CONTENTS

Authors . . . . . **xv**

Foreword . . . . . **xvii**

Some Notations and Remarks . . . . . **xix**

1. Parabolic Equations with One Space Variable **1**

1.1. Equations with Power-Law Nonlinearities 1
1.1.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw + cw^2$ 1
1.1.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3$ 2
1.1.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(w)$ 6
1.1.4. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 9
1.1.5. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(w)\frac{\partial w}{\partial x} + g(w)$ 9
1.1.6. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$ 13
1.1.7. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\big(\frac{\partial w}{\partial x}\big)^2 + f(x,t,w)$ 15
1.1.8. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 17
1.1.9. Equations of the Form $\frac{\partial w}{\partial t} = aw^k\frac{\partial^2 w}{\partial x^2} + f\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 18
1.1.10. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w^m\frac{\partial w}{\partial x}\big)$ 25
1.1.11. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w^m\frac{\partial w}{\partial x}\big) + bw^k$ 32
1.1.12. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w^m\frac{\partial w}{\partial x}\big) + bw + c_1 w^{k_1} + c_2 w^{k_2} + c_3 w^{k_3}$ 37
1.1.13. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + g(w)$ 40
1.1.14. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 42
1.1.15. Other Equations 46

1.2. Equations with Exponential Nonlinearities 52
1.2.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b_0 + b_1 e^{\lambda w} + b_2 e^{2\lambda w}$ 52
1.2.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(e^{\lambda w}\frac{\partial w}{\partial x}\big) + f(w)$ 53
1.2.3. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + g(w)$ 56
1.2.4. Other Equations Explicitly Independent of x and t 58
1.2.5. Equations Explicitly Dependent on x and/or t 61

1.3. Equations with Hyperbolic Nonlinearities 62
1.3.1. Equations Involving Hyperbolic Cosine 62
1.3.2. Equations Involving Hyperbolic Sine 63
1.3.3. Equations Involving Hyperbolic Tangent 63
1.3.4. Equations Involving Hyperbolic Cotangent 64

1.4. Equations with Logarithmic Nonlinearities 64
1.4.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 64
1.4.2. Other Equations 66

1.5. Equations with Trigonometric Nonlinearities 68
1.5.1. Equations Involving Cosine 68
1.5.2. Equations Involving Sine 69
1.5.3. Equations Involving Tangent 70
1.5.4. Equations Involving Cotangent 70
1.5.5. Equations Involving Inverse Trigonometric Functions 71

1.6. Equations Involving Arbitrary Functions 71
1.6.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 71
1.6.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$ 75
1.6.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$ 78
1.6.4. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\big(\frac{\partial w}{\partial x}\big)^2 + f(x,t,w)$ 81
1.6.5. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\big(\frac{\partial w}{\partial x}\big)^2 + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$ 83
1.6.6. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\big(\frac{\partial w}{\partial x}\big)^2 + g(x,t,w)\frac{\partial w}{\partial x} + h(x,t,w)$ 84
1.6.7. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 88
1.6.8. Equations of the Form $\frac{\partial w}{\partial t} = f(x,t)\frac{\partial^2 w}{\partial x^2} + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 89
1.6.9. Equations of the Form $\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$ 92
1.6.10. Equations of the Form $\frac{\partial w}{\partial t} = (aw+b)\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\big(\frac{\partial w}{\partial x}\big)^2 + g(x,t,w)\frac{\partial w}{\partial x} + h(x,t,w)$ 94
1.6.11. Equations of the Form $\frac{\partial w}{\partial t} = aw^m\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$ 97
1.6.12. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w\frac{\partial w}{\partial x}\big) + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$ 98
1.6.13. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w^m\frac{\partial w}{\partial x}\big) + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$ 100
1.6.14. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(e^{\lambda w}\frac{\partial w}{\partial x}\big) + f(x,t,w)$ 102
1.6.15. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 104
1.6.16. Equations of the Form $\frac{\partial w}{\partial t} = f(x,w)\frac{\partial^2 w}{\partial x^2}$ 111
1.6.17. Equations of the Form $\frac{\partial w}{\partial t} = f(x,t,w)\frac{\partial^2 w}{\partial x^2} + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 113
1.6.18. Equations of the Form $\frac{\partial w}{\partial t} = f\big(x,w,\frac{\partial w}{\partial x}\big)\frac{\partial^2 w}{\partial x^2} + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 120
1.6.19. Nonlinear Equations of the Thermal (Diffusion) Boundary Layer 124

1.7. Nonlinear Schrödinger Equations and Related Equations 125
1.7.1. Equations of the Form $i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + f(|w|)w = 0$ Involving Arbitrary Parameters 125
1.7.2. Equations of the Form $i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\big(x^n\frac{\partial w}{\partial x}\big) + f(|w|)w = 0$ Involving Arbitrary Parameters 128
1.7.3. Other Equations Involving Arbitrary Parameters 130
1.7.4. Equations with Cubic Nonlinearities Involving Arbitrary Functions 131
1.7.5. Equations of General Form Involving Arbitrary Functions of a Single Argument 134
1.7.6. Equations of General Form Involving Arbitrary Functions of Two Arguments . . 137

2. Parabolic Equations with Two or More Space Variables 141

2.1. Equations with Two Space Variables Involving Power-Law Nonlinearities 141
2.1.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + aw^p$ 141
2.1.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(w^n\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(w^k\frac{\partial w}{\partial y}\big)$ 142
2.1.3. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(w)\frac{\partial w}{\partial y}\big] + h(w)$ 147
2.1.4. Other Equations 149

2.2. Equations with Two Space Variables Involving Exponential Nonlinearities 154
2.2.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + ae^{\lambda w}$ 154
2.2.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\big(e^{\beta w}\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(e^{\lambda w}\frac{\partial w}{\partial y}\big) + f(w)$ 155

2.3. Other Equations with Two Space Variables Involving Arbitrary Parameters 157
2.3.1. Equations with Logarithmic Nonlinearities 157
2.3.2. Equations with Trigonometrical Nonlinearities 158

2.4. Equations Involving Arbitrary Functions 159
2.4.1. Heat and Mass Transfer Equations in Quiescent or Moving Media with Chemical Reactions 159
2.4.2. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + h(w)$ 161
2.4.3. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + h(t,w)$ 162
2.4.4. Other Equations Linear in the Highest Derivatives 165
2.4.5. Nonlinear Diffusion Boundary Layer Equations 168

2.5. Equations with Three or More Space Variables 169
2.5.1. Equations of Mass Transfer in Quiescent or Moving Media with Chemical Reactions 169
2.5.2. Heat Equations with Power-Law or Exponential Temperature-Dependent Thermal Diffusivity 173
2.5.3. Equations of Heat and Mass Transfer in Anisotropic Media 174
2.5.4. Other Equations with Three Space Variables 177
2.5.5. Equations with n Space Variables 179

2.6. Nonlinear Schrödinger Equations 186
2.6.1. Two-Dimensional Equations 186
2.6.2. Three and n-Dimensional Equations 189

3. Hyperbolic Equations with One Space Variable 191

3.1. Equations with Power-Law Nonlinearities 191
3.1.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{2n-1}$ 191
3.1.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 193
3.1.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 196
3.1.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(x)\frac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 198
3.1.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = aw^n\frac{\partial^2 w}{\partial x^2} + f(x,w)$ 202
3.1.6. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + f(w)$ 204
3.1.7. Other Equations 209

3.2. Equations with Exponential Nonlinearities 213
3.2.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + be^{\beta w} + ce^{\gamma w}$ 213
3.2.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 215
3.2.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(x)\frac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 217
3.2.4. Other Equations 222

3.3. Other Equations Involving Arbitrary Parameters 225
3.3.1. Equations with Hyperbolic Nonlinearities 225
3.3.2. Equations with Logarithmic Nonlinearities 226
3.3.3. Sine-Gordon Equation and Other Equations with Trigonometric Nonlinearities . 227
3.3.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$ 230
3.3.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + f(w)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[g(w)\frac{\partial w}{\partial x}\right]$ 232

3.4. Equations Involving Arbitrary Functions 234
3.4.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)$ 234
3.4.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 239
3.4.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(x)\frac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 245
3.4.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(w)\frac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 250
3.4.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(x,w)\frac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\frac{\partial w}{\partial x}\right)$ 257

3.4.6. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = f(t,w)\frac{\partial^2 w}{\partial x^2} + g\big(x,t,w,\frac{\partial w}{\partial x}\big)$ 259
3.4.7. Other Equations Linear in the Highest Derivatives 260

3.5. Equations of the Form $\frac{\partial^2 w}{\partial x \partial y} = F\big(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\big)$ 266

3.5.1. Equations Involving Arbitrary Parameters of the Form $\frac{\partial^2 w}{\partial x \partial y} = f(w)$ 266
3.5.2. Other Equations Involving Arbitrary Parameters 270
3.5.3. Equations Involving Arbitrary Functions 271

4. Hyperbolic Equations with Two or Three Space Variables 275

4.1. Equations with Two Space Variables Involving Power-Law Nonlinearities 275
4.1.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + aw^p$ 275
4.1.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\big(w^n\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(w^k\frac{\partial w}{\partial y}\big)$ 277
4.1.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(w)\frac{\partial w}{\partial y}\big]$ 285
4.1.4. Other Equations 290

4.2. Equations with Two Space Variables Involving Exponential Nonlinearities 292
4.2.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + ae^{\lambda w}$ 292
4.2.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\big(e^{\beta w}\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(e^{\lambda w}\frac{\partial w}{\partial y}\big)$ 294
4.2.3. Other Equations 299

4.3. Nonlinear Telegraph Equations with Two Space Variables 299
4.3.1. Equations Involving Power-Law Nonlinearities 299
4.3.2. Equations Involving Exponential Nonlinearities 303

4.4. Equations with Two Space Variables Involving Arbitrary Functions 305
4.4.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + h(w)$ 305
4.4.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(w)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(w)\frac{\partial w}{\partial y}\big] + h(w)$ 308
4.4.3. Other Equations 313

4.5. Equations with Three Space Variables Involving Arbitrary Parameters 317
4.5.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + \frac{\partial}{\partial z}\big[h(z)\frac{\partial w}{\partial z}\big] + aw^p$ 317
4.5.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[g(y)\frac{\partial w}{\partial y}\big] + \frac{\partial}{\partial z}\big[h(z)\frac{\partial w}{\partial z}\big] + ae^{\lambda w}$ 318
4.5.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\big(w^n\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(w^m\frac{\partial w}{\partial y}\big) + c\frac{\partial}{\partial z}\big(w^k\frac{\partial w}{\partial z}\big) + sw^p$ 320
4.5.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\big(e^{\lambda_1 w}\frac{\partial w}{\partial x}\big) + b\frac{\partial}{\partial y}\big(e^{\lambda_2 w}\frac{\partial w}{\partial y}\big) + c\frac{\partial}{\partial z}\big(e^{\lambda_3 w}\frac{\partial w}{\partial z}\big) + se^{\beta w}$ 327

4.6. Equations with Three Space Variables Involving Arbitrary Functions 334
4.6.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f_1(x)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[f_2(y)\frac{\partial w}{\partial y}\big] + \frac{\partial}{\partial z}\big[f_3(z)\frac{\partial w}{\partial z}\big] + g(w)$ 334
4.6.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[f_1(w)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[f_2(w)\frac{\partial w}{\partial y}\big] + \frac{\partial}{\partial z}\big[f_3(w)\frac{\partial w}{\partial z}\big] + g(w)$ 338
4.6.3. Other Equations 344

5. Elliptic Equations with Two Space Variables 347

5.1. Equations with Power-Law Nonlinearities 347
5.1.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = aw + bw^n + cw^{2n-1}$ 347
5.1.2. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y,w)$ 349
5.1.3. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + a\frac{\partial^2 w}{\partial y^2} = F\big(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\big)$ 350
5.1.4. Equations of the Form $\frac{\partial}{\partial x}\big[f_1(x,y)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[f_2(x,y)\frac{\partial w}{\partial y}\big] = g(w)$ 351
5.1.5. Equations of the Form $\frac{\partial}{\partial x}\big[f_1(w)\frac{\partial w}{\partial x}\big] + \frac{\partial}{\partial y}\big[f_2(w)\frac{\partial w}{\partial y}\big] = g(w)$ 353
5.1.6. Other Equations Involving Arbitrary Parameters 358

5.2. Equations with Exponential Nonlinearities 364
5.2.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=a+be^{\beta w}+ce^{\gamma w}$ 364
5.2.2. Equations of the Form $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y,w)$ 367
5.2.3. Equations of the Form $\frac{\partial}{\partial x}\left[f_1(x,y)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(x,y)\frac{\partial w}{\partial y}\right]=g(w)$ 367
5.2.4. Equations of the Form $\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(w)\frac{\partial w}{\partial y}\right]=g(w)$ 370
5.2.5. Other Equations Involving Arbitrary Parameters 373
5.3. Equations Involving Other Nonlinearities 376
5.3.1. Equations with Hyperbolic Nonlinearities 376
5.3.2. Equations with Logarithmic Nonlinearities 377
5.3.3. Equations with Trigonometric Nonlinearities 380
5.4. Equations Involving Arbitrary Functions 382
5.4.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=F(x,y,w)$ 382
5.4.2. Equations of the Form $a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}=F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)$ 387
5.4.3. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]=h(w)$ 391
5.4.4. Equations of the Form $\frac{\partial}{\partial x}\left[f(x,y,w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(x,y,w)\frac{\partial w}{\partial y}\right]=h(x,y,w)$ 393
5.4.5. Other Equations 399

6. Elliptic Equations with Three or More Space Variables 405

6.1. Equations with Three Space Variables Involving Power-Law Nonlinearities 405
6.1.1. Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right]=aw^p$ 405
6.1.2. Equations of the Form $\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[g(w)\frac{\partial w}{\partial z}\right]=0$ 408
6.2. Equations with Three Space Variables Involving Exponential Nonlinearities 413
6.2.1. Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right]=ae^{\lambda w}$ 413
6.2.2. Equations of the Form $a_1\frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right)+a_2\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right)+a_3\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right)=be^{\beta w}$ 416
6.3. Three-Dimensional Equations Involving Arbitrary Functions 420
6.3.1. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[f_3(z)\frac{\partial w}{\partial z}\right]=g(w)$ 420
6.3.2. Heat and Mass Transfer Equations with Complicating Factors 423
6.3.3. Other Equations 426
6.4. Equations with n Independent Variables 428
6.4.1. Equations of the Form $\frac{\partial}{\partial x_1}\left[f_1(x_1)\frac{\partial w}{\partial x_1}\right]+\cdots+\frac{\partial}{\partial x_n}\left[f_n(x_n)\frac{\partial w}{\partial x_n}\right]=g(x_1,\ldots,x_n,w)$ 428
6.4.2. Other Equations 430

7. Equations Involving Mixed Derivatives and Some Other Equations 433

7.1. Equations Linear in the Mixed Derivative 433
7.1.1. Calogero Equation 433
7.1.2. Khokhlov–Zabolotskaya Equation 435
7.1.3. Equation of Unsteady Transonic Gas Flows 440
7.1.4. Equations of the Form $\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=F\left(x,y,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)$ 443
7.1.5. Other Equations with Two Independent Variables 445
7.1.6. Other Equations with Three Independent Variables 448
7.2. Equations Quadratic in the Highest Derivatives 449
7.2.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=F(x,y)$ 449
7.2.2. Monge–Ampère equation $\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=F(x,y)$ 451
7.2.3. Equations of the Form $\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)$ 461

7.2.4. Equations of the Form $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x,y)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x,y)$ 465
7.2.5. Other Equations 469

7.3. Bellman Type Equations and Related Equations 472
7.3.1. Equations with Quadratic Nonlinearities 472
7.3.2. Equations with Power-Law Nonlinearities 475

8. Second-Order Equations of General Form 479

8.1. Equations Involving the First Derivative in t 479
8.1.1. Equations of the Form $\frac{\partial w}{\partial t} = F\left(w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$ 479
8.1.2. Equations of the Form $\frac{\partial w}{\partial t} = F\left(t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$ 486
8.1.3. Equations of the Form $\frac{\partial w}{\partial t} = F\left(x, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$ 490
8.1.4. Equations of the Form $\frac{\partial w}{\partial t} = F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$ 494
8.1.5. Equations of the Form $F\left(x, t, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) = 0$ 499
8.1.6. Equations with Three Independent Variables 500

8.2. Equations Involving Two or More Second Derivatives 501
8.2.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = F\left(w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$ 501
8.2.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}\right)$ 505
8.2.3. Equations Linear in the Mixed Derivative 508
8.2.4. Equations with Two Independent Variables, Nonlinear in Two or More Highest Derivatives 509
8.2.5. Equations with n Independent Variables 512

9. Third-Order Equations 515

9.1. Equations Involving the First Derivative in t 515
9.1.1. Korteweg–de Vries Equation $\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + bw\frac{\partial w}{\partial x} = 0$ 515
9.1.2. Cylindrical, Spherical, and Modified Korteweg–de Vries Equations 521
9.1.3. Generalized Korteweg–de Vries Equation $\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + f(w)\frac{\partial w}{\partial x} = 0$ 524
9.1.4. Equations Reducible to the Korteweg–de Vries Equation 526
9.1.5. Equations of the Form $\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + f\left(w, \frac{\partial w}{\partial x}\right) = 0$ 529
9.1.6. Equations of the Form $\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + F\left(x, t, w, \frac{\partial w}{\partial x}\right) = 0$ 530
9.1.7. Burgers–Korteweg–de Vries Equation and Other Equations 532

9.2. Equations Involving the Second Derivative in t 536
9.2.1. Equations with Quadratic Nonlinearities 536
9.2.2. Other Equations 539

9.3. Hydrodynamic Boundary Layer Equations 540
9.3.1. Steady Hydrodynamic Boundary Layer Equations for a Newtonian Fluid 540
9.3.2. Steady Boundary Layer Equations for Non-Newtonian Fluids 547
9.3.3. Unsteady Boundary Layer Equations for a Newtonian Fluid 553
9.3.4. Unsteady Boundary Layer Equations for Non-Newtonian Fluids 564
9.3.5. Related Equations 568

9.4. Equations of Motion of Ideal Fluid (Euler Equations) 570
9.4.1. Stationary Equations 570
9.4.2. Nonstationary Equations 574

9.5. Other Third-Order Nonlinear Equations 580
9.5.1. Equations Involving Second-Order Mixed Derivatives 580
9.5.2. Equations Involving Third-Order Mixed Derivatives 583
9.5.3. Equations Involving $\frac{\partial^3 w}{\partial x^3}$ and $\frac{\partial^3 w}{\partial y^3}$ 587

10. Fourth-Order Equations **589**

10.1. Equations Involving the First Derivative in t 589
10.1.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^4 w}{\partial x^4} + F\big(x, t, w, \frac{\partial w}{\partial x}\big)$ 589
10.1.2. Other Equations 593

10.2. Equations Involving the Second Derivative in t 595
10.2.1. Boussinesq Equation and Its Modifications 595
10.2.2. Equations with Quadratic Nonlinearities 600
10.2.3. Other Equations 603

10.3. Equations Involving Mixed Derivatives 605
10.3.1. Kadomtsev–Petviashvili Equation 605
10.3.2. Stationary Hydrodynamic Equations (Navier–Stokes Equations) 607
10.3.3. Nonstationary Hydrodynamic Equations (Navier–Stokes equations) 616
10.3.4. Other Equations 628

11. Equations of Higher Orders **631**

11.1. Equations Involving the First Derivative in t and Linear in the Highest Derivative 631
11.1.1. Fifth-Order Equations 631
11.1.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(x, t, w)$ 633
11.1.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(w)\frac{\partial w}{\partial x}$ 635
11.1.4. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(x, t, w)\frac{\partial w}{\partial x} + g(x, t, w)$ 637
11.1.5. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + F\big(x, t, w, \frac{\partial w}{\partial x}\big)$ 640
11.1.6. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^{n-1} w}{\partial x^{n-1}}\big)$ 645
11.1.7. Equations of the Form $\frac{\partial w}{\partial t} = aw\frac{\partial^n w}{\partial x^n} + f(x, t, w)\frac{\partial w}{\partial x} + g(x, t, w)$ 647
11.1.8. Other Equations 648

11.2. General Form Equations Involving the First Derivative in t 651
11.2.1. Equations of the Form $\frac{\partial w}{\partial t} = F\big(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 651
11.2.2. Equations of the Form $\frac{\partial w}{\partial t} = F\big(t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 656
11.2.3. Equations of the Form $\frac{\partial w}{\partial t} = F\big(x, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 659
11.2.4. Equations of the Form $\frac{\partial w}{\partial t} = F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 662

11.3. Equations Involving the Second Derivative in t 666
11.3.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + f(x, t, w)$ 666
11.3.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + F\big(x, t, w, \frac{\partial w}{\partial x}\big)$ 667
11.3.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^{n-1} w}{\partial x^{n-1}}\big)$ 671
11.3.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^n w}{\partial x^n} + f(x, t, w)\frac{\partial w}{\partial x} + g(x, t, w)$ 673
11.3.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 675

11.4. Other Equations 676
11.4.1. Equations Involving Mixed Derivatives 676
11.4.2. Equations Involving $\frac{\partial^n w}{\partial x^n}$ and $\frac{\partial^m w}{\partial y^m}$ 680

Supplements. Exact Methods for Solving Nonlinear Partial Differential Equations **683**

S.1. Classification of Second-Order Semilinear Partial Differential Equations in Two Independent Variables 683
S.1.1. Types of Equations. Characteristic Equation 683
S.1.2. Canonical Form of Parabolic Equations 683
S.1.3. Canonical Form of Hyperbolic Equations 684
S.1.4. Canonical Form of Elliptic Equations 684

S.2. Transformations of Equations of Mathematical Physics 685
S.2.1. Point Transformations 685
S.2.2. Hodograph Transformation 686
S.2.3. Contact Transformations. Legendre and Euler Transformations 688
S.2.4. Bäcklund Transformations. Differential Substitutions 690

S.3. Traveling-Wave Solutions and Self-Similar Solutions. Similarity Methods 693
S.3.1. Preliminary Remarks 693
S.3.2. Traveling-Wave Solutions. Invariance of Equations Under Translations 694
S.3.3. Self-Similar Solutions. Invariance of Equations Under Scaling Transformations 695
S.3.4. Exponential Self-Similar Solutions. Equations Invariant Under Combined Translation and Scaling 696

S.4. Method of Generalized Separation of Variables 698
S.4.1. Introduction 698
S.4.2. Structure of Generalized Separable Solutions 700
S.4.3. Solution of Functional-Differential Equations by Differentiation 701
S.4.4. Solution of Functional-Differential Equations by Splitting 705
S.4.5. Simplified Scheme for Constructing Generalized Separable Solutions 709
S.4.6. Titov–Galaktionov Method 710

S.5. Method of Functional Separation of Variables 713
S.5.1. Structure of Functional Separable Solutions 713
S.5.2. Special Functional Separable Solutions 713
S.5.3. Differentiation Method 718
S.5.4. Splitting Method. Reduction to a Functional Equation with Two Variables 721
S.5.5. Solutions of Some Nonlinear Functional Equations and Their Applications 723

S.6. Generalized Similarity Reductions of Nonlinear Equations 728
S.6.1. Clarkson–Kruskal Direct Method: a Special Form for Similarity Reduction 728
S.6.2. Clarkson–Kruskal Direct Method: the General Form for Similarity Reduction . . 731
S.6.3. Some Modifications and Generalizations 732

S.7. Group Analysis Methods 735
S.7.1. Classical Method for Symmetry Reductions 735
S.7.2. Nonclassical Method for Symmetry Reductions 744

S.8. Differential Constraints Method 747
S.8.1. Description of the Method 747
S.8.2. First-Order Differential Constraints 749
S.8.3. Second- and Higher-Order Differential Constraints 754
S.8.4. Connection Between the Differential Constraints Method and Other Methods . . 756

S.9. Painlevé Test for Nonlinear Equations of Mathematical Physics 758
S.9.1. Movable Singularities of Solutions of Ordinary Differential Equations 758
S.9.2. Solutions of Partial Differential Equations with a Movable Pole. Description of the Method 760
S.9.3. Examples of the Painlevé Test Applications 761

S.10. Inverse Scattering Method 764
S.10.1. Lax Pair Method 764
S.10.2. Method Based on the Compatibility Condition for Two Linear Equations 766
S.10.3. Method Based on Linear Integral Equations 767

S.11. Conservation Laws 769
S.11.1. Basic Definitions and Examples 769
S.11.2. Equations Admitting Variational Formulation. Noetherian Symmetries 770

S.12. Hyperbolic Systems of Quasilinear Equations 772
S.12.1. Conservation Laws. Some Examples 772
S.12.2. Cauchy Problem, Riemann Problem, and Initial-Boundary Value Problem . . . 773
S.12.3. Characteristic Lines. Hyperbolic Systems. Riemann Invariants 773
S.12.4. Self-Similar Continuous Solutions. Rarefaction Waves 777
S.12.5. Shock Waves. Rankine–Hugoniot Jump Conditions 779
S.12.6. Evolutionary Shocks. Lax Condition (Various Formulations) 780
S.12.7. Solutions for the Riemann Problem 782
S.12.8. Initial-Boundary Value Problems of Special Form 786
S.12.9. Examples of Nonstrict Hyperbolic Systems 786

References **791**

Index **809**