models.

2. The idea of the method of finite-difference invariants springs from the invariant representation of differential equations. In the continuous case for the group G_n we can find the complete set of functionally independent differential invariants $J = (J_1, J_2, \ldots, J_k)$ in the specified space which contains dependent and independent variables as well as the set of derivatives up to the highest derivatives involved in the formulation of the PDE [15]. For the heat equation (1.1) we consider the space $M \sim (t, x, u, u_t, u_x, u_{xx})$. We prolong the operator (2.2) on the variables of the space M

$$\mathbf{pr}\ X = X + \zeta^t \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{xx} \frac{\partial}{\partial u_{xx}},$$

with

$$\zeta^t = D_t(\eta) - u_t D_t(\xi^t) - u_x D_t(\xi^x), \qquad \zeta^x = D_x(\eta) - u_t D_x(\xi^t) - u_x D_x(\xi^x),$$
$$\zeta^{xx} = D_x(\zeta^x) - u_{tx} D_x(\xi^t) - u_{xx} D_x(\xi^x),$$

where D_t and D_x are the total derivative operators for time and space correspondingly. Differential invariants are solutions of the system of linear equations

$$\mathbf{pr}\ X_i\ \Phi(t, x, u, u_t, u_x, u_{xx}) = 0, \qquad i = 1, \ldots, n,$$

and can be solved by standard procedure (see [15]).

Then we represent the invariant differential equation in terms of these invariants

$$\tilde{F}(J_1, J_2, \ldots, J_k) = 0.$$

The obtained equation is invariant with respect to the group G_n.

3. In the discrete case the situation is more complicated. Any given differential equation can be approximated by means of infinitely many difference equations and meshes, which have the original differential equation as its continuous limit. The requirement of preservation of Lie group properties of the differential equation in its discrete counterpart still leaves some freedom in the approximations. Thus, as it can be seen recently, at some point we have to make a chose among the general family of invariant meshes.

The structure of the admitted group essentially effects on construction of equations and meshes. Group transformations can break the geometric structure of the difference mesh that influences approximation and other properties of difference equations. First steps to the construction of the difference grids geometry based on the symmetries of the initial difference model were done in [5, 6, 7, 9]. There were found classes of transformations that conserve uniformity, orthogonality and other properties of the grids.

It was shown [5, 6, 7, 9] that a transformation defined by (2.2) conserves uniformity of a mesh in t and x directions if and only if

$$\underset{+\tau}{D}\underset{-\tau}{D}(\xi^t) = 0, \tag{2.4}$$

$$\underset{+h}{D}\underset{-h}{D}(\xi^x) = 0, \tag{2.5}$$

where $\underset{\pm\tau}{D}$ and $\underset{\pm h}{D}$ denote total difference derivatives in the time and space directions with steps τ and h correspondingly.

For an orthogonal mesh to be conserved under the transformation, it is necessary and sufficient that

$$\underset{+h}{D}(\xi^t) = -\underset{+\tau}{D}(\xi^x). \tag{2.6}$$

When condition (2.6) is not satisfied for a given group, the flatness of the layer of a grid in some direction is rather important. For evolution equations it is significant to have flat time layers. There is a simple criterion of the invariance of flat time layers under the action of a given operator (2.2):

$$\underset{\pm h}{D}\underset{+\tau}{D}(\xi^t) = 0. \tag{2.7}$$

These condition specify invariant geometry of grids for the given Lie group symmetries.

If the operator coefficients ξ^t, ξ^x do not depend on solution, then we can choose the invariant mesh as any solution of corresponding condition (2.4)–(2.7). Otherwise the conditions (2.4)–(2.7) should hold on the solutions of the considered difference model. In that case we can figure a mesh out starting from the set of difference invariants.

Further we choose a stencil which is sufficient to approximate all derivatives which appear in the equation. We will consider six-point stencils which have three points on each of two time layers. Such stencils allow us to write down both explicit and implicit difference schemes. For different transformation groups we will consider different meshes: orthogonal mesh which is uniform in space, orthogonal mesh which is nonuniform in space and nonorthogonal in time-space mesh, i.e. moving mesh. The corresponding stencils are different. Furthermore, the corresponding spaces of discrete variables are of different dimensions so that they have different number of difference invariants $I = (I_1, I_2, \ldots, I_l)$ for the same Lie group G_n.

For example, let us take an orthogonal mesh which is uniform in space. (We will describe later for which groups this mesh can be considered.) The stencil of this mesh is shown in Fig. 1.

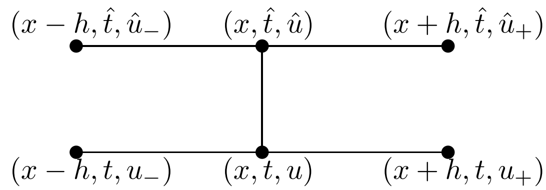

Figure 1. The stencil of the orthogonal mesh.

The corresponding discrete subspace is ten-dimensional: $M \sim (t,x,\tau,h,u,u_-,u_+,\hat{u}, \hat{u}_-,\hat{u}_+)$, where $\tau = \hat{t} - t$. The prolonged operator (2.2) in this subspace has the form

$$\mathbf{pr}\ X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + (\hat{\xi}^t - \xi^t)\frac{\partial}{\partial \tau} + (\xi^x_+ - \xi^x)\frac{\partial}{\partial h} + \eta \frac{\partial}{\partial u} + \eta_- \frac{\partial}{\partial u_-} + \eta_+ \frac{\partial}{\partial u_+} + \hat{\eta}\frac{\partial}{\partial \hat{u}} + \hat{\eta}_- \frac{\partial}{\partial \hat{u}_-} + \hat{\eta}_+ \frac{\partial}{\partial \hat{u}_+},$$

where we use time and space shifts notations $\hat{f} = f(t+\tau,x,u)$, $f_- = f(t,x-h,u)$, $f_+ = f(t,x+h,u)$. The number of functionally independent invariants is given by

$$l = \dim M - \operatorname{rank} Z, \qquad l \geq 0, \tag{2.8}$$

with $\dim M = 10$ and the matrix Z composed by the coefficients of the prolonged on the space M operators

$$Z = \begin{pmatrix} \xi^t_1 & \xi^x_1 & (\hat{\xi}^x_1 - \xi^x_1) & ((\xi^x_1)_+ - \xi^x_1) & \eta_1 & \cdots & (\hat{\eta}_1)_+ \\ \vdots & & & & & & \\ \xi^t_n & \xi^x_n & (\hat{\xi}^x_n - \xi^x_n) & ((\xi^x_n)_+ - \xi^x_n) & \eta_n & \cdots & (\hat{\eta}_n)_+ \end{pmatrix}.$$

Having found the finite-difference invariants as the solutions of system of linear equations

$$\mathbf{pr}\ X_i\ \Phi(t,x,\tau,h,u,u_-,u_+,\hat{u},\hat{u}_-,\hat{u}_+) = 0, \qquad i = 1,\ldots,n,$$

we can use them to approximate the differential invariants

$$J_j = f_j(I_1, I_2, \ldots, I_l) + O(\tau^\alpha, h^\beta), \qquad j = 1,\ldots,k,$$

where α and β define some fixed order of approximation. Notice, that approximation error $O(\tau^\alpha, h^\beta)$ is invariant together with other terms in the above representation. *Substitution of difference invariants I_i instead of differential ones J_i into the function $\tilde{F}$ provides us with an invariant difference scheme.* Practically we can often omit the representation of the differential equation in terms of its invariants and just approximate the original differential equation by the finite-difference invariants. The use of finite difference invariants is the main point in both ways.

So, the first step in the invariant approximation is the choice of the invariant mesh. The last step is the choice of the invariant discretization of the original equation on the invariant mesh.

The described above method is algorithmic. We would like to stress that the invariant approximation in our way is still not unique. For example extending the stencil (means enlarging the number of mesh points involved in approximation) we can find invariant approximations of any higher order.

3 An arbitrary heat transfer coefficient $K(u)$

Now we start to develop invariant schemes going throw all cases of the Lie group classification [4]. Let us note that the group classification of the equation (1.1) was done in [4] (see also [1]) up to equivalent transformations:

$$\bar{t} = at + e, \qquad \bar{x} = bx + f, \qquad \bar{u} = cu + g, \qquad \bar{K} = \frac{b^2}{a} K, \qquad \bar{Q} = \frac{c}{a} Q, \tag{3.1}$$

where a, b, c, e, f and g are arbitrary constants, $abc \neq 0$. These transformations do not change differential structure of the equation (1.1), transforming an admitted group into a similar group of point transformations.

1. We start from general case, when the coefficients $K(u)$ and $Q(u)$ are arbitrary. Then the equation (1.1) admits a two parameter group of translations only. This group is defined by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \tag{3.2}$$

which generate the translations of independent variables. In this case almost no limits are imposed on a mesh and a difference equation. In particular, we can use an orthogonal grid in the plane (x, t) which is regular in both directions, as the conditions (2.4)–(2.6) are valid for the operators (3.2).

The group with operators (3.2) in the subspace $(x, t, h, \tau, u, u_-, u_+, \hat{u}, \hat{u}_-, \hat{u}_+)$ corresponding to the stencil shown in Fig. 1 has eight invariants:

$$\tau, \quad h, \quad u, \quad u_+, \quad u_-, \quad \hat{u}, \quad \hat{u}_-, \quad \hat{u}_+.$$

That is why any difference approximation of the equation (1.1) by the above invariants could give difference equation which admits the operators (3.2). For example, the explicit model

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}\left(K\left(\frac{u_+ + u}{2}\right) u_{\underset{h}{x}} - K\left(\frac{u + u_-}{2}\right) u_{\underset{h}{\bar{x}}} \right) + Q(u), \tag{3.3}$$

where $K(u)$ and $Q(u)$ represent any approximation of the corresponding coefficients by invariants and $u_{\underset{h}{x}} = \frac{u_+ - u}{h}$, $u_{\underset{h}{\bar{x}}} = \frac{u - u_-}{h}$ are right and left difference derivatives, admits the operators (3.2).

2. If $K(u)$ is arbitrary function and $Q(u) \equiv 0$, the equation

$$u_t = (K(u) u_x)_x \tag{3.4}$$

admits a three-parameter algebra of operators (see [17]):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{3.5}$$

This case is almost analogous to the previous one. The operators (3.5) do not violate conditions of invariant orthogonality (2.6) and invariant uniformity of a grid (2.4), (2.5). Thus, in this case we could use the orthogonal grid shown in Fig. 1. Any approximation of the equation (3.4) by the seven invariants

$$\frac{h^2}{\tau}, \quad u, \quad u_+, \quad u_-, \quad \hat{u}, \quad \hat{u}_-, \quad \hat{u}_+$$

gives an invariant model for the equation (3.4). In particular the explicit scheme (3.3) with $Q \equiv 0$:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(K\left(\frac{u_+ + u}{2}\right)\underset{h}{u}_x - K\left(\frac{u+u_-}{2}\right)\underset{h}{u}_{\bar{x}}\right) \tag{3.6}$$

can be used.

4 The exponential heat transfer coefficient $K = e^u$

In this paragraph we consider three cases of group classification for $K = e^u$, in accordance with [4] and [17].

1. If $Q = 0$ then the equation

$$u_t = (e^u u_x)_x \tag{4.1}$$

admits a four-dimensional algebra of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}. \tag{4.2}$$

As in the cases considered above, conditions of invariant uniformity and invariant orthogonality are valid. A difference model for the equation (4.1) can be constructed by approximation of the differential equation with the help of difference invariants:

$$e^u\frac{\tau}{h^2}, \quad (\hat{u}-u), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-).$$

An example is the simple explicit difference model:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right)\underset{h}{u}_x - \exp\left(\frac{u+u_-}{2}\right)\underset{h}{u}_{\bar{x}}\right), \tag{4.3}$$

but one has a lot of freedom to construct invariant schemes using the finite-difference invariants.

2. For $Q = \delta = \pm 1$ we have a possibility to exclude the constant source from the equation

$$u_t = (e^u u_x)_x + \delta \tag{4.4}$$

by change of variables:

$$\bar{u} = u - \delta t, \qquad \bar{t} = \delta(e^{\delta t} - 1). \tag{4.5}$$

The equation (4.4) will be transformed into the equation (4.1) by this change, but the uniformity of the grid in the t-direction is destroyed. The equation (4.4) admits the four-dimensional algebra of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{-\delta t}\frac{\partial}{\partial t} + \delta e^{-\delta t}\frac{\partial}{\partial u}, \qquad X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}, \tag{4.6}$$

and we can easily see that the operator X_3 does not conserve uniformity of a grid in the time direction. The finite-difference invariants:

$$\frac{e^u(e^{\delta\tau} - 1)}{h^2}, \quad (\hat{u} - u - \delta\tau), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-)$$

permit us to construct the following variant of difference model for the equation (4.4):

$$\frac{\delta(\hat{u} - u) - \tau}{e^{\delta\tau} - 1} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right) \underset{h}{u}_x - \exp\left(\frac{u + u_-}{2}\right) \underset{h}{u}_{\bar{x}}\right). \tag{4.7}$$

Let us note that the change of variables (4.5) transforms the model (4.7) considered on the orthogonal grid with the time interval $[0, T]$, given by the formula

$$t_n = \delta \ln\left(1 + \frac{n}{k}(e^{\delta T} - 1)\right), \qquad n = 0, \ldots, k, \tag{4.8}$$

where k is the number of time steps of the grid, into the model (4.3) with uniform time grid on the time interval $\left[0, \delta\left(e^{\delta T} - 1\right)\right]$.

3. If $Q = \pm e^{\alpha u}$, $\alpha \neq 0$ the equation

$$u_t = (e^u u_x)_x \pm e^{\alpha u} \tag{4.9}$$

admits 3 infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2\alpha t\frac{\partial}{\partial t} + (\alpha - 1)x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}. \tag{4.10}$$

These operators satisfy the conditions of orthogonality and uniformity of invariant grids and we will consider the stencil of Fig. 1. Any approximation of the equation (4.9) by the finite-difference invariants

$$\frac{\tau^{\frac{\alpha-1}{2\alpha}}}{h}, \quad e^{\alpha u}\tau, \quad (\hat{u} - u), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-)$$

gives a variant of a difference model for the equation (4.9), admitting the symmetries (4.10), for example, we obtain the following model:

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right) \underset{h}{u}_x - \exp\left(\frac{u + u_-}{2}\right) \underset{h}{u}_{\bar{x}}\right) \pm e^{\alpha u}. \tag{4.11}$$

4. In accordance with group classification [4] we will also consider the case $Q = \pm e^u + \delta$, $\delta = \pm 1$. As in the case 2 we have the possibility to exclude the constant source from the equation

$$u_t = (e^u u_x)_x \pm e^u + \delta \tag{4.12}$$

by the change of variables (4.5). The equation (4.12) will be transformed into the equation (4.9). The equation (4.12) admits the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{-\delta t}\frac{\partial}{\partial t} + \delta e^{-\delta t}\frac{\partial}{\partial u}. \tag{4.13}$$

Finite-difference invariants of (4.13)

$$e^u(e^{\delta\tau} - 1), \quad h, \quad (\hat{u} - u - \delta\tau), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-)$$

permit to construct the following variant of the difference model

$$\frac{\delta(\hat{u} - u) - \tau}{e^{\delta\tau} - 1} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right) \underset{h}{u}_x - \exp\left(\frac{u + u_-}{2}\right) \underset{h}{u}_{\bar{x}}\right) \pm e^u \tag{4.14}$$

on the invariant grid (4.8). The model for the considered equation can be obtained from the model (4.11) with the help of the transformation (4.5).

5 The power heat transfer coefficient: $K = u^\sigma$, $\sigma \neq 0, -\frac{4}{3}$

For $K = u^\sigma$ further classification depends on the source.

1. Let us start with the simplest case $Q \equiv 0$:

$$u_t = (u^\sigma u_x)_x. \tag{5.1}$$

Symmetries of the equation (5.1) are described by the four-dimensional algebra of infinitesimal operators (see [17]):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_4 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{5.2}$$

With any σ the operators (5.2) conserve uniformity and orthogonality of a grid. The finite-difference invariants corresponding to the stencil of Fig. 1 are

$$u^\sigma \frac{\tau}{h^2}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}.$$

They permit us to write, for example, the following variant of the difference model on the orthogonal uniform mesh in both directions:

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^\sigma \underset{h}{u}_x - \left(\frac{u + u_-}{2}\right)^\sigma \underset{h}{u}_{\bar{x}}\right). \tag{5.3}$$

Orthogonal grid is not the only possible way for discrete modeling. On the example of the equation (5.1) we will show how to introduce a moving mesh of the form shown in Fig. 2.

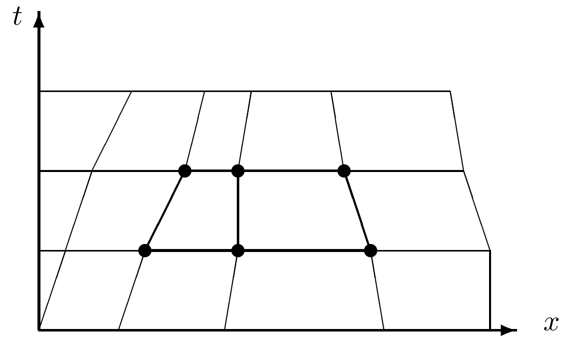

Figure 2. A moving mesh with flat time-layers.

One can use an adaptive grid defined by an evolution equation (see also [3])

$$\frac{dx}{dt} = \varphi(t, x, u, u_x). \tag{5.4}$$

In this case the heat transfer equation will take the form

$$\frac{du}{dt} = (u^\sigma u_x)_x + \varphi(t, x, u, u_x)u_x. \tag{5.5}$$

Different requirements could be imposed on the function φ. If we require invariance of the equation (5.4) with respect to the whole set of the operators (5.2), our freedom to chose φ is limited by the function

$$\varphi = Cu^{\sigma-1}u_x, \qquad C = \text{const}.$$

Below we show how to introduce Lagrangian type of evolution $\frac{dx}{dt}$. Let us note that the equation (5.1) has a form of the conservation law that presents the conservation of heat. Hence we can search for a moving mesh of Lagrange type which evolves in accordance with heat diffusion. We should find an evolution $\frac{dx}{dt}$ which satisfies the equation

$$\frac{d}{dt}\int_{x1(t)}^{x2(t)} u\,dx = 0.$$

Since

$$\frac{d}{dt}\int_{x1}^{x2} u\,dx = \int_{x1}^{x2} \frac{\partial u}{\partial t}\,dx + \left[u\frac{dx}{dt}\right]_{x1}^{x2} = \left[u^\sigma u_x + u\frac{dx}{dt}\right]_{x1}^{x2}$$

we obtain the evolution $\frac{dx}{dt} = -u^{\sigma-1}u_x$. Note that this evolution is invariant with respect to the operators (5.2). Our initial differential equation (5.1) can now be presented in the form of the system

$$\frac{dx}{dt} = -u^{\sigma-1}u_x, \qquad \frac{du}{dt} = u^\sigma u_{xx} + (\sigma-1)u^{\sigma-1}u_x^2. \tag{5.6}$$

Let us mention that the equation (5.1) has two conservation laws

$$u_t = (u^\sigma u_x)_x \qquad \text{and} \qquad (xu)_t = \left(xu^\sigma u_x - \frac{u^{\sigma+1}}{\sigma+1} \right)_x .$$

For the evolution system (5.6) it is convenient to present the conservation laws in the integral form

$$\frac{d}{dt}\int_{x1(t)}^{x2(t)} u\,dx = 0, \qquad \frac{d}{dt}\int_{x1(t)}^{x2(t)} xu\,dx = -\left.\frac{u^{\sigma+1}}{\sigma+1}\right|_{x_1}^{x_2}. \tag{5.7}$$

For difference modeling of the system (5.6) we can consider the stencil shown in Fig. 3.

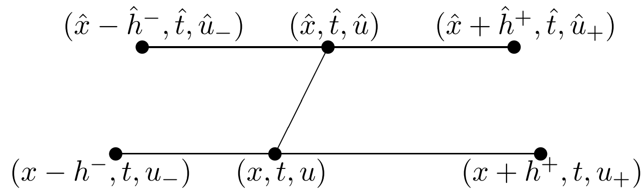

Figure 3. The stencil of the evolutionary mesh.

In the space of the variables $(t, x, \tau, h^+, h^-, \hat{h}^+, \hat{h}^-, \Delta x, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ corresponding to this stencil there are ten finite-difference invariants:

$$u^\sigma \frac{\tau}{h^{+2}}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}, \quad \frac{h^-}{h_+}, \quad \frac{\hat{h}^-}{h^+}, \quad \frac{\hat{h}^+}{h^+}, \quad \frac{\Delta x}{h^+}.$$

Approximating the system (5.6) by these invariants we can get, for example, the system of two equations:

$$\frac{\Delta x}{\tau} = -\frac{1}{2\sigma}\left(\frac{u_+^\sigma - u^\sigma}{h^+} + \frac{u^\sigma - u_-^\sigma}{h^-} \right), \qquad \frac{\hat{u} + \hat{u}_+}{2}\hat{h}_+ = \frac{u + u_+}{2}h_+, \tag{5.8}$$

where we approximated the heat conservation law to obtain the equation for the solution u.

The first equation of system (5.6) shows that the evolution of x depends on the solution. The system (5.8) may be inconvenient for computations because steplength will be changed automatically and the nature of this process is not clear. In order to avoid this indeterminacy we introduce a new space variable which values characterize the evolution trajectories of x. Let us consider the variable s defined by the system:

$$s_t = u^\sigma u_x, \qquad s_x = u.$$

It is easy to see that each trajectory of x is prescribed by a fixed value of s since

$$\frac{ds}{dt} = s_t + s_x \frac{dx}{dt} = u^\sigma u_x - u\frac{1}{\sigma}(u^\sigma)_x = 0.$$

In a new coordinate system with the independent variables (t, s) the equation (5.1) has the form

$$\left(\frac{1}{u}\right)_t = -(u^\sigma u_s)_s \tag{5.9}$$

and the former space variable x satisfies

$$x_t = -u^\sigma u_s, \qquad x_s = \frac{1}{u}. \tag{5.10}$$

For discrete modeling of the equation (5.1) one can use the equation (5.9) in the new independent variables (t, s) to describe the diffusion process and the first equation of the system (5.10) to trace the evolution of the coordinate x. The equation (5.9) considered together with the system (5.10) admits the following symmetries

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial s}, \qquad X_4 = 2t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + x\frac{\partial}{\partial x}, \\ X_5 = (\sigma+2)s\frac{\partial}{\partial s} + \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{gathered} \tag{5.11}$$

In the new variables (t, s) the stencil becomes orthogonal so that there is no need to consider a nonuniform grid in the variable s. There are following invariants for this set of operators in the space $(t, \tau, s, h_s, x, h_x^+, h_x^-, \hat{h}_x^+, \hat{h}_x^-, \Delta x, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ corresponding to the orthogonal stencil in (t, s) extended by additional dependent variable x:

$$u^\sigma \frac{\tau}{h_x^{+2}}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}, \quad \frac{h_x^-}{h_x^+}, \quad \frac{\hat{h}_x^-}{h_x^+}, \quad \frac{\hat{h}_x^+}{h_x^+}, \quad \frac{\Delta x}{h_x^+}, \quad \frac{h_s}{h_x^+}.$$

By means of these invariants we get an approximation of (5.9) which has the form of a conservation law

$$\begin{aligned} \frac{1}{\tau}\left(\frac{1}{\hat{u}} - \frac{1}{u}\right) = &-\frac{\alpha}{\sigma+1}\left(\frac{u_+^{\sigma+1} - 2u^{\sigma+1} + u_-^{\sigma+1}}{h_s^2}\right) \\ &- \frac{1-\alpha}{\sigma+1}\left(\frac{\hat{u}_+^{\sigma+1} - 2\hat{u}^{\sigma+1} + \hat{u}_-^{\sigma+1}}{h_s^2}\right), \end{aligned} \tag{5.12}$$

where $0 \le \alpha \le 1$. Note that in the coordinates (t, s) variable x is introduced by the system (5.10) as a potential for the equation (5.9). Similarly we can introduce x as a discrete potential with the help of the system

$$\begin{gathered} \frac{\Delta x}{\tau} = -\frac{\alpha}{\sigma+1}\left(\frac{u_+^{\sigma+1} - u_-^{\sigma+1}}{2h_s}\right) - \frac{1-\alpha}{\sigma+1}\left(\frac{\hat{u}_+^{\sigma+1} - \hat{u}_-^{\sigma+1}}{2h_s}\right). \\ \frac{h_x^+}{h_s} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u_+}\right). \end{gathered} \tag{5.13}$$

In computations only the equation (5.12) and the first equation of (5.13) are needed. The second equation of the system (5.13) is needed only to establish the connection between solutions $u(x)$ and $u(s)$ for a fixed time. Given some initial data $u(0, x) = u_0(x)$, we choose an appropriate steplength h_s for the Lagrangian mass coordinate s. Then we can introduce the mesh points x_i in the original coordinates satisfying

$$\frac{x_{i+1} - x_i}{h_s} = \frac{1}{2}\left(\frac{1}{u_0(x_i)} + \frac{1}{u_0(x_{i+1})}\right),$$

i.e., we use this equation to establish difference relation between the original space coordinate x and the Lagrangian mass coordinate s. Computing the solution with the help of the numerical scheme (5.12) and the first equation of (5.13), we preserve the relation

$$\frac{x_{i+1}-x_i}{h_s} = \frac{1}{2}\left(\frac{1}{u_i} + \frac{1}{u_{i+1}}\right).$$

Introducing the mass type variable s, we can rewrite the conservation laws (5.7) as

$$\frac{\partial}{\partial t}\int_{s1}^{s2} ds = 0, \qquad \frac{\partial}{\partial t}\int_{s1}^{s2} x\,ds = -\left.\frac{u^{\sigma+1}}{\sigma+1}\right|_{s_1}^{s_2}.$$

The proposed discrete model possesses difference analogs of these conservation laws

$$\sum_{i=1}^{N-1} h_s = \text{const},$$

$$\sum_{i=1}^{N-1}\frac{\hat{x}_i+\hat{x}_{i+1}}{2}\,h_s - \sum_{i=1}^{N-1}\frac{x_i+x_{i+1}}{2}\,h_s = -\frac{\alpha}{\sigma+1}\left(\frac{u_{N+1}^{\sigma+1}+u_N^{\sigma+1}}{2}\right)$$
$$-\frac{1-\alpha}{\sigma+1}\left(\frac{\hat{u}_{N+1}^{\sigma+1}+\hat{u}_N^{\sigma+1}}{2}\right) + \frac{\alpha}{\sigma+1}\left(\frac{u_{-1}^{\sigma+1}+u_0^{\sigma+1}}{2}\right) + \frac{1-\alpha}{\sigma+1}\left(\frac{\hat{u}_{-1}^{\sigma+1}+\hat{u}_0^{\sigma+1}}{2}\right).$$

Let us mention that for computations we need to propose some method for the boundary points.

2. $Q=\delta u$, $\delta=\pm 1$. In this case the symmetry of the equation

$$u_t = (u^\sigma u_x)_x + \delta u \tag{5.14}$$

is described by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u},$$
$$X_4 = e^{-\delta\sigma t}\frac{\partial}{\partial t} + \delta e^{-\delta\sigma t}u\frac{\partial}{\partial u}. \tag{5.15}$$

By the change of variables

$$\bar{u} = ue^{-\delta t}, \qquad \bar{t} = \frac{\delta}{\sigma}(e^{\delta\sigma t}-1) \tag{5.16}$$

the equation (5.14) is transformed into the equation (5.1).

The finite-difference invariants

$$\frac{u^\sigma(e^{\delta\sigma\tau}-1)}{h^2}, \quad \left(\delta\ln\frac{\hat{u}}{u}-\tau\right), \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}$$

give the following possibility for an explicit difference model

$$\frac{\sigma u}{e^{\delta\sigma\tau}-1}\left(\delta\ln\frac{\hat{u}}{u}-\tau\right) = \frac{1}{h}\left(\left(\frac{u_++u}{2}\right)^\sigma \underset{h}{u}_x - \left(\frac{u+u_-}{2}\right)^\sigma \underset{h}{u}_{\bar{x}}\right). \tag{5.17}$$

Let us remark that the change of variables (5.16) transforms this equation considered on the orthogonal mesh with time layers

$$t_n = \frac{\delta}{\sigma} \ln \left(1 + \frac{n}{k}(e^{\delta\sigma T} - 1)\right), \qquad n = 0, \ldots, k, \tag{5.18}$$

which fill the time interval $[0, T]$, into the equation (5.3) on the uniform grid on the time interval $\left[0, \frac{\delta}{\sigma}(e^{\delta\sigma T} - 1)\right]$.

3. $Q = \pm u^{\sigma+1} + \delta u^n$, $\delta = \pm 1$. The equation

$$u_t = (u^\sigma u_x)_x \pm u^n, \qquad \sigma, n = \text{const}, \tag{5.19}$$

admits a three-parameter symmetry group. A possible representation of this group is by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2(n-1)t\frac{\partial}{\partial t} + (n - \sigma - 1)x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{5.20}$$

The set (5.20) satisfies all the conditions (2.4)–(2.6). So we can use an orthogonal grid that is uniform in both t and x directions. By considering the set of the operators (5.20) in the space $(t, \hat{t}, x, h^+, h^-, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ that corresponds to the stencil shown in Fig. 1 we find 7 difference invariants of the Lie algebra:

$$\frac{\tau^{\frac{n-\sigma-1}{2(n-1)}}}{h}, \quad \tau u^{n-1}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}.$$

The small number of symmetry operators provides us with a large number of difference invariants. Thus we are left with some additional degrees of freedom in invariant difference modeling of (5.19). By means of the discrete invariants we obtain the following explicit scheme:

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^\sigma \underset{h}{u}_x - \left(\frac{u + u_-}{2}\right)^\sigma \underset{h}{u}_{\bar{x}}\right) \pm u^n, \tag{5.21}$$

where $\underset{h}{u}_x = \frac{u_+ - u}{h}$, $\underset{h}{u}_{\bar{x}} = \frac{u - u_-}{h}$.

4. $Q = \pm u^{\sigma+1} + \delta u$, $\delta = \pm 1$. The equation

$$u_t = (u^\sigma u_x)_x \pm u^{\sigma+1} + \delta u \tag{5.22}$$

is connected with the equation (5.19) by the transformation (5.16). The infinitesimal operators admitted by the equation are

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{-\delta\sigma t}\frac{\partial}{\partial t} + \delta e^{-\delta\sigma t} u\frac{\partial}{\partial u}. \tag{5.23}$$

With the help of invariants for the operators (5.23)

$$u^\sigma(e^{\delta\sigma\tau} - 1), \quad h, \quad \left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}$$

we have the following example of an explicit difference model:

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^\sigma \underset{h}{u}_x - \left(\frac{u + u_-}{2}\right)^\sigma \underset{h}{u}_{\bar{x}}\right) \pm u^{\sigma+1}. \tag{5.24}$$

This equation considered on the grid (5.18) is transformed into the equation (5.21) on a uniform time grid by the variable change (5.16).

6 The special case of power heat transfer coefficient: $K = u^{-4/3}$

1. If $Q \equiv 0$, then the symmetry of the equation

$$u_t = \left(u^{-4/3}u_x\right)_x \tag{6.1}$$

is described by the five-dimensional algebra of infinitesimal operators (see [17]):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},$$
$$X_4 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \qquad X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}. \tag{6.2}$$

These operators conserve orthogonality and uniformity of a grid in the time direction. The operator X_5 conserve uniformity in the t-direction, but does not conserve uniformity of the grid in the x-direction; however orthogonality is not disturbed. We will consider the stencil shown in Fig. 4.

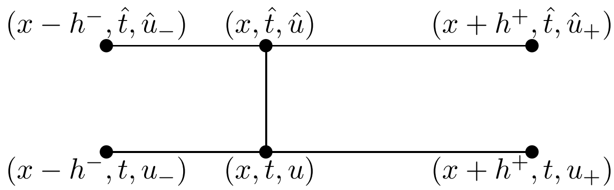

Figure 4. The stencil of nonuniform mesh.

The finite-difference invariants corresponding to this stencil

$$\frac{\hat{u}}{u}, \quad \frac{\hat{u}_+}{u_+}, \quad \frac{\hat{u}_-}{u_-}, \quad u_+^{1/3}u^{1/3}\frac{h^+}{\sqrt{\tau}}, \quad u_-^{1/3}u^{1/3}\frac{h^-}{\sqrt{\tau}}, \quad \frac{u^{2/3}}{\sqrt{\tau}}\left(\frac{h^+h^-}{h^++h^-}\right)$$

give among others the explicit difference model

$$\frac{\hat{u}-u}{\tau} = -\frac{h^++h^-}{6h^+h^-}\left(\frac{u_+^{-1/3}-u^{-1/3}}{h^+} - \frac{u^{-1/3}-u_-^{-1/3}}{h^-}\right). \tag{6.3}$$

Remark. Let us note that we can not propose a space mesh $h_+ = f(x, h_-)$ which is preserved under all transformations of the group (6.2). It can be clearly seen from the absence of difference invariants in the space (x, h_-, h_+). For example, if we take a solution of the difference scheme (6.3) on a regular mesh $h_- = h_+$, a general group transformation corresponding to (6.2) will transform the solution into another solution of this difference scheme but possibly on a nonuniform mesh. This remark is also valid for the cases 2, 4 and 5 of this section.

2. In the case $Q = \delta u$, $\delta = \pm 1$ equation

$$u_t = \left(u^{-4/3}u_x\right)_x + \delta u \tag{6.4}$$

admits operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u},$$
$$X_4 = e^{\frac{4\delta t}{3}}\frac{\partial}{\partial t} + \delta e^{\frac{4\delta t}{3}} u\frac{\partial}{\partial u}, \qquad X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}. \tag{6.5}$$

By the change of variables (5.16) this equation can be transformed into the equation (6.1). Let us write out the difference invariants for the set of the operators (6.5):

$$\left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad u^{2/3}\left(\frac{h^+h^-}{h^+ + h^-}\right)\frac{1}{\sqrt{(e^{\delta\sigma\tau} - 1)}},$$
$$\frac{u_+^{1/3}u^{1/3}h^+}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{u_-^{1/3}u^{1/3}h^-}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{\hat{u}_+^{1/3}\hat{u}^{1/3}h^+}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{\hat{u}_-^{1/3}\hat{u}^{1/3}h^-}{\sqrt{(e^{\delta\sigma\tau} - 1)}}.$$

These invariants can be used for construction of a difference model for the equation (6.4). We show the explicit variant of the difference model:

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) = -\frac{h^+ + h^-}{6h^+h^-}\left(\frac{u_+^{-1/3} - u^{-1/3}}{h^+} - \frac{u^{-1/3} - u_-^{-1/3}}{h^-}\right). \tag{6.6}$$

3. $Q = \pm u^n$, $n \neq -\frac{1}{3}$. The equation

$$u_t = \left(u^{-4/3}u_x\right)_x \pm u^n \tag{6.7}$$

admits infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2(n-1)t\frac{\partial}{\partial t} + (n + \frac{1}{3})x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{6.8}$$

Although this equation is specified in group classification (see [4]), it is a particular case of the equation (5.19), — there is no extension of the admitted group. That's why as an invariant difference model for the equation (6.7) we can use the model (5.21) with parameter $\sigma = -\frac{4}{3}$, corresponding to the given equation.

4. If $Q = \alpha u^{-1/3}$, $\alpha = \pm 1$, then the variant of the difference model for the equation

$$u_t = \left(u^{-4/3}u_x\right)_x \pm u^{-1/3} \tag{6.9}$$

depends on the sign of the coefficient α. The equation (6.9) admits a five-dimensional algebra of infinitesimal operators, namely

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{4}{3}t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},$$
$$X_4 = e^{2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} - \sqrt{3\alpha}e^{2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u},$$
$$X_5 = e^{-2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} + \sqrt{3\alpha}e^{-2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}. \tag{6.10}$$

a.) The case $\alpha = 1$. By the change of variables

$$\bar{u} = u\cosh^3\left(\frac{x}{\sqrt{3}}\right), \qquad \bar{x} = \sqrt{3}\tanh\left(\frac{x}{\sqrt{3}}\right) \tag{6.11}$$

we transfer the considered equation into the equation (6.1) (see [1]). With the help of difference invariants

$$\frac{\hat{u}}{u},\quad \frac{\hat{u}_+}{u_+},\quad \frac{\hat{u}_-}{u_-},\quad \sqrt{\tau}u^{-2/3}\left(\frac{1}{\tanh\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{\tanh\left(\frac{h^-}{\sqrt{3}}\right)}\right),$$
$$\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}\sinh\left(\frac{h^+}{\sqrt{3}}\right),\quad \frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}\sinh\left(\frac{h^-}{\sqrt{3}}\right)$$

we can construct a difference model. Let us write out one of the possible variants of the difference model, namely an explicit model:

$$\frac{\hat{u}-u}{\tau}=-\frac{1}{18}\left(\frac{1}{\tanh\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{\tanh\left(\frac{h^-}{\sqrt{3}}\right)}\right)$$
$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}\cosh\left(\frac{h^+}{\sqrt{3}}\right)}{\sinh\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}\cosh\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{\sinh\left(\frac{h^-}{\sqrt{3}}\right)}\right). \tag{6.12}$$

b.) The case $\alpha=-1$. By the change of variables

$$\bar{u}=u\cos^3\left(\frac{x}{\sqrt{3}}\right),\qquad \bar{x}=\sqrt{3}\tan\left(\frac{x}{\sqrt{3}}\right) \tag{6.13}$$

we can transfer the given equation into the equation (6.1) (see [1]). The set of finite-difference invariants:

$$\frac{\hat{u}}{u},\quad \frac{\hat{u}_+}{u_+},\quad \frac{\hat{u}_-}{u_-},\quad \sqrt{\tau}u^{-2/3}\left(\frac{1}{\tan\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{\tan\left(\frac{h^-}{\sqrt{3}}\right)}\right),$$
$$\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}\sin\left(\frac{h^+}{\sqrt{3}}\right),\quad \frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}\sin\left(\frac{h^-}{\sqrt{3}}\right)$$

provides us with a possibility to construct an invariant difference scheme. For example, one can use an explicit difference model:

$$\frac{\hat{u}-u}{\tau}=-\frac{1}{18}\left(\frac{1}{\tan\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{\tan\left(\frac{h^-}{\sqrt{3}}\right)}\right)$$
$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}\cos\left(\frac{h^+}{\sqrt{3}}\right)}{\sin\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}\cos\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{\sin\left(\frac{h^-}{\sqrt{3}}\right)}\right). \tag{6.14}$$

We stress that the obtained difference models (6.12) and (6.14) are connected with the difference model (6.3) for the equation (6.1) by the changes of variables (6.11) and (6.13) correspondingly as the initial differential equations.

5. $Q=\alpha u^{-1/3}+\delta u$, $|\alpha|=|\delta|=1$. As in the previous point, two cases of parameter α in the equation

$$u_t=(u^\sigma u_x)_x\pm u^{\sigma+1}+\delta u \tag{6.15}$$

should be considered separately and two difference models should be constructed. Let us write out the infinitesimal operators, admitted by the equation (6.15):

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{\frac{4\delta t}{3}}\frac{\partial}{\partial t} + \delta e^{\frac{4\delta t}{3}} u\frac{\partial}{\partial u},\\
&X_4 = e^{2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} - \sqrt{3\alpha}e^{2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u},\\
&X_5 = e^{-2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} + \sqrt{3\alpha}e^{-2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}.
\end{aligned} \tag{6.16}$$

a.) The case $\alpha = 1$. The change of variables (5.16) transforms the considered equation into the equation (6.9) and the change (6.11) into the equation (6.1).

We write out the set of finite-difference invariants for the equation (6.15) with $\alpha = 1$:

$$\begin{aligned}
&\left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad \sqrt{(e^{\delta\sigma\tau}-1)}u^{-2/3}\left(\frac{1}{\tanh\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{\tanh\left(\frac{h^-}{\sqrt{3}}\right)}\right),\\
&\frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sinh\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sinh\left(\frac{h^-}{\sqrt{3}}\right),\\
&\frac{\hat{u}^{1/3}\hat{u}_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sinh\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{\hat{u}^{1/3}\hat{u}_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sinh\left(\frac{h^-}{\sqrt{3}}\right).
\end{aligned}$$

The explicit variant of the difference model for the equation (6.15) on the time grid (5.18) has the form:

$$\begin{aligned}
\frac{\sigma u}{e^{\delta\sigma\tau}-1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) &= -\frac{1}{18}\left(\frac{1}{\tanh\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{\tanh\left(\frac{h^-}{\sqrt{3}}\right)}\right)\\
&\times\left(\frac{u_+^{-1/3} - u^{-1/3}\cosh\left(\frac{h^+}{\sqrt{3}}\right)}{\sinh\left(\frac{h^+}{\sqrt{3}}\right)} - \frac{u^{-1/3}\cosh\left(\frac{h^-}{\sqrt{3}}\right) - u_-^{-1/3}}{\sinh\left(\frac{h^-}{\sqrt{3}}\right)}\right).
\end{aligned} \tag{6.17}$$

b.) The case $\alpha = -1$ Using the change of variables (6.13) we can transfer this equation into the equation (6.4) and by the change (5.16) into the equation (6.9). Difference model for the equation (6.15) can be obtained with the help of the invariants

$$\begin{aligned}
&\left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad \sqrt{(e^{\delta\sigma\tau}-1)}u^{-2/3}\left(\frac{1}{\tan\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{\tan\left(\frac{h^-}{\sqrt{3}}\right)}\right),\\
&\frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^-}{\sqrt{3}}\right),\\
&\frac{\hat{u}^{1/3}\hat{u}_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{\hat{u}^{1/3}\hat{u}_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^-}{\sqrt{3}}\right).
\end{aligned}$$

One of possible difference models for the equation (6.15) is

$$\frac{\sigma u}{e^{\delta\sigma\tau}-1}\left(\delta\ln\frac{\hat{u}}{u}-\tau\right)=-\frac{1}{18}\left(\frac{1}{\tan\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{\tan\left(\frac{h^-}{\sqrt{3}}\right)}\right)$$
$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}\cos\left(\frac{h^+}{\sqrt{3}}\right)}{\sin\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}\cos\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{\sin\left(\frac{h^-}{\sqrt{3}}\right)}.\right). \tag{6.18}$$

The difference models (6.17) and (6.18) are connected with the model (6.6) by changes of variables (6.11) and (6.13) correspondingly. The variable change (5.16) transforms the difference models obtained in this point into model of the point 4 for corresponding values of the parameter α. This example shows that in invariant difference modeling it is possible to get consistent models which are connected with each other by the same point transformations as their original differential counterparts.

7 Linear heat conductivity with a source

In this section we consider the semilinear heat transfer equation

$$u_t=u_{xx}+Q(u) \tag{7.1}$$

with different types of a source.

1. With $Q=\pm e^u$ the equation becomes

$$u_t=u_{xx}\pm e^u. \tag{7.2}$$

It admits a three-dimensional algebra of infinitesimal operators

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2\frac{\partial}{\partial u}. \tag{7.3}$$

It is easy to check that for the operators the conditions of orthogonality and uniformity conservation of a grid hold. Approximation of the equation by the invariants

$$\frac{h^2}{\tau},\quad \tau e^u,\quad (\hat{u}-u),\quad (u_+-u),\quad (u-u_-),\quad (\hat{u}_+-\hat{u}),\quad (\hat{u}-\hat{u}_-)$$

will give different types of difference models. An explicit one is

$$\frac{\hat{u}-u}{\tau}=\frac{1}{h}(\underset{h}{u}_x-\underset{h}{u}_{\bar{x}})\pm e^u. \tag{7.4}$$

2. $Q=\pm u^n$. The equation

$$u_t=u_{xx}\pm u^n \tag{7.5}$$

admits the following infinitesimal operators:

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=2(n-1)t\frac{\partial}{\partial t}+(n-1)x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u}. \tag{7.6}$$

The operators satisfy to conditions (2.4)–(2.6) and the finite-difference invariants

$$\frac{h^2}{\tau}, \quad \tau u^{n-1}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}$$

permit us construct, for example, the following difference scheme:

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}(\underset{h}{u}_x - \underset{h}{u}_{\bar{x}}) \pm u^n. \tag{7.7}$$

3. $Q = \delta u \ln u$, $\delta = \pm 1$. The semilinear heat transfer equation

$$u_t = u_{xx} + \delta u \ln u, \qquad \delta = \pm 1 \tag{7.8}$$

admits the four-parameter Lie symmetry group of point transformations [4] corresponding to the following set of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2e^{\delta t}\frac{\partial}{\partial x} - \delta e^{\delta t} x u \frac{\partial}{\partial u}, \qquad X_4 = e^{\delta t} u \frac{\partial}{\partial u}. \tag{7.9}$$

Before constructing a difference equation and a grid that approximate (7.8) and inherit the whole Lie algebra (7.9) we should first check condition (2.6) for the invariance of orthogonality. The operators X_1, X_2 and X_4 conserve orthogonality, while X_3 does not: the condition (2.6) is not true for operator X_3. Consequently an orthogonal mesh can not be used for the invariant modeling of (7.8). Conditions (2.7) are true for the complete set of operators, so it is possible to use a nonorthogonal grid with flat time layers and we will use the grid shown in Fig. 2.

A possible reformulation of equation (7.8) by using the four differential invariants in the subspace $(t, x, u, u_x, u_{xx}, dt, dx, du)$:

$$J_1 = dt, \qquad J_2 = \left(\frac{u_x}{u}\right)^2 - \frac{u_{xx}}{u}, \qquad J_3 = 2\frac{u_x}{u} + \frac{dx}{dt},$$
$$J_4 = \frac{du}{udt} - \delta \ln u + \frac{1}{4}\left(\frac{dx}{dt}\right)^2$$

is given by the system

$$J_3 = 0, \qquad J_4 = J_2$$

that is

$$\frac{dx}{dt} = -2\frac{u_x}{u}, \qquad \frac{du}{dt} = u_{xx} + \delta u \ln u - 2\frac{u_x^2}{u}. \tag{7.10}$$

So, the structure of the admitted group suggests the use of two evolution equations.

As the next step, we will find difference invariants for the set X_1–X_4 of the group (7.9). These invariants are necessary for the approximation of system (7.10). We will use the six-point difference stencil of Fig. 3 on which we will approximate system (7.10). The stencil defines the difference subspace $(t,\hat{t},x,\hat{x},h^+,h^-,\hat{h}^+,\hat{h}^-,u,u_+,u_-,\hat{u},\hat{u}_+,\hat{u}_-)$ and the group (7.9) has the following difference invariants

$$I_1=\tau,\quad I_2=h^+,\quad I_3=h^-,\quad I_4=\hat{h}^+,\quad I_5=\hat{h}^-,$$
$$I_6=(\ln u)_x-(\ln u)_{\bar{x}},\quad I_7=(\ln\hat{u})_x-(\ln\hat{u})_{\bar{x}},$$
$$I_8=\delta\Delta x+2(e^{\delta\tau}-1)\left(\frac{h^-}{h^++h^-}(\ln u)_x+\frac{h^+}{h^++h^-}(\ln u)_{\bar{x}}\right),$$
$$I_9=\delta\Delta x+2(1-e^{-\delta\tau})\left(\frac{\hat{h}^-}{\hat{h}^++\hat{h}^-}(\ln\hat{u})_x+\frac{\hat{h}^+}{\hat{h}^++\hat{h}^-}(\ln\hat{u})_{\bar{x}}\right),$$
$$I_{10}=\delta(\Delta x)^2+4(1-e^{-\delta\tau})\left(\ln\hat{u}-e^{\delta\tau}\ln u\right),$$

where $\Delta x=\hat{x}-x$, $(\ln u)_x=\frac{\ln u_+-\ln u}{h^+}$, $(\ln u)_{\bar{x}}=\frac{\ln u-\ln u_-}{h^-}$.

An explicit model can be chosen

$$I_8=0,\qquad I_{10}=\frac{8}{\delta}\frac{(e^{\delta I_1}-1)^2}{I_2+I_3}I_6,$$

i.e.

$$\delta\Delta x+2(e^{\delta\tau}-1)\left(\frac{h^-}{h^++h^-}(\ln u)_x+\frac{h^+}{h^++h^-}(\ln u)_{\bar{x}}\right)=0,$$
$$\delta(\Delta x)^2+4(1-e^{-\delta\tau})\left(\ln\hat{u}-e^{\delta\tau}\ln u\right)=\frac{8}{\delta}\frac{(e^{\delta\tau}-1)^2}{h^++h^-}[(\ln u)_x-(\ln u)_{\bar{x}}].\tag{7.11}$$

One invariant solution of this scheme is given in [2].

4. A linear heat equation without a source ($Q=0$).

4.1. Preliminary consideration. The linear heat transfer equation

$$u_t=u_{xx}\tag{7.12}$$

admits a six-parameter Lie symmetry group of point transformations, corresponding to the infinitesimal operators

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=2t\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u},\qquad X_4=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},$$
$$X_5=4t^2\frac{\partial}{\partial t}+4tx\frac{\partial}{\partial x}-(x^2+2t)u\frac{\partial}{\partial u},\qquad X_6=u\frac{\partial}{\partial u}\tag{7.13}$$

and an infinite-dimensional symmetry

$$X^*=a(x,t)\frac{\partial}{\partial u},$$

where $a(t,x)$ in an arbitrary solution of equation (7.12). Symmetry X^* represents linearity of the equation (7.12).

Probably, the simplest approximation of the linear equation is the explicit scheme

$$\frac{\hat{u}-u}{\tau}=\frac{u_+-2u+u_-}{h^2}\tag{7.14}$$

considered on a uniform orthogonal mesh. This equation is invariant with respect to the operators X_1, X_2, X_4 and X_6 of the set (7.13). Since the equation is linear it possesses a superposition principle that is reflected in the invariance with respect to the operator

$$X_h^* = a_h(x,t)\frac{\partial}{\partial u},$$

where $a_h(x,t)$ is an arbitrary solution of equation (7.14). In [2] it was shown how to construct a discrete model which admits the six-dimensional group (7.13).

To preserve the Galilean operator X_3 and the projective operator X_5 it is necessary to introduce a moving mesh.

4.2. Heat transfer system of equations and superposition principle. With the help of the differential invariants of the operators (7.13) in the space $(t, x, u, u_x, u_{xx}, dt, dx, du)$

$$J_1 = \frac{dx + 2\frac{u_x}{u}dt}{dt^{1/2}}, \qquad J_2 = \frac{du}{u} + \frac{1}{4}\frac{dx}{dt}^2 + \left(-\frac{u_{xx}}{u} + \frac{u_x^2}{u^2}\right) dt$$

we can represent the heat equation (7.12) as the system

$$J_1 = 0, \qquad J_2 = 0$$

that is

$$\frac{dx}{dt} = -2\frac{u_x}{u}, \qquad \frac{du}{dt} = u_{xx} - 2\frac{u_x^2}{u}. \tag{7.15}$$

By construction this system is invariant with respect to the six-dimensional group generated by the operators (7.13). The system also inherits the superposition principle of the linear heat transfer equation. The superposition principle has the form of summing two solutions of the system (7.12), but it also acts on the trajectories on which the variable x evolves. For two arbitrary solutions $U_1(t,x)$ with the trajectories $X_1(t)$:

$$\frac{dX_1}{dt} = -\frac{2U_{1x}}{U_1},$$

and $U_2(t,x)$ with the trajectories $X_2(t)$:

$$\frac{dX_2}{dt} = -\frac{2U_{2x}}{U_2},$$

their linear combination

$$U = \alpha U_1 + \beta U_2, \qquad \alpha, \beta = \text{const}, \tag{7.16}$$

is also the solution of (7.15). However, this linear combination has its own trajectories satisfying

$$\frac{dX}{dt} = \frac{\alpha U_1}{\alpha U_1 + \beta U_2}\frac{dX_1}{dt} + \frac{\beta U_2}{\alpha U_1 + \beta U_2}\frac{dX_2}{dt}. \tag{7.17}$$

Therefore, the superposition principle can be presented in the following form:

$$\begin{pmatrix} U(t,x) \\ \dfrac{dX}{dt} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \dfrac{\alpha U_1}{\alpha U_1 + \beta U_2} \end{pmatrix} \begin{pmatrix} U_1(t,x) \\ \dfrac{dX_1}{dt} \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \dfrac{\beta U_2}{\alpha U_1 + \beta U_2} \end{pmatrix} \begin{pmatrix} U_2(t,x) \\ \dfrac{dX_2}{dt} \end{pmatrix}. \tag{7.18}$$

Let us show the superposition principle for system (7.15) by an example. The solution

$$U_1 = \frac{1}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)} \right)$$

has the trajectories

$$x = a + (x_0 - a)\frac{t+t_1}{t_1},$$

while the solution

$$U_2 = \frac{1}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)} \right)$$

exists on the trajectories

$$x = b + (x_0 - b)\frac{t+t_2}{t_2}.$$

The linear combination (7.16) of these two solutions is also the solution of system (7.15). Its trajectories are

$$\frac{dX}{dt} = \frac{\frac{\alpha}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)}\right)\left(\frac{x-a}{t+t_1}\right) + \frac{\beta}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)}\right)\left(\frac{x-b}{t+t_2}\right)}{\frac{\alpha}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)}\right) + \frac{\beta}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)}\right)}.$$

Examples of evolution of grid points and corresponding solutions are shown on Figs. 5–8 (for computations we used discrete model (7.19) which will be introduced in point 4.3 of this section).

If we consider Lagrangian derivatives of the solution U, we could get the superposition principle

$$\left(\frac{dU}{dt} - U_x\frac{dX}{dt}\right) = \alpha\left(\frac{dU_1}{dt} - U_{1x}\frac{dX_1}{dt}\right) + \beta\left(\frac{dU_2}{dt} - U_{2x}\frac{dX_2}{dt}\right),$$

i.e. the superposition principle $U_t = \alpha U_{1t} + \beta U_{2t}$ of the linear heat equation expressed in terms of the of total derivatives of U and X.

4.3. Invariant schemes on moving meshes. For the difference modeling of system (7.15) we need the whole set of difference invariants of Lie symmetry group (7.13) in

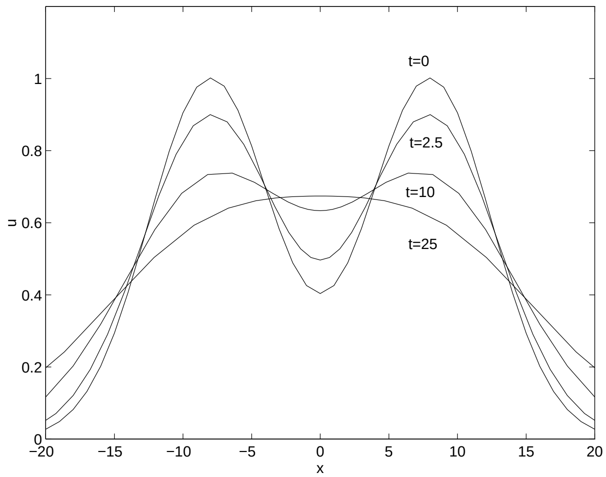

Figure 5. Evolution of solution (7.16): $\alpha = \beta = 1$, $t_1 = t_2 = 10$, $a = -8$, $b = 8$.

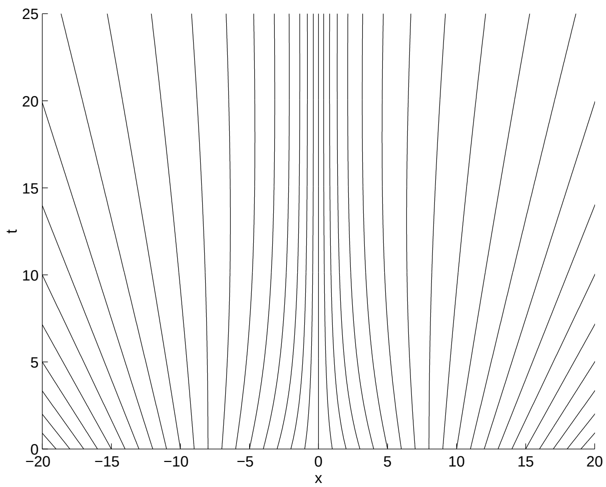

Figure 6. Mesh trajectories for the solution shown in Fig. 5.

the difference space corresponding to the chosen stencil $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, u, \hat{u}, u_+, u_-, \hat{u}_+, \hat{u}_-)$:

$$I_1 = \frac{h^+}{h^-}, \quad I_2 = \frac{\hat{h}^+}{\hat{h}^-}, \quad I_3 = \frac{\hat{h}^+ h^+}{\tau}, \quad I_4 = \frac{\tau^{1/2}}{h^+}\frac{\hat{u}}{u}\exp\left(\frac{1}{4}\frac{(\Delta x)^2}{\tau}\right),$$

$$I_5 = \frac{1}{4}\frac{h^{+2}}{\tau} - \frac{h^{+2}}{h^+ + h^-}\left(\frac{1}{h^+}\ln\frac{u_+}{u} + \frac{1}{h^-}\ln\frac{u_-}{u}\right),$$

$$I_6 = \frac{1}{4}\frac{\hat{h}^{+2}}{\tau} + \frac{\hat{h}^{+2}}{\hat{h}^+ + \hat{h}^-}\left(\frac{1}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{1}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),$$

$$I_7 = \frac{\Delta x h^+}{\tau} + \frac{2h^+}{h^+ + h^-}\left(\frac{h^-}{h^+}\ln\frac{u_+}{u} - \frac{h^+}{h^-}\ln\frac{u_-}{u}\right),$$

$$I_8 = \frac{\Delta x \hat{h}^+}{\tau} + \frac{2\hat{h}^+}{\hat{h}^+ + \hat{h}^-}\left(\frac{\hat{h}^-}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} - \frac{\hat{h}^+}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right).$$

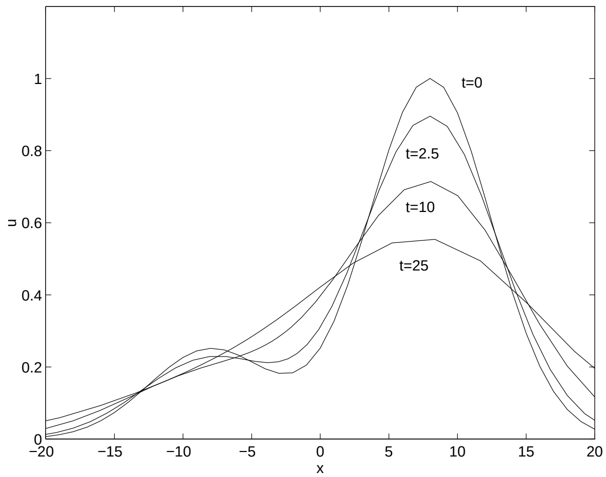

Figure 7. Evolution of solution (7.16): $\alpha = 0.25$, $\beta = 1$, $t_1 = t_2 = 10$, $a = -8$, $b = 8$.

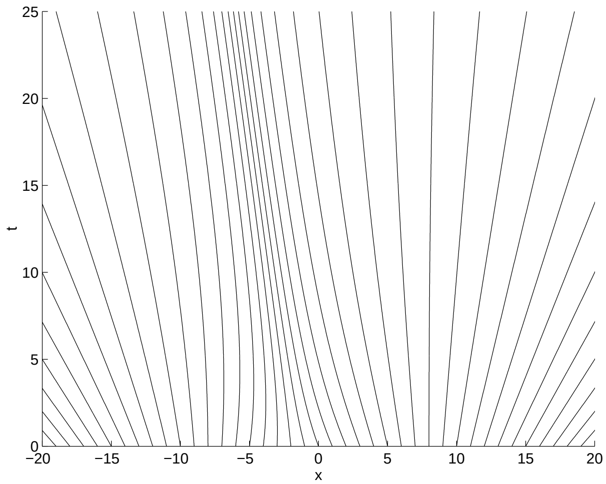

Figure 8. Mesh trajectories for the solution shown in Fig. 7.

Approximating system (7.15) by the invariants, we obtain a system of difference evolution equations. As an example, we present here an invariant difference model which has explicit equations for the solution u and the trajectory of x:

$$\Delta x = \frac{2\tau}{h^+ + h^-}\left(-\frac{h^-}{h^+}\ln\frac{u_+}{u} + \frac{h^+}{h^-}\ln\frac{u_-}{u}\right),$$
$$\left(\frac{u}{\hat{u}}\right)^2 \exp\left(-\frac{1}{2}\frac{(\Delta x)^2}{\tau}\right) = 1 - \frac{4\tau}{h^+ + h^-}\left(\frac{1}{h^+}\ln\frac{u_+}{u} + \frac{1}{h^-}\ln\frac{u_-}{u}\right). \tag{7.19}$$

We also can write out an implicit model

$$\Delta x = \frac{2\tau}{\hat{h}^+ + \hat{h}^-}\left(-\frac{\hat{h}^-}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{\hat{h}^+}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),$$
$$\left(\frac{\hat{u}}{u}\right)^2 \exp\left(\frac{1}{2}\frac{\Delta x^2}{\tau}\right) = 1 + \frac{4\tau}{\hat{h}^+ + \hat{h}^-}\left(\frac{1}{\hat{h}^+}\ln\frac{\hat{u}_-}{\hat{u}} + \frac{1}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right).$$

It is also possible to combine an explicit equation for the mesh and an implicit approximation of the PDE or vice versa. Other ways to approximate the system (7.15) are also possible.

4.4. Optimal system of subalgebras and reduced systems. Among all invariant solutions there is a minimal set of such solutions, called the optimal system of invariant solutions. From this set of invariant solutions any invariant solution can be obtained by an appropriate group transformation. The difference model (7.19) is a system of two evolution equations. To find its invariant solutions we need to provide a time mesh which is invariant with respect to the considered operator. An invariant time mesh giving flat time layers can be represented by the equation

$$\tau_i = g(t_i), \qquad i = 0, 1, 2, \dots . \tag{7.20}$$

We request this equation to be invariant with respect to the considered symmetry. Since for the operators (7.13) the coefficients ξ^t do not depend on x and u we can propose an invariant time mesh for any symmetry. In the case $\xi^t = 0$ the function g can be taken arbitrary. For example, we can choose the uniform mesh $t_j = j\tau$, $\tau = \text{const}$. Thus, different invariant solutions may have different time meshes.

The adjoint action of the Lie group transforms an invariant solution into another one [15, 14]. In our case it also transforms the time mesh equation (7.20). Thus the adjoint action gives us a new invariant solution with a corresponding invariant mesh.

On the example of the difference model (7.19) we will construct the optimal system of solutions which are invariant with respect to one-parameter groups. The optimal system of one-dimensional subalgebras of the algebra of symmetries for the linear heat equation consists of algebras corresponding to the operators (see [14]):

$$\begin{aligned}
&Y_1 = X_2 = \frac{\partial}{\partial x}, \qquad Y_2 = X_6 = u\frac{\partial}{\partial u}, \qquad Y_3 = X_1 + cX_6 = \frac{\partial}{\partial t} + cu\frac{\partial}{\partial u},\\
&Y_4 = X_1 - X_3 = \frac{\partial}{\partial t} - 2t\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \qquad Y_5 = X_4 + 2cX_6 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2cu\frac{\partial}{\partial u},\\
&Y_6 = X_1 + X_5 + cX_6 = (4t^2+1)\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + (c - x^2 - 2t)u\frac{\partial}{\partial u}.
\end{aligned}$$

Let us find invariant solutions corresponding to these one-dimensional subalgebras.

1) The subalgebra corresponding to the operator Y_1 has only constant solutions $u = C$, $C = \text{const}$ considered on an orthogonal mesh $\Delta x = 0$.

2) The subalgebra corresponding to the operator Y_2 does not have invariant solutions (the necessary condition of existence of invariant solutions does not hold [15]).

3) The operator Y_3 has the following invariants: $u\exp(-ct)$, τ and Δx. The time step τ is invariant, so we can consider a uniform time mesh. We will seek a solution of the difference model in the form

$$u = \exp(ct) f(x).$$

Substituting this invariant form of the solution into system (7.19), we get:

$$\Delta x = \frac{2\tau}{h^+ + h^-}\left(-\frac{h^+}{h^-}\ln\left(\frac{f(x+h^+)}{f(x)}\right) + \frac{h^+}{h^-}\ln\left(\frac{f(x-h^-)}{f(x)}\right)\right),$$

$$\left(\frac{f(x)}{f(x+\Delta x)}\right)^2 \exp\left(-2c\tau - \frac{1}{2}\frac{\Delta x^2}{\tau}\right)$$
$$= 1 - \frac{4\tau}{h^+ + h^-}\left(\frac{1}{h^+}\ln\left(\frac{f(x+h^+)}{f(x)}\right) + \frac{1}{h^-}\ln\left(\frac{f(x-h^-)}{f(x)}\right)\right). \tag{7.21}$$

System (7.21) will become a system of two ordinary difference equations if we project it into the invariants space. To project the system we have to impose

$$\Delta x = -h^-, \ 0, \ \text{or } h^+. \tag{7.22}$$

A solution of system (7.21) with one of conditions (7.22) provides the solution of system (7.19) which is invariant with respect to the operator Y_3.

4) The operator Y_4 has invariants: $u\exp\left(-xt - \frac{2}{3}t^3\right)$, $x + t^2$, τ and $\frac{\Delta x}{2\tau} - t$. Let us search a solution of the difference model (7.19) in the form

$$u = \exp\left(tx + \frac{2}{3}t^3\right) f(x + t^2).$$

By means of variables

$$y = x + t^2, \qquad y - h_y^- = x - h^- + t^2,$$
$$y + h_y^+ = x + h^+ + t^2, \qquad y + \Delta y = x + \Delta x + (t+\tau)^2$$

we get the following system for the invariant solution of system (7.19):

$$\Delta y - \tau^2 = \frac{2\tau}{h_y^+ + h_y^-}\left(-\frac{h_y^-}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{h_y^+}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right),$$

$$\left(\frac{f(y)}{f(y+\Delta y)}\right)^2 \exp\left(-\frac{1}{2\tau}\Delta y^2 - \tau(2y+\Delta y) + \frac{1}{6}\tau^3\right)$$
$$= 1 - \frac{4\tau}{h_y^+ + h_y^-}\left(\frac{1}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{1}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right),$$

where Δy can have one of the following values

$$\Delta y = -h_y^-, \ 0 \text{ or } h_y^+. \tag{7.23}$$

A solution of the above system with one of conditions (7.23) let us find the invariant solution for the operator Y_4.

5) Expressions $\frac{x}{\sqrt{t}}$, $t^{-c}u$, $\frac{\tau}{t}$ and $\frac{\Delta x}{x}$ are invariants of the operator Y_5. Let us search a solution of the difference model in the form

$$u = t^c f\left(\frac{x}{\sqrt{t}}\right).$$

In variables

$$y = \frac{x}{\sqrt{t}}, \qquad y - h_y^- = \frac{x - h^-}{\sqrt{t}}, \qquad y + h_y^+ = \frac{x + h^+}{\sqrt{t}}, \qquad y + \Delta y = \frac{x + \Delta x}{\sqrt{t + \tau}}$$

we get the following system of equations:

$$\sqrt{1+a}(y + \Delta y) - y = \frac{2a}{h_y^+ + h_y^-}\left(-\frac{h_y^-}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{h_y^+}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right),$$

$$(1+a)^{-2c}\left(\frac{f(y)}{f(y+\Delta y)}\right)^2 \exp\left(-\frac{1}{2}\left((y+\Delta y)\sqrt{\frac{1+a}{a}} - y\frac{1}{\sqrt{a}}\right)^2\right)$$
$$= 1 - \frac{4a}{h_y^+ + h_y^-}\left(\frac{1}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{1}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right).$$

Here Δy can have one of the values determined by conditions (7.23) and a is a constant from the condition $a = \frac{\tau}{t}$ which determines an invariant time spacing. This condition can be found if we look for a time spacing $\tau = g(t)$ which is invariant with respect to the considered operator Y_5.

6) For the operator Y_6 we have the following invariants:

$$\frac{x}{\sqrt{4t^2+1}}, \qquad (4t^2+1)^{1/4} u \exp\left(\frac{tx^2}{4t^2+1} + \frac{c}{2}\arctan(2t)\right),$$
$$\frac{4t^2+1}{\tau} + 4t, \qquad \frac{\Delta x}{x}\frac{4t^2+1}{\tau} - 4t.$$

We look for a solution of the difference model in the form

$$u = (4t^2+1)^{-1/4}\exp\left(-\frac{tx^2}{4t^2+1} - \frac{c}{2}\arctan(2t)\right) f\left(\frac{x}{\sqrt{4t^2+1}}\right).$$

Involving new variables

$$y = \frac{x}{\sqrt{4t^2+1}}, \qquad y - h_y^- = \frac{x - h^-}{\sqrt{4t^2+1}},$$
$$y + h_y^+ = \frac{x + h^+}{\sqrt{4t^2+1}}, \qquad y + \Delta y = \frac{x + \Delta x}{\sqrt{4(t+\tau)^2+1}},$$

the system of equations (7.19) can be presented in the form:

$$\sqrt{b^2+1}(y+\Delta y) - by = \frac{1}{h_y^+ + h_y^-}\left(-\frac{h_y^-}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{h_y^+}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right),$$

$$\sqrt{b^2+1}\left(\frac{f(y)}{f(y+\Delta y)}\right)^2$$
$$\times \exp\left(c\arctan\left(\frac{1}{b}\right) - b(y^2 + (y+\Delta y)^2) + 2\sqrt{b^2+1}\,y(y+\Delta y)\right)$$
$$= b - \frac{2}{h_y^+ + h_y^-}\left(\frac{1}{h_y^+}\ln\left(\frac{f(y+h_y^+)}{f(y)}\right) + \frac{1}{h_y^-}\ln\left(\frac{f(y-h_y^-)}{f(y)}\right)\right).$$

where Δy has one of the values of (7.23), b is the constant from necessary condition of invariant grid existence

$$2b = 4t + \frac{4t^2+1}{\tau}.$$

Therefore, the obtained reduced systems of equations determine the optimal system of invariant solutions for the difference model of the liner heat transfer equation. It means that each invariant solution can be found by transformation of a solution from the optimal system with the help of the corresponding element of the group. As we mentioned before the invariant time mesh for the new solution is obtained from the time mesh of the solution from the optimal system with the help of the same group transformation. For example, the transformation corresponding to the operator X_1 with the value of the parameter $-t_0$ gives shift in time $\hat{t} = t - t_0$. Since

$$\mathrm{Ad}\,[\exp(-t_0 X_1)]Y_5 = X_* = Y_5 + 2t_0 X_1,$$

the action of this transformation transfers the invariant solution with respect to the operator Y_5 into the solution which is invariant with respect to the operator X_*. By this transformation the spacing $\frac{\tau}{t} = a$ is transformed into the spacing $\frac{\tau}{t+t_0} = a$.

Example of an exact solution. Among all group invariant solutions for the difference model (7.19) there is one interesting solution which can be integrated exactly [2]. This is the solution invariant with respect to the operator

$$2t_0 X_2 + X_3, \qquad t_0 = \mathrm{const}, \tag{7.24}$$

namely the solution

$$u(x,t) = C\left(\frac{t_0}{t+t_0}\right)^{1/2} \exp\left(-\frac{x^2}{4(t+t_0)}\right), \tag{7.25}$$

considered on the mesh

$$x_i = x_i^0\left(\frac{t+t_0}{t_0}\right), \tag{7.26}$$

where x_i^0 are space mesh point at $t = t_0$. In the case $t_0 = 0$ we get the well known fundamental solution of the linear heat equation. Note that it has a "singular" mesh.

Let us show how this solution can be obtained form the optimal system of the invariant solutions. From

$$\mathrm{Ad}\,[\exp(\varepsilon X_5)]Y_1 = X_2 + 2\varepsilon X_3$$

we see that the solution (7.25) can be obtained from the solution invariant with respect to operator Y_1 by the transformation $\mathrm{Ad}\,[\exp(\varepsilon X_5)]$ with $\varepsilon = \frac{1}{4t_0}$. If we take the original solution on the orthogonal mesh which is uniform in space and has the following special time spacing on the interval $[0, t_0]$:

$$u_i^j = C, \qquad x_i = ih, \qquad i = 0, \pm 1, \pm 2, \ldots, \qquad t_j = \frac{j\tau t_0}{t_0 + j\tau} i, \qquad j = 0, 1, 2, \ldots$$

then the proposed transformation provides us the solution (7.25) on the uniform space mesh (7.26) and uniform time mesh $t_j = j\tau$.

Thus, we see that difference model (7.19) inherits both the group admitted by the original differential equation and the ability to be integrated on a subgroup.

4.5. The way to stop a moving mesh. The obtained difference models have adaptive nonorthogonal grids. We can find a way to stop the moving mesh, i.e., an exchange of variables which orthogonalizes the mesh. The differentiation operator of Lagrange type $\frac{d}{dt}$ can be presented in the following form

$$\frac{d}{dt} = D_t - 2\frac{u_x}{u}D_x,$$

where

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \cdots, \qquad D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + \cdots.$$

The operator $\frac{d}{dt}$ in contrast to the operators D_t and D_x does not commute with the operators of total differentiation with respect to t and x:

$$\left[\frac{d}{dt}, D_t\right] = 2\left(\frac{u_{xt}}{u} - \frac{u_x u_t}{u^2}\right)D_x, \qquad \left[\frac{d}{dt}, D_x\right] = 2\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2}\right)D_x.$$

It is necessary to find an operator of total differentiation with respect to a new space variable s such that

$$\left[\frac{d}{dt}, D_s\right] = 0. \tag{7.27}$$

The last commuting property is possible if we involve a new dependent variable $\rho > 0$ (density) [16]. The operator $D_s = \frac{1}{\rho}D_x$ satisfies (7.27) if ρ holds the equation

$$\rho_t - 2\rho\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2}\right) - 2\frac{u_x}{u}\rho_x = 0.$$

The new space variable s is introduced with the help of equations

$$s_t = 2\rho\frac{u_x}{u}, \qquad s_x = \rho.$$

For convenience we can put initial data $\rho(0,x) \equiv 1$. Then, $s = x$ for $t = 0$.

In the variables (t, s) the heat transfer equation will get a form of the system

$$u_t = \rho^2\left(u_{ss} - 2\frac{u_s^2}{u}\right) + \rho\rho_s u_s, \qquad \rho_t = 2\rho^3\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) + 2\rho^2\rho_s\frac{u_s}{u} \tag{7.28}$$

which can be rewritten in the form of conservation laws

$$\left(\frac{1}{\rho}\right)_t = \left(-2\rho\frac{u_s}{u}\right)_s, \qquad \left(\frac{u}{\rho}\right)_t = (-\rho u_s)_s. \tag{7.29}$$

The space coordinate x is defined by the system of equations

$$x_t = -2\rho\frac{u_s}{u}, \qquad x_s = \frac{1}{\rho}. \tag{7.30}$$

System (7.28) in the space of independent variables (t, s) and extended set of dependent variables (u, ρ, x) admits a group of point transformations determined by the following infinitesimal operators:

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},\\
&X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s}, \quad X_5 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2+2t)u\frac{\partial}{\partial u} - 4t\rho\frac{\partial}{\partial \rho},\\
&X_6 = u\frac{\partial}{\partial u}, \qquad X^* = f(s)\frac{\partial}{\partial s} + \rho f'(s)\frac{\partial}{\partial \rho},
\end{aligned}
\tag{7.31}
$$

where $f(s)$ is an arbitrary function of s.

In the independent variables (t, s) operators X_1–X_6 are operators of Lie algebra factorized by the operator X^*. Condition (2.6) of grid orthogonality and condition (2.5) of space grid uniformity hold and it gives an opportunity to construct a difference model which is invariant with respect to operators X_1–X_6 on the orthogonal grid.

Let us write system (7.28), (7.30) in the form of differential invariants. In the space of variables $(t, x, s, u, \rho, dt, dx, ds, du, d\rho, u_s, \rho_s, x_s, u_{ss})$ there are five invariants:

$$
\begin{aligned}
&J_1 = x_s\rho, \qquad J_2 = \frac{\rho}{ds}\left(dx + 2\rho\frac{u_s}{u}dt\right), \qquad J_3 = \frac{(ds)^2}{\rho^2 dt},\\
&J_4 = \frac{(ds)^2}{\rho^3}\left(\frac{d\rho}{dt} - \frac{\rho_s ds}{dt} - 2\rho^3\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) - 2\rho^2\rho_s\frac{u_s}{u}\right),\\
&J_5 = \left(\frac{ds}{\rho}\right)^2\left(-\frac{2}{u}\frac{du}{dt} - \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + 2\rho^2\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) + 2\rho\rho_s\frac{u_s}{u}\right).
\end{aligned}
$$

With the help of these invariants we rewrite system (7.28), (7.30) as

$$
\begin{aligned}
&u_t = \rho^2\left(u_{ss} - \frac{u_s^2}{u}\right) + \rho\rho_s u_s,\\
&\rho_t = 2\rho^3\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) + 2\rho^2\rho_s\frac{u_s}{u}, \qquad x_t = -2\rho\frac{u_s}{u}
\end{aligned}
\tag{7.32}
$$

with the constraint equation $x_s = \frac{1}{\rho}$.

Now we can find a system of equations which approximates (7.32) and is invariant with respect to the set of operators (7.31). We can use the six-point invariant stencil (Fig. 4). We have the following invariants for the set of operators (7.31) where the operator X^* is changed on its difference analog

$$
X_h^* = f(s)\frac{\partial}{\partial s} + \rho \underset{+s}{D}(f(s))\frac{\partial}{\partial \rho}
$$

in the corresponding to the chosen stencil space $(t, \hat{t}, s, h_s^+, h_s^-, x, \hat{x}, h_x^+, h_x^-, \hat{h}_x^+, \hat{h}_x^-, u, \hat{u},$

$u_+, u_-, \hat{u}_+, \hat{u}_-, \rho, \hat{\rho}, \rho_+, \rho_-, \hat{\rho}_+, \hat{\rho}_-$):

$$
\begin{aligned}
&I_1 = \frac{h_x^+}{h_x^-}, \quad I_2 = \frac{\hat{h}_x^+}{\hat{h}_x^-}, \quad I_3 = \frac{h_x^+ \hat{h}_x^+}{\tau}, \quad I_4 = \frac{\tau^{1/2}}{h_x^+}\frac{\hat{u}}{u}\exp\left(\frac{1}{4}\frac{\Delta x^2}{\tau}\right),\\
&I_5 = \frac{1}{4}\frac{h_x^{+2}}{\tau} - \frac{h_x^{+2}}{h^+ + h^-}\left(\frac{1}{h_x^+}\ln\frac{u_+}{u} + \frac{1}{h_x^-}\ln\frac{u_-}{u}\right),\\
&I_6 = \frac{1}{4}\frac{\hat{h}_x^{+2}}{\tau} + \frac{\hat{h}_x^{+2}}{\hat{h}_x^+ + \hat{h}_x^-}\left(\frac{1}{\hat{h}_x^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{1}{\hat{h}_x^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),\\
&I_7 = \frac{\Delta x h_x^+}{\tau} + \frac{2h_x^+}{h_x^+ + h_x^-}\left(\frac{h_x^-}{h_x^+}\ln\frac{u_+}{u} - \frac{h_x^+}{h_x^-}\ln\frac{u_-}{u}\right),\\
&I_8 = \frac{\Delta x \hat{h}_x^+}{\tau} + \frac{2\hat{h}_x^+}{\hat{h}_x^+ + \hat{h}_x^-}\left(\frac{\hat{h}_x^-}{\hat{h}_x^+}\ln\frac{\hat{u}_+}{\hat{u}} - \frac{\hat{h}_x^+}{\hat{h}_x^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),\\
&I_9 = \frac{\hat{\rho}_-}{\rho_-}, \quad I_{10} = \frac{\hat{\rho}}{\rho}, \quad I_{11} = \frac{\hat{\rho}_+}{\rho_+}, \quad I_{12} = \frac{h_s^+}{\rho h_x^+}, \quad I_{13} = \frac{h_s^-}{\rho_- h_x^-}.
\end{aligned}
$$

With the help of these invariants we can write difference model in the form of the following system of evolution difference equations (we present here only one invariant difference model which corresponds to the system (7.19) in variables (t, x)):

$$
\begin{aligned}
&\Delta x = 2\tau\frac{-\frac{h_s^-}{h_s^+}\frac{\rho}{\rho_-}\ln\frac{u_+}{u} + \frac{h_s^+}{h_s^-}\frac{\rho_-}{\rho}\ln\frac{u_-}{u}}{\frac{h_s^+}{\rho} + \frac{h_s^-}{\rho_-}},\\
&\left(\frac{u}{\hat{u}}\right)^2\exp\left(-\frac{1}{2}\frac{\Delta x^2}{\tau}\right) = 1 - 4\tau\frac{\frac{\rho}{h_s^+}\ln\frac{u_+}{u} + \frac{\rho_-}{h_s^-}\ln\frac{u_-}{u}}{\frac{h_s^+}{\rho} + \frac{h_s^-}{\rho_-}},\\
&\hat{\rho} = \rho\frac{h_x^+}{\hat{h}_x^+}.
\end{aligned}
$$

In the case of the uniform grid ($h_s^+ = h_s^- = h_s$) this model could be simplified as follows:

$$
\begin{aligned}
&\Delta x = 2\tau\frac{-\rho^2\ln\frac{u_+}{u} + \rho_-^2\ln\frac{u_-}{u}}{h_s(\rho + \rho_-)},\\
&\left(\frac{u}{\hat{u}}\right)^2\exp\left(-\frac{1}{2}\frac{\Delta x^2}{\tau}\right) = 1 - \frac{4\tau\rho\rho_-}{h_s^2(\rho + \rho_-)}\left(\rho\ln\frac{u_+}{u} + \rho_-\ln\frac{u_-}{u}\right),\\
&\hat{\rho} = \rho\frac{h_x^+}{\hat{h}_x^+}.
\end{aligned}
$$

The system (7.28) has only two dependent variables u and ρ and it can be approximated without involvement of the space variable x. However, Galilean symmetry X_3 and projective symmetry X_5 are nonlocal in the coordinate system (t, s) and we need to consider the dependent variable x in order to have these symmetries. Constructing an invariant with respect to the set of operators (7.31) difference model, we inevitably involve x into the difference equations.

It is important to notice that in all cases moving meshes could be stopped by using the Lagrange type coordinate system (for an introduction of Lagrange type coordinate systems see, for example, [16]).

5. If $Q = \delta u$, $\delta = \pm 1$, the equation

$$u_t = u_{xx} + \delta u \tag{7.33}$$

can be transformed into equation (7.12) by the change of variables

$$\bar{u} = u e^{-\delta t}. \tag{7.34}$$

Reversing this transformation, one can get an invariant model for equation (7.33) from an invariant model for the heat transfer equation without a source.

6. $Q = \delta =$ const. The equation has the form:

$$u_t = u_{xx} + \delta. \tag{7.35}$$

The case $Q = 0$, which was considered in point 4 of this section, is a partial case for $Q = \delta =$ const, but the constant source can be excluded by the evident transformation

$$\bar{u} = u - \delta t. \tag{7.36}$$

It means that we can get a difference model for (7.35) from the model for (7.12).

8 Concluding remarks

In the paper [10] the entire set of invariant schemes for ordinary difference equations of the second order was developed. There were shown that for some equations and symmetries it is necessary to involve nontrivial lattices, which are not uniform in a space of independent variable and should depend on solution. Such schemes are self adapted for any solution and they are as much exactly integrable, as its continuous counterpart.

In the papers [2, 3, 8, 9, 11] several examples of invariant schemes for PDEs with two independent variables (KdV equation, nonlinear Schrödinger equation etc.) were constructed. Again, there were shown that for some equations and symmetries it is necessary to involve nontrivial two-dimensional meshes, which are not uniform and rectangular in a space of two independent variables and should depend on solution. Such schemes are self-adapted for any solution, the meshes are evolutionary in time and these schemes have as many exact invariant solutions as their continuous counterparts. Moreover, for invariant variational cases invariant schemes have difference conservation laws as well as for continuous case.

Thus, the applications of symmetry to difference equations led to the evolution of idea of possible meshes: from simple regular stationary meshes to self-adapted moving meshes, explicitly depending on solutions. Notice, that this idea evolution well corresponds to the big changes in numerical analysis, where the idea of self-adaptivity of schemes and meshes is in a broad fashion now.

In this paper we developed the entire set of invariant difference schemes for the heat transfer equation

$$u_t = (K(u)u_x)_x + Q(u), \tag{8.1}$$

for arbitrary coefficients $K(u)$, $Q(u)$ and for all special cases of the coefficients which extend the symmetry group admitted by equation (8.1).

The main conclusion is that we have presented an algorithmic way to construct the invariant difference schemes (i.e. a difference equation and a mesh it is defined on) for all cases of underlying heat equation.

Other conclusion is that symmetry preservation in difference schemes led to essential different discrete models. For different cases of coefficients $K(u)$ and $Q(u)$ taken in accordance with the group classification of equation (8.1) we have obtained different discrete models: for some cases $K(u)$ and $Q(u)$ we had to construct discrete models not for equation (8.1), but for the equivalent Lagrangian system:

$$\frac{dx}{dt} = \varphi(u, u_x), \qquad \frac{du}{dt} = \psi(u, u_x, u_{xx}). \tag{8.2}$$

The consideration of the equation (8.1) in the form of system (8.2) at the very beginning would provide us with the classification by functions φ and ψ. In that case there is one-to-one correspondence between the coordinate systems for continuous and discrete cases of the group classification. In particular, if the symmetry of the equation (8.1) does not require application of Lagrangian type moving mesh (evolving in time mesh), then we have $\varphi \equiv 0$. In that case the classification of the system (8.2) and corresponding difference equations on orthogonal mesh can be carried out by means of $\psi(u, u_x, u_{xx})$.

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References

[1] Ames W F, Anderson R L, Dorodnitsyn V A, Ferapontov E V, Gazizov R K, Ibragimov N H and Svirshevskii S R, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. I: Symmetries, Exact Solutions and Conservation Laws, CRC Press, Boca Raton, FL, 1994.

[2] Bakirova M I, Dorodnitsyn V A and Kozlov R V, Symmetry-Preserving Difference Schemes for Some Heat Transfer Equations, *J. Phys. A: Math. Gen* **30**, Nr. 23 (1997), 8139–8155.

[3] Budd C and Collins G, An Invariant Moving Mesh Scheme for the Nonlinear Diffusion Equation, *Appl. Numer. Math.* **26**, Nr. 1–2 (1998), 23–39.

[4] Dorodnitsyn V A, Invariant Solutions of the Nonlinear Heat Equation with a Source,*Zh. Vychisl. Mat. i Mat. Fiz.* **22**, Nr. 6 (1982), 1393–1400, (in Russian).

[5] Dorodnitsyn V A, Transformation Groups in Difference Spaces, *J. Sov. Math.* **55**, Nr. 1 (1991), 1490–1517.

[6] Dorodnitsyn V A, Finite-Difference Models Entirely Inheriting Symmetry of Original Differential Equations, in Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Kluwer Academic Publishers Group, Dordrecht, 1993, 191–201.

[7] Dorodnitsyn V A, Finite-Difference Analog of Noether's Theorem, *Dokl. Akad. Nauk S.S.S.R.* **328**, Nr. 6 (1993), 678–682 (in Russian);
Dorodnitsyn V, Noether-Type Theorems for Difference Equations, *Appl. Numer. Math.* **39**, Nr. 3–4 (2001), 307–321.

[8] Dorodnitsyn V, Invariant Discrete Models for the Korteweg-de Vries Equation, Preprint CRM–2187, Universite de Montreal, 1994.

[9] Dorodnitsyn V and Winternitz P, Lie Point Symmetry Preserving Discretizations for Variable Coefficient Korteweg-de Vries Equations, *Nonlinear Dynamics* **22**, Nr. 1 (2000), 49–59.

[10] Dorodnitsyn V, Kozlov R and Winternitz P, Lie Group Classification of Second-Order Ordinary Difference Equations, *J. Math. Phys.* **41**, Nr. 1 (2000), 480–504.

[11] Budd C and Dorodnitsyn V, Symmetry-Adapted Moving Mesh Scheme for the Nonlinear Schrödinger Equation, *J. Phys. A: Math. Gen.* **34**, Nr. 48 (2001), 10387–10400.

[12] Dorodnitsyn V, The Lie Group Properties of Difference Equations, Fizmatlit, Moscow, 2001.

[13] Ibragimov N H, Transformation Groups Applied to Mathematical Physics, D. Reidel, Dovdrecht, 1985.

[14] Olver P J, Applications of Lie Groups to the Differential Equations, Springer-Verlag, New York, 1986.

[15] Ovsiannikov L V, Group Analysis of Differential Equations, Academic Press, New York, 1982.

[16] Ovsiannikov L V, Lections on Gas Dynamics, Nauka, Moscow, 1981 (in Russian).

[17] Ovsiannikov L V, Group Relations of the Equation of Non-linear Heat Conductivity, *Dokl. Akad. Nauk S.S.S.R.* **125**, Nr. 3 (1959), 492–495 (in Russian).

[18] Proceedings of Workshop on Symmetries and Integrability of Difference Equations (SIDE-4), *J. Phys. A: Math. Gen.* **34** (2001), 10337-10774

[19] Levi D, Tremblay S and Winternitz P, Lie Symmetries of Multidimensional Difference Equations, *J. Phys. A: Math. Gen.* **34** (2001), 9507–9524.