COLLOCATION METHODS FOR THE SOLUTION OF EIGENVALUE PROBLEMS FOR SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We demonstrate that eigenvalue problems for ordinary differential equations can be recast in a formulation suitable for the solution by polynomial collocation. It is shown that the well-posedness of the two formulations is equivalent in the regular as well as in the singular case. Thus, a collocation code equipped with asymptotically correct error estimation and adaptive mesh selection can be successfully applied to compute the eigenvalues and eigenfunctions efficiently and with reliable control of the accuracy. Numerical examples illustrate this claim.

1. Introduction

We discuss the numerical solution of eigenvalue problems for singular ODEs. To keep the presentation simple, we will focus on linear first order problems

$$z'(t) - A(t)z(t) = \lambda z(t), \quad t \in (0,1], \tag{1}$$

$$B_0 z(0) + B_1 z(1) = 0. \tag{2}$$

The problem is to determine the *eigenvalues* $\lambda \in \mathbb{C}$ such that a nontrivial vector-valued *eigenfunction* $z \in C[0,1]$, $z(t) \in \mathbb{C}^n$, satisfying (1) and (2) exists. For the uniqueness of the eigenfunctions the normalization condition

$$\int_0^1 |z(\tau)|^2 \, d\tau = 1 \tag{3}$$

is imposed, which serves the purpose in the case that the *eigenspace* associated with λ has dimension one. We will restrict ourselves to problems satisfying this assumption, which are most common in applications, see also Section 2.

Our main interest is in singular problems, where

$$A(t) = M(t)/t^\alpha, \quad \alpha \geq 1. \tag{4}$$

In the case of $\alpha = 1$, the problem has a *singularity of the first kind*, while for $\alpha > 1$ we speak of an *essential singularity* or *singularity of the second kind.* For a discussion of the eigenvalue problem (1), (2), particularly in the singular case, see Section 2.

For the numerical computation of the eigenvalues and eigenfunctions, we rewrite the problem by introducing the following auxiliary quantities: We formally interpret λ as a function of t and add the auxiliary differential equation

$$\lambda'(t) = 0, \tag{5}$$

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and define

$$x(t) := \int_0^t |z(\tau)|^2 \, d\tau, \tag{6}$$

whence we have a further differential equation involving a quadratic nonlinearity, and two additional boundary conditions:

$$x'(t) = |z(t)|^2, \quad x(0) = 0, \quad x(1) = 1. \tag{7}$$

The resulting augmented system is a boundary value problem in standard form for the set of unknowns $z(t)$, $\lambda(t)$ and $x(t)$ without any further unknown parameters, see also Section 2. This system is subsequently solved by polynomial collocation. In this way, at some extra cost, we can make use of the elaborate theory and practical usefulness of these methods, particularly for singular problems, and use a code developed by the authors featuring asymptotically correct error estimation and adaptive mesh selection for an efficient solution of the problem, see Section 3. Numerical results demonstrating the success of this approach are given in Section 4.

Remark: Our treatment can easily be extended to *Sturm-Liouville problems* of second order,

$$y''(t) - A_1(t)y'(t) - A_0(t)y(t) = \lambda g(t)y(t), \quad t \in (0,1], \tag{8}$$

$$B_0(y(0), y'(0))^T + B_1(y(1), y'(1))^T = 0. \tag{9}$$

Transformation to a first order system yields a problem with a more general dependence on λ. Our approach naturally incorporates such problems as well, in fact the approach is applicable without modification to any problem with an unknown parameter,

$$z'(t) = f(t, z(t); \lambda), \quad t \in (0,1], \tag{10}$$

$$b(z(0), z(1)) = 0. \tag{11}$$

Since the sufficient conditions backing application of our solution approach are most readily formulated for the linear eigenvalue problem (1), (2) with normalization (3), we will restrict our attention to this case. However, numerical examples in Section 4 also comprise more general situations, particularly (8), (9).

2. Eigenvalue problems in ODEs

There is an abundant literature on the theory and numerical solution of eigenvalue problems for ODEs, particularly for the practically relevant case of Sturm-Liouville problems (8), (9). For a comprehensive overview, see for example the monograph [17], which also includes a discussion of the singular case. We do not attempt to give a complete picture here, but rather cite two results which apply directly to first order problems (1), (2) with singularity (4). In [11, Theorem 10.1] and [12, Theorem 7.1], the following result is proven for a generalized eigenvalue problem with a singularity of the first and of the second kind, respectively:

Theorem 2.1. *Consider the generalized eigenvalue problem*

$$\mathcal{L}z = z'(t) - M(t)/t^\alpha = \lambda G(t)z(t), \quad t \in (0,1], \tag{12}$$

$$B_0 z(0) + B_1 z(1) = 0, \tag{13}$$

where the matrices $M(0)$ and B_0, B_1 are such that $\mathcal{L}z = 0$ has a unique, smooth solution. Then

- *The spectrum Λ has no finite limit point. For $\lambda \notin \Lambda$, $(\mathcal{L} - \lambda G)^{-1}$ exists and is compact.*

- *Let us define*

$$P_{\lambda_0} := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\mathcal{L} - \lambda G)^{-1} G \, d\lambda,$$

where $\lambda_0 \in \Lambda$, $\Gamma = \{\lambda : |\lambda - \lambda_0| = \delta\}$ *and* δ *is so small that there is no* $\lambda_1 \in \Lambda$ *with* $|\lambda_1 - \lambda_0| \leq \delta$. *Then* P_{λ_0} *is a projection with a finite-dimensional range which is invariant under the mapping* $(\mathcal{L} - \lambda G)^{-1} G$, $\lambda \notin \Lambda$.

Remarks:

- The formulation as generalized eigenvalue problem (12) also includes cases resulting from the transformation of eigenvalue problems of higher order like (8), (9) to the first order form, see [11].
- The Fredholm theory for the operator $\mathcal{L}$ and smoothness results for the solutions, depending on the eigenvalues of $M(0)$, are derived in [11] and [12]. Also, corresponding smoothness results for the eigenfunctions are formulated. We do not give details here.
- The result for the spectral projection P_{λ_0} means that the linear space of generalized eigenfunctions associated with the eigenvalue λ_0 has finite dimension. The more restricted assumption that this space has dimension one, which underlies normalization condition (3) yielding a unique eigenfunction, cannot be concluded from Theorem 2.1. Rather, this has to be verified separately for each particular problem. However, some results are available for particular problem types, see for instance Theorem 2.2 below.
- In [11], the numerical solution of singular eigenvalue problems (12), (13) is also discussed, and *matrix methods* based on finite difference schemes as described and analyzed in [15] are considered, see also [17]. It is shown representatively for the box scheme that the eigenvalues of the discrete system converge to the solution of the analytical system.

As an example of a theoretical result backing our approach, consider the following assertion which readily follows from the results cited in Section 4 of [9] (see also [16]):

Theorem 2.2. *Consider the self-adjoint Sturm-Liouville problem with real coefficient functions and separated boundary conditions,*

$$(\mathcal{L}y)(t) = -(py')'(t) + q(t)y(t) = \lambda g(t)y(t), \quad t \in (0,1], \tag{14}$$

$$a_0 y(0) + b_0 (py)'(0) = 0, \quad a_1 y(1) + b_1 (py)'(1) = 0, \tag{15}$$

$a_0^2 + b_0^2 > 0$, $a_1^2 + b_1^2 > 0$. *Assume that* $p, q > 0$ *on* $(0,1]$ *and* $1/p$, q *and* g *are continuous functions satisfying* $1/p$, q, $g \in L^1[0, \alpha)$ *for some* $\alpha > 0$. *Then there exists an infinite, countable set of isolated real eigenvalues* λ_k, *and the associated eigenfunctions* $y_k(t)$ *are unique to constant multiples, i.e., each eigenspace has dimension one.*

Theorem 2.2 describes a standard situation where the coefficient functions are admitted to show a weakly singular behavior, such that $t = 0$ is a ‘regular endpoint’ in the terminology of [9]. For $p(t) = 1$ and $q(t) = t^{-\alpha}$, for instance, 0 is a regular endpoint for $\alpha < 1$. For the case of singular endpoints, the corresponding theory involves additional assumptions and a distinction of different types of boundary conditions. For details we refer the reader to [8]–[10].

3. Augmented Systems and Collocation Methods

We propose to solve eigenvalue problem (1)–(3) by using the augmented system (1), (2), (5), and (7). This is a first order, explicit, nonlinear boundary value problem. We will first demonstrate that this problem is well-posed if and only if the original eigenvalue problem is well-posed. We first consider the regular case. To this end, we show that the solution of eigenvalue problem (1)–(3) is isolated in the sense of [13] if and only if the solution of the augmented system has this property. This means that the numerical solution can be safely computed for sufficiently accurate starting values. Clearly, since the solutions of the two formulations are equivalent, we conclude that the augmented system represents a valid alternative for the computation of the eigenvalues and eigenvectors of (1), (2).

To discuss whether the solutions of the original and the augmented problem are isolated, we show that unique solvability of the linearization is equivalent for the two formulations [13]. We first rewrite the problems as operator equations

$$F(z,\lambda)=0, \tag{16}$$

where

$$F:\ \mathcal{B}_1\to\mathcal{B}_2,$$
$$F(z(\cdot),\lambda)(t)=\left(\begin{array}{c} z'(t)-(A(t)+\lambda I)z(t)\\ \int_0^1 \overline{z}^T(\tau)z(\tau)\,d\tau-1\\ B_0z(0)+B_1z(1)\end{array}\right),$$
$$\mathcal{B}_1=C^1[0,1]\times\mathbb{C},\quad \mathcal{B}_2=C[0,1]\times\mathbb{C}\times\mathbb{C}^n,$$

and

$$\hat F(z,\lambda,x)=0, \tag{17}$$

where

$$\hat F:\ \hat{\mathcal{B}}_1\to\hat{\mathcal{B}}_2,$$
$$\hat F(z(\cdot),\lambda(\cdot),x(\cdot))(t)=\left(\begin{array}{c} z'(t)-(A(t)+\lambda(t)I)z(t)\\ \lambda'(t)\\ x'(t)-\overline{z}^T(t)z(t)\\ B_0z(0)+B_1z(1)\\ x(0)\\ x(1)-1\end{array}\right),$$
$$\hat{\mathcal{B}}_1=C^1[0,1]\times C^1[0,1]\times C^1[0,1],$$
$$\hat{\mathcal{B}}_2=C[0,1]\times C[0,1]\times C[0,1]\times\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}.$$

It is readily observed that the corresponding homogeneous linearized equations are given by

$$DF(z(\cdot),\lambda)\left(\begin{array}{c} h(\cdot)\\ \mu\end{array}\right)(t)=\left(\begin{array}{c} h'(t)-(A(t)+\lambda I)h(t)-\mu z(t)\\ \int_0^1 2\,\Re\big((\overline{z}^T(\tau)h(\tau)\big)\,d\tau\\ B_0h(0)+B_1h(1)\end{array}\right)=0, \tag{18}$$

and
(19)

$$D\hat{F}(z(\cdot),\lambda(\cdot),x(\cdot))\begin{pmatrix} h(\cdot)\\ \mu(\cdot)\\ v(\cdot)\end{pmatrix}(t) = \begin{pmatrix} h'(t)-(A(t)+\lambda(t)I)h(t)-\mu(t)z(t)\\ \mu'(t)\\ v'(t)-2\,\Re\big(\overline{z}^T(t)h(t)\big)\\ B_0h(0)+B_1h(1)\\ v(0)\\ v(1)\end{pmatrix} = 0,$$

respectively. It is easy to see that the question of unique solvability of the linearized equations is equivalent for both formulations.

The argument also applies to the singular case. This is also reflected when we consider the sufficient conditions given in [11, Theorem 3.1] and [12, Theorem 3.2], respectively, for the Fredholm alternative of the involved operators. We do not carry out the argument in detail to avoid overboarding notation, but sketch the proof for the case of a singularity of the first kind. Let S denote the projection onto the invariant subspace associated with the eigenvalues with positive real part of the matrix $M(0)$ from (4), R the projection onto the nullspace of that matrix, and $P = S + R$. If

(20) $$\operatorname{rank}[B_0R, B_1] = \operatorname{rank}(P),$$

then boundary value problem (1), (2) has a unique solution for every fixed λ (note that with slight abuse of notation the linearized problem (18) has a similar structure). In the augmented system for $D\hat{F}$, the matrix $[B_0R, B_1]$ is augmented by two linearly independent rows. Likewise, the rank of P is increased by two, and thus the relation corresponding to (20) is equivalent to its original version for DF. A similar argument applies for the condition formulated in [12, Theorem 3.2].

To solve problem (1), (2), (5), and (7) numerically, we use *polynomial collocation*. This is a common and well-established solution method for boundary value problems in ODEs, see for example [2]. Collocation means that the solution is approximated by a continuous, piecewise polynomial function $p(t)$ satisfying the augmented ODE system in a pointwise sense at a certain number of collocation nodes $t_{i,j} \in (0,1]$, together with the associated boundary conditions. Many standard implementations of these methods exist on different platforms [1, 3, 18].

The collocation approach is particularly suited for the solution of singular problems [5, 6, 14]. In the implementation which we use for the purpose of solving eigenvalue problems [3], the efficient and reliable approximation of the solution is guaranteed by adaptive mesh selection [7] based on asymptotically correct estimation of the global error [4, 6, 14]. From the theoretical results it is clear that this solution approach will work well for boundary value problem (1), (2), (5), and (7). We will demonstrate in Section 4 that with this approach we are able to compute the eigenvalues and eigenfunctions of problem (1)–(3) efficiently and reliably to high accuracy given by prescribed tolerance requirements.

4. Numerical Results

In this section we illustrate the performance of our approach. As proposed in Section 3, we solve the original eigenvalue problem (1)–(3) by computing the solution of the augmented system (1), (2), (5), and (7). For the numerical treatment we used our Matlab code `sbvp`, see [3], which is available from `http://www.mathworks.com/matlabcentral/fileexchange`. This code was designed to solve efficiently boundary value problems with singular endpoints of the type arising in all model problems discussed in this section.

For our tests we have selected some model problems discussed in the relevant literature, cf. in particular [9], [19]. We first consider the well-known Bessel equation,

$$-y''(t) + \frac{c}{t^2}y(t) = \lambda y(t), \quad t \in (0,\pi], \tag{21}$$

$$y(0) = 0, \quad y(\pi) = 0, \tag{22}$$

with $c \in \mathbb{R}$. For $c = 0$, the exact solution reads $\lambda_k^\star = k^2$, $y_k(t) = \sin(kt)$, $k \in \mathbb{N}$. For $c \neq 0$, the Bessel equation is singular with a singularity of the first kind. In order to derive the associated first order system we apply the standard transformation $(z_1(t), z_2(t))^T := (y(t), y'(t))^T$ to (21). This, together with $z_3(t) := \lambda(t)$ and $z_4(t) := x(t)$, cf. (5) and (7), respectively, yields the augmented system in first order form,

$$z'(t) = \frac{1}{t^2}\begin{pmatrix} 0 & t^2 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ -z_1(t)z_3(t) \\ 0 \\ z_1^2(t) \end{pmatrix}, \quad t \in (0,\pi], \tag{23}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} z(\pi) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{24}$$

Note that this first order system is essentially singular. The numerical results for different values of c are given in Tables 1–3 and Figure 1 below. In all tables we use $\lambda_k^{(0)}$ to denote the starting value for the approximation λ_k. Moreover, N_k is the number of points in the final grid which was necessary to satisfy the prescribed tolerance requirements. The approximations for the eigenvalues λ_k and eigenfunctions[1] $y_k(t)$ were computed using default tolerances `absTOL` $= 10^{-6}$ and `relTOL` $= 10^{-3}$. For $c = 0$, the exact solution has been used for the calculation of the absolute and relative errors. In order to determine the errors in case of $c = 3$ and $c = 4$, we also computed related reference solutions using stricter tolerances, `absTOL` $=$ `relTOL` $= 10^{-8}$. Here, N_k^{ref} is the respective number of grid points in the final mesh.

$\lambda_k^{(0)}$	λ_k	$\lambda_k^\star$	abs. error	rel. error	N_k
2.00	9.99999979 e−01	1.00000000 e+00	2.086878 e−08	2.0869 e−08	32
5.00	4.00000001 e+00	4.00000000 e+00	6.622438 e−09	1.6556 e−09	32
10.00	9.00000041 e+00	9.00000000 e+00	4.145668 e−07	4.6063 e−08	32
20.00	1.60000002 e+01	1.60000000 e+01	1.702868 e−07	1.0643 e−08	32
30.00	2.50000010 e+01	2.50000000 e+01	9.929543 e−07	3.9718 e−08	32

TABLE 1. Bessel equation, $c = 0$

[1]Note that the eigenfunction $y_k(t)$ is the first component of the vector $z(t)$ associated with the eigenvalue λ_k, so the more precise notation would be $z_{1,k}(t)$.

$\lambda_k^{(0)}$	λ_k	λ_k^{ref}	abs. error	rel. error	N_k	N_k^{ref}
5.00	2.41710617 e+00	2.41710621 e+00	3.955774 e−08	1.6366 e−08	32	766
8.00	6.72365318 e+00	6.72365302 e+00	1.534105 e−07	2.2817 e−08	32	513
20.00	1.30275016 e+01	1.30275009 e+01	7.543607 e−07	5.7905 e−08	32	663
30.00	2.13307309 e+01	2.13307282 e+01	2.609459 e−06	1.2233 e−07	32	819

TABLE 2. Bessel equation, $c = 3$

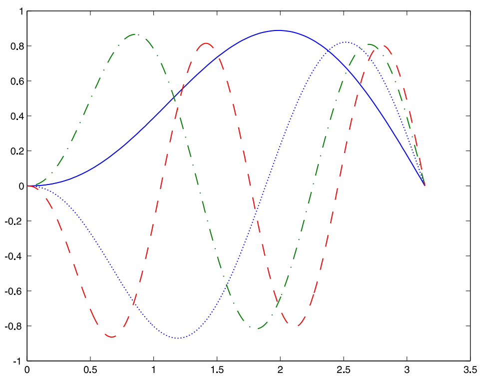

FIGURE 1. Eigenfunctions for the Bessel equation, $c = 3$: y_1 – solid line, y_2 – dotted line, y_3 – dashed-dotted line, y_4 – dashed line

$\lambda_k^{(0)}$	λ_k	λ_k^{ref}	abs. error	rel. error	N_k	N_k^{ref}
5.00	2.75408474 e+00	2.75408479 e+00	4.798895 e−08	1.7425 e−08	32	339
10.00	7.32285253 e+00	7.32285252 e+00	8.971130 e−09	1.2251 e−09	32	820
20.00	1.38865475 e+01	1.38865474 e+01	9.957818 e−08	7.1708 e−09	32	600
26.50	2.24490247 e+01	2.24490241 e+01	6.039615 e−07	2.6904 e−08	32	765

TABLE 3. Bessel equation, $c = 4$

The next model equation, cf. [19], has the form

$$\varrho''(r) + \frac{n-1}{r}\varrho'(r) = \lambda\varrho(r), \quad r \in [0, a], \tag{25}$$

$$\varrho(a) = 0, \ \varrho'(0) = 0, \tag{26}$$

with $n = 3$ and $a = 1$. Here, the exact eigenvalues are known to satisfy $\lambda_k^\star = -(k\pi)^2$, $k \in \mathbb{N}$. The problem is singular with a singularity of the first kind. In order to derive the associated first order system we apply the so-called Euler transformation $(z_1(r), z_2(r))^T := (\varrho(r), r\varrho'(r))^T$ to (25). Together with $z_3(r) := \lambda(r)$

and $z_4(r) := x(r)$ we obtain the augmented first order system,

$$z'(r) = \frac{1}{r}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(r) + \begin{pmatrix} 0 \\ r\, z_3(r) z_1(r) \\ 0 \\ z_1^2(r) \end{pmatrix}, \quad r \in (0,1],$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The approximations for the eigenvalues are displayed in Table 4, the associated eigenfunctions can be found in Figure 2.

$\lambda_k^{(0)}$	λ_k	$\lambda_k^\star$	abs. error	rel. error	N_k
−10.00	−9.86960440 e+00	−9.86960440 e+00	9.848122 e−12	9.9782 e−13	32
−39.48	−3.94784177 e+01	−3.94784176 e+01	1.010882 e−07	2.5606 e−09	32
−90.00	−8.88264376 e+01	−8.88264396 e+01	2.052493 e−06	2.3107 e−08	32
−158.00	−1.57913672 e+02	−1.57913670 e+02	1.179932 e−06	7.4720 e−09	32
−245.00	−2.46740123 e+02	−2.46740110 e+02	1.255438 e−05	5.0881 e−08	32

TABLE 4. Model problem from [19]: $a = 1$, $n = 3$

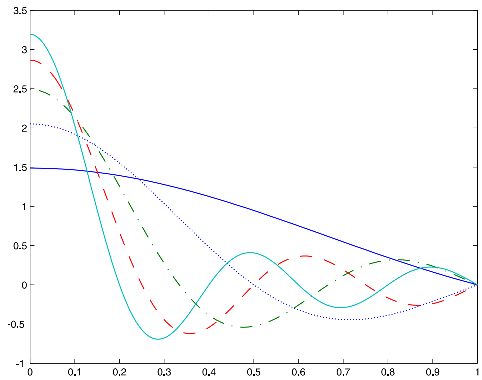

FIGURE 2. Eigenfunctions for the model problem from [19]: y_1 – solid line, y_2 – dotted line, y_3 – dashed-dotted line, y_4 – dashed line, y_5 – fine dotted line

The final test problem is the so-called Boyd equation, see [9],

$$-y''(t) - \frac{1}{t}y(t) = \lambda y(t), \quad t \in (0,1], \tag{27}$$

$$y(0) = 0, \quad y(1) = 0. \tag{28}$$

The augmented first order formulation now reads:

$$z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ -t\, z_3(t) z_1(t) \\ 0 \\ z_1^2(t) \end{pmatrix}, \quad t \in (0,1],$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The numerical results are very similar to those given before, cf. Table 5 and Figure 3.

$\lambda_k^{(0)}$	λ_k	λ_k^{ref}	abs. error	rel. error	N_k	N_k^{ref}
1.00	7.37398502 e+00	7.37398502 e+00	3.559406 e−09	4.8270 e−10	32	153
40.00	3.63360196 e+01	3.63360196 e+01	5.270783 e−08	1.4506 e−09	32	267
80.00	8.52925811 e+01	8.52925821 e+01	9.478771 e−07	1.1113 e−08	32	425
155.00	1.54098619 e+02	1.54098624 e+02	4.293315 e−06	2.7861 e−08	32	583
250.00	2.42705545 e+02	2.42705559 e+02	1.420079 e−05	5.8510 e−08	32	741

TABLE 5. Boyd equation

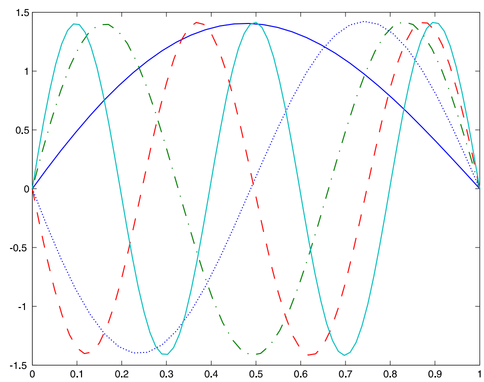

FIGURE 3. Eigenfunctions for the Boyd equation: y_1 – solid line, y_2 – dotted line, y_3 – dashed-dotted line, y_4 – dashed line, y_5 – fine dotted line

Finally, in Table 6 we list the empirical order of convergence obtained for a collocation solution of order $p = 4$, collocating at equidistant collocation points. In this case the eigenvalues and eigenfunctions were computed on equidistant grids with decreasing stepsizes h. To obtain a reference solution, we executed our program using the tolerances $\texttt{absTOL} = \texttt{relTOL} = 10^{-10}$, utilizing error estimate and grid adaptivity. The norm of the absolute error for an eigenfunction $y_h(t)$, $\|y_h - y^{ref}\|_\Delta$, has been calculated by taking a discrete maximum of $|y_h(t) - y^{ref}(t)|$ from its evaluation at 1,000 equidistantly spaced points in the interval of integration. In order to estimate the error constant c and the convergence order p, we have assumed that the stepsize h is small enough to justify the following asymptotic behavior:

$$|\lambda_h - \lambda^\star| = c_\lambda h^{p_\lambda}, \quad \|y_h - y^\star\|_\infty = c_y h^{p_y}.$$

Using the data associated with two consecutive grids, we were able to provide the approximations for the values c_λ, p_λ and c_y, p_y. In Table 6, the order $p = 4$ both for the convergence towards the eigenvalues and the eigenfunctions can be clearly observed. Note that the accuracy of the reference solution constitutes a limitation for the range of observability of the convergence order.

h	$\lvert\lambda_h - \lambda^{ref}\rvert$	$\lVert y_h - y^{ref}\rVert_\Delta$	c_λ	p_λ	c_y	p_y
$\lambda \approx 7.3740$						
2.50 e−01	1.029 e−03	1.213 e−04				
1.25 e−01	4.854 e−05	9.160 e−06	2.983 e−03	4.41	2.986 e−04	3.73
6.25 e−02	2.317 e−06	6.388 e−07	2.936 e−03	4.39	3.322 e−04	3.84
3.13 e−02	5.944 e−08	4.279 e−08	1.262 e−02	5.29	3.652 e−04	3.90
1.56 e−02	5.992 e−08	3.017 e−09	5.786 e−08	−0.01	3.078 e−04	3.83
7.81 e−03	6.667 e−08	1.283 e−09	3.767 e−08	−0.15	1.242 e−07	1.23
$\lambda \approx 36.334$						
2.50 e−01	8.215 e−02	2.638 e−03				
1.25 e−01	4.744 e−03	1.934 e−04	2.220 e−01	4.11	6.558 e−03	3.77
6.25 e−02	2.476 e−04	1.691 e−05	2.544 e−01	4.26	5.169 e−03	3.52
3.13 e−02	1.146 e−05	2.767 e−06	3.369 e−01	4.43	1.189 e−03	2.61
1.56 e−02	1.128 e−06	1.886 e−06	2.697 e−02	3.35	9.988 e−06	0.55
7.81 e−03	1.839 e−06	1.853 e−06	1.350 e−07	−0.70	2.034 e−06	0.03
$\lambda \approx 85.293$						
2.50 e−01	6.566 e−01	1.304 e−02				
1.25 e−01	5.976 e−02	1.102 e−03	1.514 e+00	3.46	3.086 e−02	3.56
6.25 e−02	3.453 e−03	9.645 e−05	2.793 e+00	4.11	2.947 e−02	3.52
3.13 e−02	1.967 e−04	7.432 e−06	2.888 e+00	4.13	3.969 e−02	3.70
1.56 e−02	1.193 e−05	5.062 e−07	2.341 e+00	4.04	5.997 e−02	3.88
7.81 e−03	3.675 e−06	7.690 e−08	1.999 e−03	1.70	1.834 e−03	2.72
$\lambda \approx 154.10$						
2.50 e−01	6.702 e−01	1.853 e−01				
1.25 e−01	3.205 e−01	3.934 e−03	8.667 e−01	1.06	7.099 e−01	5.56
6.25 e−02	2.097 e−02	3.270 e−04	1.267 e+01	3.93	1.126 e−01	3.59
3.13 e−02	1.231 e−03	2.659 e−05	1.634 e+01	4.09	1.186 e−01	3.62
1.56 e−02	7.127 e−05	2.349 e−06	1.716 e+01	4.11	8.983 e−02	3.50
7.81 e−03	2.202 e−05	2.110 e−06	1.177 e−02	1.69	3.743 e−06	0.15
$\lambda \approx 242.71$						
2.50 e−01	4.800 e+00	1.264 e−01				
1.25 e−01	1.058 e+00	9.736 e−03	8.132 e+00	2.18	3.089 e−01	3.70
6.25 e−02	8.186 e−02	8.353 e−04	3.336 e+01	3.69	2.671 e−01	3.54
3.13 e−02	5.005 e−03	7.005 e−05	5.798 e+01	4.03	2.817 e−01	3.58
1.56 e−02	3.031 e−04	8.057 e−06	5.991 e+01	4.05	9.784 e−02	3.12
7.81 e−03	2.051 e−05	4.462 e−06	3.695 e+01	3.89	1.052 e−04	0.85

TABLE 6. Convergence order for the collocation solution to the Boyd equation

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