# Contents

Authors ........................................................................................................ xv

Foreword ........................................................................................................ xvii

Some Notations and Remarks ..................................................................... xix

1. Parabolic Equations with One Space Variable ..................................... 1
   1.1. Equations with Power-Law Nonlinearities ........................................ 1
       1.1.1. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + bw + cw^2 \) .................................................................. 1
       1.1.2. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_3 w^3 \) ......................................................... 2
       1.1.3. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(w) \) .............................................................................................. 6
       1.1.4. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \) ....................................................................................... 9
       1.1.5. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \frac{\partial w}{\partial x} + g(w) \) ......................................................... 9
       1.1.6. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w) \) .............................................. 13
       1.1.7. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b \left( \frac{\partial w}{\partial x} \right)^2 + f(x, t, w) \) .................................................. 15
       1.1.8. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w, \frac{\partial w}{\partial x}) \) ................................................................. 17
       1.1.9. Equations of the Form \( \frac{\partial w}{\partial t} = a w^k \frac{\partial^2 w}{\partial x^2} + f(x, t, w, \frac{\partial w}{\partial x}) \) ....................................................... 18
       1.1.10. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( w^m \frac{\partial w}{\partial x} \right) \) ................................................................. 25
       1.1.11. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( w^m \frac{\partial w}{\partial x} \right) + bw^k \) .............................................................. 32
       1.1.12. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( w^m \frac{\partial w}{\partial x} \right) + bw + c_1 w^{k_1} + c_2 w^{k_2} + c_3 w^{k_3} \) .................. 37
       1.1.13. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( f(w) \frac{\partial w}{\partial x} \right) + g(x, t, w, \frac{\partial w}{\partial x}) \) .................................................. 40
       1.1.14. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( f(w) \frac{\partial w}{\partial x} \right) + g(x, t, w, \frac{\partial w}{\partial x}) \) .................................................. 42
       1.1.15. Other Equations ........................................................................ 46

   1.2. Equations with Exponential Nonlinearities ..................................... 52
       1.2.1. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b_0 + b_1 e^{\lambda w} + b_2 e^{2\lambda w} \) ............................................... 52
       1.2.2. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( e^{\lambda w} \frac{\partial w}{\partial x} \right) + f(w) \) ................................................................. 53
       1.2.3. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left[ f(w) \frac{\partial w}{\partial x} \right] + g(w) \) ................................................................. 56
       1.2.4. Other Equations Explicitly Independent of \( x \) and/or \( t \) ............ 58
       1.2.5. Equations Explicitly Dependent on \( x \) and/or \( t \) ....................... 61

   1.3. Equations with Hyperbolic Nonlinearities ..................................... 62
       1.3.1. Equations Involving Hyperbolic Cosine .................................... 62
       1.3.2. Equations Involving Hyperbolic Sine ....................................... 63
       1.3.3. Equations Involving Hyperbolic Tangent .................................. 63
       1.3.4. Equations Involving Hyperbolic Cotangent ............................... 64

   1.4. Equations with Logarithmic Nonlinearities ................................... 64
       1.4.1. Equations of the Form \( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \) ................................................................. 64
       1.4.2. Other Equations .................................................................. 66

   1.5. Equations with Trigonometric Nonlinearities ............................... 68
       1.5.1. Equations Involving Cosine .................................................... 68
       1.5.2. Equations Involving Sine ...................................................... 69
       1.5.3. Equations Involving Tangent .................................................. 70
       1.5.4. Equations Involving Cotangent .............................................. 70
       1.5.5. Equations Involving Inverse Trigonometric Functions ............. 71
1.6. Equations Involving Arbitrary Functions .................................................. 71
  1.6.1. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w)$ ........ 71
  1.6.2. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t) \frac{\partial w}{\partial x} + g(x, t, w)$ ... 75
  1.6.3. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w)$ 78
  1.6.4. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(x, t, w)$ ... 81
  1.6.5. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w)$ $\frac{\partial w}{\partial x} + h(x, t, w)$ 84
  1.6.6. Equations of the Form $\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w)$ $\frac{\partial w}{\partial x} + g(x, t, w)$ 87

1.6.7. Equations of the Form $\frac{\partial w}{\partial t} = (aw + b) \frac{\partial w}{\partial x} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w)$ $\frac{\partial w}{\partial x} + h(x, t, w)$ 94
  1.6.8. Equations of the Form $\frac{\partial w}{\partial t} = f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w, \frac{\partial w}{\partial x})$ .... 98
  1.6.9. Equations of the Form $\frac{\partial w}{\partial t} = f(x, t, w) \frac{\partial^2 w}{\partial x^2}$ ......................... 101
  1.6.10. Equations of the Form $\frac{\partial w}{\partial t} = f(x, t, w) \frac{\partial^2 w}{\partial x^2} + g(x, t, w, \frac{\partial w}{\partial x})$ .... 102

1.7. Nonlinear Schrödinger Equations and Related Equations .............................. 125
  1.7.1. Equations of the Form $i \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + f(|w|)w = 0$ Involving Arbitrary Parameters 125
  1.7.2. Equations of the Form $i \frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left(e^{i\phi} \frac{\partial w}{\partial x}\right) + f(|w|)w = 0$ Involving Arbitrary Parameters 128

1.7.3. Other Equations Involving Arbitrary Parameters ........................................ 130
  1.7.4. Equations with Cubic Nonlinearities Involving Arbitrary Functions ............. 131
  1.7.5. Equations of General Form Involving Arbitrary Functions of a Single Argument 134
  1.7.6. Equations of General Form Involving Arbitrary Functions of Two Arguments .... 137

2. Parabolic Equations with Two or More Space Variables .................................. 141
  2.1. Equations with Two Space Variables Involving Power-Law Nonlinearities .......... 141
    2.1.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[f(x) \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y} \left[g(y) \frac{\partial w}{\partial y}\right] + aw^p$ ... 141
    2.1.2. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[f(w) \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y} \left[g(w) \frac{\partial w}{\partial y}\right] + h(w)$ 142
    2.1.3. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[w^n \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y} \left[w^m \frac{\partial w}{\partial y}\right] + a e^w$ 147
    2.1.4. Other Equations ................................................................. 149

2.2. Equations with Two Space Variables Involving Exponential Nonlinearities ........ 154
    2.2.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[f(x) \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y} \left[g(y) \frac{\partial w}{\partial y}\right] + ae^w$ 154
    2.2.2. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[w^n \frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y} \left[w^m \frac{\partial w}{\partial y}\right] + f(w)$ 155

2.3. Other Equations with Two Space Variables Involving Arbitrary Parameters .......... 157
    2.3.1. Equations with Logarithmic Nonlinearities .................................. 157
    2.3.2. Equations with Trigonometrical Nonlinearities ................................ 158
2.4. Equations Involving Arbitrary Functions ................................................................. 159
  2.4.1. Heat and Mass Transfer Equations in Quiescent or Moving Media with Chemical
          Reactions .................................................................................................................. 159
  2.4.2. Equations of the Form \( \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(x) \frac{\partial w}{\partial x} + \frac{\partial}{\partial y} [g(y) \frac{\partial w}{\partial y}] + h(w) \) ................. 161
  2.4.3. Equations of the Form \( \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w) \frac{\partial w}{\partial x} + \frac{\partial}{\partial y} [g(w) \frac{\partial w}{\partial y}] + h(t, w) \) .............. 162
  2.4.4. Other Equations Linear in the Highest Derivatives .............................................. 165
  2.4.5. Nonlinear Diffusion Boundary Layer Equations ................................................. 168

2.5. Equations with Three or More Space Variables ..................................................... 169
  2.5.1. Equations of Mass Transfer in Quiescent or Moving Media with Chemical
          Reactions .................................................................................................................. 169
  2.5.2. Heat Equations with Power-Law or Exponential Temperature-Dependent Thermal
          Diffusivity .................................................................................................................. 173
  2.5.3. Equations of Heat and Mass Transfer in Anisotropic Media ............................... 174
  2.5.4. Other Equations with Three Space Variables .................................................... 177
  2.5.5. Equations with \( n \) Space Variables ..................................................................... 179

2.6. Nonlinear Schrödinger Equations ............................................................................ 186
  2.6.1. Two-Dimensional Equations ............................................................................... 186
  2.6.2. Three and \( n \)-Dimensional Equations ............................................................... 189

3. Hyperbolic Equations with One Space Variable ......................................................... 191
  3.1. Equations with Power-Law Nonlinearities ............................................................ 191
     3.1.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{n-1} \) ................. 191
     3.1.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \) ........................................ 193
     3.1.3. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w, \frac{\partial w}{\partial x}) \) ................. 196
     3.1.4. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = \frac{\partial f(x, \frac{\partial w}{\partial x}) + g(x, t, w, \frac{\partial w}{\partial x})}{\partial x^2} \) .................. 198
     3.1.5. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = aw^n \frac{\partial^2 w}{\partial x^2} + f(x, w) \) ........................................ 202
     3.1.6. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} (w^n \frac{\partial w}{\partial x}^2) + f(w) \) ...................... 204
     3.1.7. Other Equations .................................................................................................. 209

3.2. Equations with Exponential Nonlinearities ............................................................. 213
     3.2.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + be^{\alpha w} + ce^{\gamma w} \) ....................... 213
     3.2.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \) ........................................ 215
     3.2.3. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(x, \frac{\partial w}{\partial x}) + g(x, t, w, \frac{\partial w}{\partial x}) \) .................. 217
     3.2.4. Other Equations .................................................................................................. 222

3.3. Other Equations Involving Arbitrary Parameters .................................................... 225
     3.3.1. Equations with Hyperbolic Nonlinearities ......................................................... 225
     3.3.2. Equations with Logarithmic Nonlinearities ....................................................... 226
     3.3.3. Sine-Gordon Equation and Other Equations with Trigonometric Nonlinearities . 227
     3.3.4. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} + a \frac{\partial w}{\partial t} = \frac{\partial f(w)}{\partial x} \) ........................................ 230
     3.3.5. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} + f(w) \frac{\partial w}{\partial x} = \frac{\partial [g(w) \frac{\partial w}{\partial x}]}{\partial x} \) ................. 232

3.4. Equations Involving Arbitrary Functions ............................................................... 234
     3.4.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w) \) ........................................ 234
     3.4.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x, t, w, \frac{\partial w}{\partial x}) \) .................... 239
     3.4.3. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(x) \frac{\partial^2 w}{\partial x^2} + g(x, t, w, \frac{\partial w}{\partial x}) \) ............... 245
     3.4.4. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(w) \frac{\partial^2 w}{\partial x^2} + g(x, t, w, \frac{\partial w}{\partial x}) \) .................. 250
     3.4.5. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(x, w) \frac{\partial^2 w}{\partial x^2} + g(x, t, w, \frac{\partial w}{\partial x}) \) ............... 257
4. Hyperbolic Equations with Two or Three Space Variables

4.1. Equations with Two Space Variables Involving Power-Law Nonlinearities

4.1.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(t, w) \frac{\partial w}{\partial t} + g(x, t, w) \frac{\partial^2 w}{\partial x^2} \) ........................................ 275

4.1.2. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = a \frac{\partial w}{\partial t} [w^n \frac{\partial w}{\partial t}] + b \frac{\partial w}{\partial t} [w^k \frac{\partial w}{\partial t}] \) ........................................ 277

4.1.3. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(w) \frac{\partial w}{\partial x} + g(w) \frac{\partial w}{\partial y} \) ........................................ 285

4.1.4. Other Equations ........................................ 290

4.2. Equations with Two Space Variables Involving Exponential Nonlinearities

4.2.1. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = a \frac{\partial w}{\partial t} [e^{bw} \frac{\partial w}{\partial t}] + b \frac{\partial w}{\partial t} [e^{cw} \frac{\partial w}{\partial t}] \) ........................................ 294

4.2.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(w) \frac{\partial w}{\partial x} + g(w) \frac{\partial w}{\partial y} \) ........................................ 299

4.3. Equations Involving Arbitrary Functions

4.3.1. Equations Involving Power-Law Nonlinearities ........................................ 299

4.3.2. Equations Involving Exponential Nonlinearities ........................................ 303

4.4. Equations with Two Space Variables Involving Arbitrary Functions

4.4.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f(t, w) \frac{\partial w}{\partial t} + g(x, t, w) \frac{\partial^2 w}{\partial x^2} \) ........................................ 305

4.4.2. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = a \frac{\partial w}{\partial t} [w^n \frac{\partial w}{\partial t}] + b \frac{\partial w}{\partial t} [w^k \frac{\partial w}{\partial t}] \) ........................................ 308

4.4.3. Other Equations ........................................ 313

4.5. Equations with Three Space Variables Involving Arbitrary Parameters

4.5.1. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = f(x) \frac{\partial w}{\partial t} + g(y) \frac{\partial w}{\partial z} + h(z) \frac{\partial w}{\partial t} \) ........................................ 317

4.5.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = a \frac{\partial w}{\partial t} [e^{bw} \frac{\partial w}{\partial t}] + b \frac{\partial w}{\partial t} [e^{cw} \frac{\partial w}{\partial t}] + c \frac{\partial w}{\partial t} [e^{bw} \frac{\partial w}{\partial t}] + sw^p \) ........................................ 320

4.5.3. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = a \frac{\partial w}{\partial t} [e^{cw} \frac{\partial w}{\partial t}] + b \frac{\partial w}{\partial t} [e^{cw} \frac{\partial w}{\partial t}] + c \frac{\partial w}{\partial t} [e^{cw} \frac{\partial w}{\partial t}] + sw^p \) ........................................ 327

4.6. Equations with Three Space Variables Involving Arbitrary Functions

4.6.1. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} = f_1(x, y) \frac{\partial w}{\partial t} + f_2(y, z) \frac{\partial w}{\partial z} + f_3(z) \frac{\partial w}{\partial t} + g(w) \) ........................................ 334

4.6.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = f_1(t, w) \frac{\partial w}{\partial x} + f_2(t, w) \frac{\partial w}{\partial y} + f_3(t, w) \frac{\partial w}{\partial z} + g(w) \) ........................................ 338

4.6.3. Other Equations ........................................ 344

5. Elliptic Equations with Two Space Variables

5.1. Equations with Power-Law Nonlinearities ........................................ 347

5.1.1. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} + \alpha w = a w^n + b w \) ........................................ 347

5.1.2. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} + \alpha w = f(x, y, w) \) ........................................ 349

5.1.3. Equations of the Form \( \frac{\partial^2 w}{\partial x^2} + \alpha \frac{\partial w}{\partial t} = F(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}) \) ........................................ 350

5.1.4. Equations of the Form \( \frac{\partial}{\partial x} \left[ f_1(x, y) \frac{\partial w}{\partial t} + f_2(x, y) \frac{\partial w}{\partial y} \right] = g(w) \) ........................................ 351

5.1.5. Equations of the Form \( \frac{\partial}{\partial x} \left[ f_1(w) \frac{\partial w}{\partial t} + f_2(w) \frac{\partial w}{\partial y} \right] = g(w) \) ........................................ 353

5.1.6. Other Equations Involving Arbitrary Parameters ........................................ 358
5.2. Equations with Exponential Nonlinearities ................................................................. 364
5.2.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a + be^{\gamma w} + ce^{-\gamma w}$ .......... 364
5.2.2. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x, y, w)$ .......... 367
5.2.3. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x, y) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_2(x, y) \frac{\partial w}{\partial y} \right] = g(w)$ ............... 367
5.2.4. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(w) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_2(w) \frac{\partial w}{\partial y} \right] = g(w)$ ............... 370
5.2.5. Other Equations Involving Arbitrary Parameters ....................................................... 373
5.3. Equations Involving Other Nonlinearities ................................................................. 376
5.3.1. Equations with Hyperbolic Nonlinearities ................................................................. 376
5.3.2. Equations with Logarithmic Nonlinearities ............................................................... 377
5.3.3. Equations with Trigonometric Nonlinearities ............................................................ 380
5.4. Equations Involving Arbitrary Functions ....................................................................... 382
5.4.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = F(x, y, w)$ .......... 382
5.4.2. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x, y) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_2(x, y) \frac{\partial w}{\partial y} \right] = g(w)$ ............... 387
5.4.3. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ g(x, y, w) \frac{\partial w}{\partial y} \right] = h(x, y, w)$ ............... 391
5.4.4. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x, y) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ g(x, y, w) \frac{\partial w}{\partial y} \right] = h(x, y, w)$ ............... 393
5.4.5. Other Equations ........................................................................................................... 399

6. Elliptic Equations with Three or More Space Variables .................................................... 405
6.1. Equations with Three Space Variables Involving Power-Law Nonlinearities ............ 405
6.1.1. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ g(y) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[ h(z) \frac{\partial w}{\partial z} \right] = u^{\beta w}$ .......... 405
6.1.2. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ g(y) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[ g(y) \frac{\partial w}{\partial z} \right] = 0$ .......... 408
6.2. Equations with Three Space Variables Involving Exponential Nonlinearities .......... 413
6.2.1. Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ g(y) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[ h(z) \frac{\partial w}{\partial z} \right] = e^{\lambda w}$ .......... 413
6.2.2. Equations of the Form $\alpha_1 \frac{\partial}{\partial x} \left( e^{\lambda_1 w} \frac{\partial w}{\partial x} \right) + \alpha_2 \frac{\partial}{\partial y} \left( e^{\lambda_2 w} \frac{\partial w}{\partial y} \right) + \alpha_3 \frac{\partial}{\partial z} \left( e^{\lambda_3 w} \frac{\partial w}{\partial z} \right) = b e^{\beta w}$ .......... 416
6.3. Three-Dimensional Equations Involving Arbitrary Functions ........................................ 420
6.3.1. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x} \left[ f_1(x, y) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[ f_2(y, w) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[ f_3(z, w) \frac{\partial w}{\partial z} \right] = g(w)$ .......... 420
6.3.2. Heat and Mass Transfer Equations with Complicating Factors ................................. 423
6.3.3. Other Equations ........................................................................................................... 426
6.4. Equations with $n$ Independent Variables ...................................................................... 428
6.4.1. Equations of the Form $\frac{\partial}{\partial x_1} \left[ f_1(x_1) \frac{\partial w}{\partial x_1} \right] + \cdots + \frac{\partial}{\partial x_n} \left[ f_n(x_n) \frac{\partial w}{\partial x_n} \right] = g(x_1, \ldots, x_n, w)$ .......... 428
6.4.2. Other Equations ........................................................................................................... 430

7. Equations Involving Mixed Derivatives and Some Other Equations .................................. 433
7.1. Equations Linear in the Mixed Derivative ......................................................................... 433
7.1.1. Calogero Equation ....................................................................................................... 433
7.1.2. Khokhlov–Zabolotskaya Equation ............................................................................... 435
7.1.3. Equation of Unsteady Transonic Gas Flows ............................................................... 440
7.1.4. Equations of the Form $\frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2 \partial y} \frac{\partial^2 w}{\partial x \partial y^2} = F(x, y, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$ .......... 443
7.1.5. Other Equations with Two Independent Variables .................................................... 445
7.1.6. Other Equations with Three Independent Variables .................................................... 448
7.2. Equations Quadratic in the Highest Derivatives ............................................................. 449
7.2.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} = F(x, y)$ .......... 449
7.2.2. Monge–Ampère equation $\left( \frac{\partial^2 w}{\partial x^2} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} = F(x, y)$ ............... 451
7.2.3. Equations of the Form $\left( \frac{\partial^2 w}{\partial x^2} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} = F(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$ .......... 461
7.2. Equations of the Form \( \left( \frac{\partial w}{\partial y} \right)^2 = f(x, y) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + g(x, y) \) ................................. 465
7.2.5. Other Equations ........................................... 469

7.3. Bellman Type Equations and Related Equations ........................................... 472
7.3.1. Equations with Quadratic Nonlinearities ........................................ 472
7.3.2. Equations with Power-Law Nonlinearities ......................................... 475

8. Second-Order Equations of General Form ........................................... 479
8.1. Equations Involving the First Derivative in \( t \) ................................. 479
8.1.1. Equations of the Form \( \frac{\partial w}{\partial t} = F \left( w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 479
8.1.2. Equations of the Form \( \frac{\partial w}{\partial t} = F \left( t, w, \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 486
8.1.3. Equations of the Form \( \frac{\partial w}{\partial t} = F \left( x, w, \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 490
8.1.4. Equations of the Form \( \frac{\partial w}{\partial t} = F \left( x, t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 494
8.1.5. Equations of the Form \( F \left( x, t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \right) = 0 \) ........................................... 499
8.1.6. Equations with Three Independent Variables ........................................ 500
8.2. Equations Involving Two or More Second Derivatives ........................................... 501
8.2.1. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = F \left( w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 501
8.2.2. Equations of the Form \( \frac{\partial^2 w}{\partial t^2} = F \left( x, t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \right) \) ........................................... 505
8.2.3. Equations Linear in the Mixed Derivative ......................................... 508
8.2.4. Equations with Two Independent Variables, Nonlinear in Two or More Highest Derivatives ................................................................. 509
8.2.5. Equations with \( n \) Independent Variables ........................................... 512

9. Third-Order Equations ........................................... 515
9.1. Equations Involving the First Derivative in \( t \) ........................................... 515
9.1.1. Korteweg–de Vries Equation \( \frac{\partial w}{\partial t} + a \frac{\partial^2 w}{\partial x^2} + bw \frac{\partial w}{\partial x} = 0 \) ........................................... 515
9.1.2. Cylindrical, Spherical, and Modified Korteweg–de Vries Equations ........................................... 521
9.1.3. Generalized Korteweg–de Vries Equation \( \frac{\partial w}{\partial t} + a \frac{\partial^2 w}{\partial x^2} + f(w) \frac{\partial w}{\partial x} = 0 \) ........................................... 524
9.1.4. Equations Reducible to the Korteweg–de Vries Equation ........................................... 526
9.1.5. Equations of the Form \( \frac{\partial w}{\partial t} + a \frac{\partial^2 w}{\partial x^2} + f \left( w, \frac{\partial w}{\partial x} \right) = 0 \) ........................................... 529
9.1.6. Equations of the Form \( \frac{\partial w}{\partial t} + a \frac{\partial^2 w}{\partial x^2} + F \left( x, t, w, \frac{\partial w}{\partial x} \right) = 0 \) ........................................... 530
9.1.7. Burgers–Korteweg–de Vries Equation and Other Equations ........................................... 532
9.2. Equations Involving the Second Derivative in \( t \) ........................................... 536
9.2.1. Equations with Quadratic Nonlinearities ........................................ 536
9.2.2. Other Equations ........................................... 539

9.3. Hydrodynamic Boundary Layer Equations ........................................... 540
9.3.1. Steady Hydrodynamic Boundary Layer Equations for a Newtonian Fluid ........................................... 540
9.3.2. Steady Boundary Layer Equations for Non-Newtonian Fluids ........................................... 547
9.3.3. Unsteady Boundary Layer Equations for a Newtonian Fluid ........................................... 553
9.3.4. Unsteady Boundary Layer Equations for Non-Newtonian Fluids ........................................... 564
9.3.5. Related Equations ........................................... 568

9.4. Equations of Motion of Ideal Fluid (Euler Equations) ........................................... 570
9.4.1. Stationary Equations ........................................... 570
9.4.2. Nonstationary Equations ........................................... 574

9.5. Other Third-Order Nonlinear Equations ........................................... 580
9.5.1. Equations Involving Second-Order Mixed Derivatives ........................................... 580
9.5.2. Equations Involving Third-Order Mixed Derivatives ........................................... 583
9.5.3. Equations Involving \( \frac{\partial^2 w}{\partial y^2} \) and \( \frac{\partial^2 w}{\partial x^2} \) ........................................... 587
10. **Fourth-Order Equations** .......................................................... 589

10.1. Equations Involving the First Derivative in t .......................... 589
10.1.1. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x}) \) ... 589
10.1.2. Other Equations ................................................................. 593
10.2. Equations Involving the Second Derivative in t ..................... 595
10.2.1. Boussinesq Equation and Its Modifications ....................... 595
10.2.2. Equations with Quadratic Nonlinearities ......................... 600
10.2.3. Other Equations ............................................................... 603
10.3. Equations Involving Mixed Derivatives .................................. 605
10.3.1. Kdovmptsev–Petviashvili Equation .................................. 605
10.3.2. Stationary Hydrodynamic Equations (Navier–Stokes Equations) .... 607
10.3.3. Nonstationary Hydrodynamic Equations (Navier–Stokes equations) .... 616
10.3.4. Other Equations ............................................................... 628
10.4. Fifth-Order Equations ........................................................... 631
10.4.1. Equations Involving Mixed Derivatives ............................. 631
10.4.2. Equations of the Form \( \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \) ............... 631
10.4.3. Equations of the Form \( \frac{\partial^{m} w}{\partial x^{m}} + g(x, t, w) \) ............... 633
10.4.4. Equations of the Form \( \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x}) \) ..... 635
11. **Equations of Higher Orders** ............................................... 631

11.1. Equations Involving the First Derivative in t and Linear in the Highest Derivative .... 631
11.1.1. Fifth-Order Equations ......................................................... 631
11.1.2. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \) ... 633
11.1.3. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + \frac{\partial w}{\partial x} \) ... 635
11.1.4. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w) \) ... 637
11.1.5. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x}) \) ...... 639
11.1.6. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x} \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) ... 645
11.1.7. Equations of the Form \( \frac{\partial w}{\partial t} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w) \) ... 647
11.1.8. Other Equations ............................................................... 648
11.2. General Form Equations Involving the First Derivative in t ........ 651
11.2.1. Equations of the Form \( \frac{\partial w}{\partial t} = F(w, \frac{\partial w}{\partial x}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 651
11.2.2. Equations of the Form \( \frac{\partial w}{\partial t} = F(t, w, \frac{\partial w}{\partial x}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 656
11.2.3. Equations of the Form \( \frac{\partial w}{\partial t} = F(x, w, \frac{\partial w}{\partial x}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 659
11.2.4. Equations of the Form \( \frac{\partial w}{\partial x} = F(x, t, w, \frac{\partial w}{\partial t}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 662
11.3. Equations Involving the Second Derivative in t ....................... 666
11.3.1. Equations of the Form \( \frac{\partial^{2} w}{\partial x^{2}} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \) .... 666
11.3.2. Equations of the Form \( \frac{\partial^{2} w}{\partial x^{2}} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x}) \) .... 667
11.3.3. Equations of the Form \( \frac{\partial^{2} w}{\partial x^{2}} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + F(x, t, w, \frac{\partial w}{\partial x}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 671
11.3.4. Equations of the Form \( \frac{\partial^{2} w}{\partial x^{2}} = a^{m} \frac{\partial^{m} w}{\partial x^{m}} + f(x, t, w) \frac{\partial w}{\partial x} + g(x, t, w) \) ... 673
11.3.5. Equations of the Form \( \frac{\partial^{2} w}{\partial x^{2}} = F(x, t, w, \frac{\partial w}{\partial x}, \ldots \frac{\partial^{m} w}{\partial x^{m}}) \) .... 675
11.4. Other Equations ............................................................... 676
11.4.1. Equations Involving Mixed Derivatives ............................. 676
11.4.2. Equations Involving \( \frac{\partial^{m} w}{\partial x^{m}} \) and \( \frac{\partial^{m} w}{\partial y^{m}} \) .................. 680

**Supplements.** **Exact Methods for Solving Nonlinear Partial Differential Equations** .... 683

S.1. Classification of Second-Order Semilinear Partial Differential Equations in Two Independent Variables .......................................................... 683
S.1.1. Types of Equations. Characteristic Equation ....................... 683
S.1.2. Canonical Form of Parabolic Equations .............................. 683
S.1.3. Canonical Form of Hyperbolic Equations ............................ 684
S.1.4. Canonical Form of Elliptic Equations ............................... 684
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.2.</td>
<td>Transformations of Equations of Mathematical Physics</td>
<td>685</td>
</tr>
<tr>
<td>S.2.1.</td>
<td>Point Transformations</td>
<td>685</td>
</tr>
<tr>
<td>S.2.2.</td>
<td>Hodograph Transformation</td>
<td>686</td>
</tr>
<tr>
<td>S.2.3.</td>
<td>Contact Transformations. Legendre and Euler Transformations</td>
<td>688</td>
</tr>
<tr>
<td>S.2.4.</td>
<td>Bäcklund Transformations. Differential Substitutions</td>
<td>690</td>
</tr>
<tr>
<td>S.3.</td>
<td>Traveling-Wave Solutions and Self-Similar Solutions. Similarity Methods</td>
<td>693</td>
</tr>
<tr>
<td>S.3.1.</td>
<td>Preliminary Remarks</td>
<td>693</td>
</tr>
<tr>
<td>S.3.2.</td>
<td>Traveling-Wave Solutions. Invariance of Equations Under Translations</td>
<td>694</td>
</tr>
<tr>
<td>S.3.3.</td>
<td>Self-Similar Solutions. Invariance of Equations Under Scaling Transformations</td>
<td>695</td>
</tr>
<tr>
<td>S.3.4.</td>
<td>Exponential Self-Similar Solutions. Equations Invariant Under Combined Translation and Scaling</td>
<td>696</td>
</tr>
<tr>
<td>S.4.</td>
<td>Method of Generalized Separation of Variables</td>
<td>698</td>
</tr>
<tr>
<td>S.4.1.</td>
<td>Introduction</td>
<td>698</td>
</tr>
<tr>
<td>S.4.2.</td>
<td>Structure of Generalized Separable Solutions</td>
<td>700</td>
</tr>
<tr>
<td>S.4.3.</td>
<td>Solution of Functional-Differential Equations by Differentiation</td>
<td>701</td>
</tr>
<tr>
<td>S.4.4.</td>
<td>Solution of Functional-Differential Equations by Splitting</td>
<td>705</td>
</tr>
<tr>
<td>S.4.5.</td>
<td>Simplified Scheme for Constructing Generalized Separable Solutions</td>
<td>709</td>
</tr>
<tr>
<td>S.4.6.</td>
<td>Titov–Galaktionov Method</td>
<td>710</td>
</tr>
<tr>
<td>S.5.</td>
<td>Method of Functional Separation of Variables</td>
<td>713</td>
</tr>
<tr>
<td>S.5.1.</td>
<td>Structure of Functional Separable Solutions</td>
<td>713</td>
</tr>
<tr>
<td>S.5.2.</td>
<td>Special Functional Separable Solutions</td>
<td>713</td>
</tr>
<tr>
<td>S.5.3.</td>
<td>Differentiation Method</td>
<td>718</td>
</tr>
<tr>
<td>S.5.4.</td>
<td>Splitting Method. Reduction to a Functional Equation with Two Variables</td>
<td>721</td>
</tr>
<tr>
<td>S.5.5.</td>
<td>Solutions of Some Nonlinear Functional Equations and Their Applications</td>
<td>723</td>
</tr>
<tr>
<td>S.6.</td>
<td>Generalized Similarity Reductions of Nonlinear Equations</td>
<td>728</td>
</tr>
<tr>
<td>S.6.1.</td>
<td>Clarkson–Kruskal Direct Method: a Special Form for Similarity Reduction</td>
<td>728</td>
</tr>
<tr>
<td>S.6.2.</td>
<td>Clarkson–Kruskal Direct Method: the General Form for Similarity Reduction</td>
<td>731</td>
</tr>
<tr>
<td>S.6.3.</td>
<td>Some Modifications and Generalizations</td>
<td>732</td>
</tr>
<tr>
<td>S.7.</td>
<td>Group Analysis Methods</td>
<td>735</td>
</tr>
<tr>
<td>S.7.1.</td>
<td>Classical Method for Symmetry Reductions</td>
<td>735</td>
</tr>
<tr>
<td>S.7.2.</td>
<td>Nonclassical Method for Symmetry Reductions</td>
<td>744</td>
</tr>
<tr>
<td>S.8.</td>
<td>Differential Constraints Method</td>
<td>747</td>
</tr>
<tr>
<td>S.8.1.</td>
<td>Description of the Method</td>
<td>747</td>
</tr>
<tr>
<td>S.8.2.</td>
<td>First-Order Differential Constraints</td>
<td>749</td>
</tr>
<tr>
<td>S.8.3.</td>
<td>Second- and Higher-Order Differential Constraints</td>
<td>754</td>
</tr>
<tr>
<td>S.8.4.</td>
<td>Connection Between the Differential Constraints Method and Other Methods</td>
<td>756</td>
</tr>
<tr>
<td>S.9.</td>
<td>Painlevé Test for Nonlinear Equations of Mathematical Physics</td>
<td>758</td>
</tr>
<tr>
<td>S.9.1.</td>
<td>Movable Singularities of Solutions of Ordinary Differential Equations</td>
<td>758</td>
</tr>
<tr>
<td>S.9.2.</td>
<td>Solutions of Partial Differential Equations with a Movable Pole. Description of the Method</td>
<td>760</td>
</tr>
<tr>
<td>S.9.3.</td>
<td>Examples of the Painlevé Test Applications</td>
<td>761</td>
</tr>
<tr>
<td>S.10.</td>
<td>Inverse Scattering Method</td>
<td>764</td>
</tr>
<tr>
<td>S.10.1.</td>
<td>Lax Pair Method</td>
<td>764</td>
</tr>
<tr>
<td>S.10.2.</td>
<td>Method Based on the Compatibility Condition for Two Linear Equations</td>
<td>766</td>
</tr>
<tr>
<td>S.10.3.</td>
<td>Method Based on Linear Integral Equations</td>
<td>767</td>
</tr>
<tr>
<td>S.11.</td>
<td>Conservation Laws</td>
<td>769</td>
</tr>
<tr>
<td>S.11.1.</td>
<td>Basic Definitions and Examples</td>
<td>769</td>
</tr>
<tr>
<td>S.11.2.</td>
<td>Equations Admitting Variational Formulation. Noetherian Symmetries</td>
<td>770</td>
</tr>
</tbody>
</table>
S.12. Hyperbolic Systems of Quasilinear Equations ........................................... 772
  S.12.2. Cauchy Problem, Riemann Problem, and Initial-Boundary Value Problem ... 773
  S.12.4. Self-Similar Continuous Solutions. Rarefaction Waves ......................... 777
  S.12.5. Shock Waves. Rankine–Hugoniot Jump Conditions ............................... 779
  S.12.6. Evolutionary Shocks. Lax Condition (Various Formulations) ................. 780
  S.12.7. Solutions for the Riemann Problem .................................................... 782
  S.12.8. Initial-Boundary Value Problems of Special Form ............................... 786
  S.12.9. Examples of Nonstrict Hyperbolic Systems ......................................... 786

References ............................................................................................................. 791

Index ..................................................................................................................... 809