2. $\boldsymbol{y(x+1) - ay(x) = f(x)}$.

First-order constant-coefficient linear nonhomogeneous difference equation.

1°. Solution:

$$y(x) = \Theta(x)a^x + \bar{y}(x),$$

where $\Theta(x) = \Theta(x+1)$ is an arbitrary periodic function with unit period, and $\bar{y}(x)$ is any particular solution of the nonhomogeneous equation.

2°. For $a = 1$ and $f(x) = \sum\limits_{k=0}^{n} b_k x^n$, the nonhomogeneous equation has a particular solution

$$y(x) = \sum_{k=0}^{n} \frac{b_k}{k+1} B_{k+1}(x),$$

where $B_k(x)$ are the Bernoulli polynomials.

The first six polynomials are:

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \quad B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x.$$

The generating function is

$$\frac{te^{xt}}{e^t - 1} \equiv \sum_{k=0}^{n} B_k(x) \frac{t^n}{n!} \qquad (|t| < 2\pi).$$

3°. For $a \neq 1$ and $f(x) = \sum\limits_{k=0}^{n} b_k x^n$, the nonhomogeneous equation has a particular solution $\bar{y}(x) = \sum\limits_{k=0}^{n} A_k x^n$, where the constants A_k are found by the method of undetermined coefficients.

4°. For $f(x) = \sum\limits_{k=1}^{n} b_k e^{\lambda_k x}$, the nonhomogeneous equation has a particular solution

$$\bar{y}(x) = \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{b_k}{e^{\lambda_k} - a} e^{\lambda_k x} & \text{if} \quad a \neq e^{\lambda_m}, \\ \displaystyle b_m x e^{\lambda_m (x-1)} + \sum_{k=1,\, k\neq m}^{n} \frac{b_k}{e^{\lambda_k} - a} e^{\lambda_k x} & \text{if} \quad a = e^{\lambda_m}, \end{cases}$$

where $m = 1, \ldots, n$.

5°. For $f(x) = \sum\limits_{k=0}^{n} b_k \cos(\beta_k x)$, the nonhomogeneous equation has a particular solution

$$\bar{y}(x) = \sum_{k=0}^{n} \frac{b_k}{a^2 + 1 - 2a\cos\beta_k} \bigl[(\cos\beta_k - a)\cos(\beta_k x) + \sin\beta_k \, \sin(\beta_k x)\bigr].$$

6°. For $f(x) = \sum\limits_{k=1}^{n} b_k \sin(\beta_k x)$, the nonhomogeneous equation has a particular solution

$$\bar{y}(x) = \sum_{k=1}^{n} \frac{b_k}{a^2 + 1 - 2a\cos\beta_k} \bigl[(\cos\beta_k - a)\sin(\beta_k x) - \sin\beta_k \, \cos(\beta_k x)\bigr].$$

References

Kuczma, M., *Functional Equations in a Single Variable*, Polish Scientific Publishers, 1968.

Mirolyubov, A. A., and Soldatov, M. A., *Linear Nonhomogeneous Difference Equations* [in Russian], Nauka, Moscow, 1986 (pages 35–36).

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.

