8. $\boldsymbol{ax^4 + bx^3 + cx^2 + dx + e = 0}$ **($\boldsymbol{a \neq 0}$).**

Quartic equation of general form.

1°. *Reduction to an incomplete equation.* The quartic equation in question is reduced to an incomplete equation

$$y^4 + py^2 + qy + r = 0. \tag{1}$$

with the change of variable

$$x = y - \frac{b}{4a}.$$

2°. *Decartes–Euler solution.* The roots of the incomplete equation (1) are given by

$$\begin{aligned} y_1 &= \tfrac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right), & y_2 &= \tfrac{1}{2}\left(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}\right), \\ y_3 &= \tfrac{1}{2}\left(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}\right), & y_4 &= \tfrac{1}{2}\left(-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}\right), \end{aligned} \tag{2}$$

where z_1, z_2, z_3 are roots of the cubic equation

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0, \tag{3}$$

which is called the cubic resolvent of equation (1). The signs of the roots in (2) are chosen so that

$$\sqrt{z_1}\sqrt{z_2}\sqrt{z_3} = -q.$$

The roots of the incomplete quartic equation (1) are determined by the roots of the cubic resolvent (3); see the table below.

TABLE
Relation between the roots of the incomplete quartic equation and the roots of its cubic resolvent

Cubic resolvent (3)	Quartic equation (1)
All roots are real and positive*	Four real roots
All roots are real, one positive and two negative*	Two pairs of complex conjugate roots
One roots is positive and two roots are complex conjugate	Two real and two complex conjugate roots

* By Vieta's theorem, the product of the roots z_1, z_2, z_3 is equal to $q^2 \geq 0$.

3°. *Ferrari's solution.* Suppose z_0 is any of the roots of the auxiliary cubic equation (3). Then the four roots of the incomplete equation (1) are found by solving two quadratic equations

$$y^2 - \sqrt{z_0}\,y + \frac{p + z_0}{2} + \frac{q}{2\sqrt{z_0}} = 0,$$
$$y^2 + \sqrt{z_0}\,y + \frac{p + z_0}{2} - \frac{q}{2\sqrt{z_0}} = 0.$$

References

Abramowitz, M. and Stegun, I. A. (Editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics, Washington, 1964.

Kurosh, A. G., *Lectures on General Algebra*, Chelsea Publ., New York, 1965.

Mishina, A. P. and Proskuryakov, I. V., *Higher Algebra*, Pergamon Press, New York, 1965.

Borwein, P. and Erdélyi, T., Quartic Equations, in *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, p. 4, 1995.

Korn, G. A. and Korn, T. M., *Mathematical Handbook for Scientists and Engineers, 2nd Edition*, Dover, New York, 2000.

Weisstein, E. W., *CRC Concise Encyclopedia of Mathematics, 2nd Edition*, Chapman & Hall/CRC, Boca Raton, 2003.

Bronshtein, I.N. and Semendyayev, K.A., *Handbook of Mathematics, 4th Edition*, Springer-Verlag, Berlin, 2004.

