3. $ax^3 + bx^2 + cx + d = 0 \qquad (a \neq 0).$

Cubic equation.

1. Incomplete cubic equation.

1°. *Cardano's solution.* The roots of the incomplete cubic equation

$$y^3 + py + q = 0 \tag{1}$$

are given by

$$y_1 = A + B, \quad y_{2,3} = -\frac{1}{2}(A+B) \pm i\frac{\sqrt{3}}{2}(A-B),$$

where

$$A = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3}, \quad B = \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}, \quad D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \quad i^2 = -1,$$

with A and B being any of the values of the respective cubic roots such that $AB = -p/3$.

The number of real roots of the cubic equation (1) depends on the sign of the discriminant D:

$D > 0$	one real and two complex conjugate roots,
$D < 0$	three real roots,
$D = 0$	one simple real and one twofold real roots
	or, if $p = q = 0$, one threefold real root.

2°. *Trigonometric solution.* If the coefficients p and q of the incomplete cubic equation (1) are real, then its roots can also be expressed with trigonometric functions as shown below.

(a) Let $p < 0$ and $D < 0$. Then

$$y_1 = 2\sqrt{-\frac{p}{3}}\cos\frac{\alpha}{3}, \quad y_{2,3} = -2\sqrt{-\frac{p}{3}}\cos\left(\frac{\alpha}{3} \pm \frac{\pi}{3}\right),$$

where the trigonometric functions are evaluated taking into account the formula

$$\cos\alpha = -\frac{q}{2\sqrt{-(p/3)^3}}.$$

(b) Let $p > 0$ and $D \geq 0$. Then

$$y_1 = 2\sqrt{\frac{p}{3}}\cot(2\alpha), \quad y_{2,3} = \sqrt{\frac{p}{3}}\left[\cot(2\alpha) \pm i\frac{\sqrt{3}}{\sin(2\alpha)}\right],$$

where the trigonometric functions are evaluated using the formulas

$$\tan\alpha = \left(\tan\frac{\beta}{2}\right)^{1/3}, \quad \tan\beta = \frac{2}{q}\left(\frac{p}{3}\right)^{3/2}, \quad |\alpha| \leq \frac{\pi}{4}, \quad |\beta| \leq \frac{\pi}{2}.$$

(c) Let $p < 0$ and $D \geq 0$. Then

$$y_1 = -2\sqrt{-\frac{p}{3}}\,\frac{1}{\sin(2\alpha)}, \quad y_{2,3} = \sqrt{-\frac{p}{3}}\left[\frac{1}{\sin(2\alpha)} \pm i\sqrt{3}\cot(2\alpha)\right],$$

where the trigonometric functions are evaluated using the formulas

$$\tan\alpha = \left(\tan\frac{\beta}{2}\right)^{1/3}, \quad \sin\beta = \frac{2}{q}\left(-\frac{p}{3}\right)^{3/2}, \quad |\alpha| \leq \frac{\pi}{4}, \quad |\beta| \leq \frac{\pi}{2}.$$

In all three cases, the real value of the cubic root is taken.

2. Complete cubic equation.

1°. The roots of the complete cubic equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (a \neq 0) \tag{2}$$

are evaluated by the formulas

$$x_k = y_k - \frac{b}{3a}, \qquad k = 1,\ 2,\ 3,$$

where the y_k are roots of the incomplete cubic equation (1) with coefficients

$$p = -\frac{1}{3}\left(\frac{b}{a}\right)^2 + \frac{c}{a}, \qquad q = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{bc}{3a^2} + \frac{d}{a}.$$

2°. Vieta's theorem for the roots of the cubic equation (2):

$$\begin{aligned} x_1 + x_2 + x_3 &= -b/a, \\ x_1x_2 + x_1x_3 + x_2x_3 &= c/a, \\ x_1x_2x_3 &= -d/a. \end{aligned}$$

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