2. $y_{n+2} + ay_{n+1} + by_n = f_n.$

This is a ***second-order linear nonhomogeneous difference equation*** defined on a discrete set of points $x = 0,\ 1,\ 2,\ \dots$ The notation $y_n = y(n)$, $f_n = f(n)$ is adopted here.

Suppose λ_1 and λ_2 are roots of the characteristic equations

$$\lambda^2 + a\lambda + b = 0. \tag{1}$$

1°. If $\lambda_1 \neq \lambda_2$, the general solution of the difference equation is expressed as

$$y_n = y_1 \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} - y_0 b \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} + \sum_{k=2}^{n} f_{n-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}, \tag{2}$$

where y_1 and y_0 are arbitrary constants (the values of the unknown function at the first two points).

In the case of complex conjugate roots in solution (2), one should separate the real and imaginary parts.

If $\lambda_1 = \lambda_2$, the general solution of the difference equation is given by

$$y_n = y_1 n \lambda_1^{n-1} - y_0 b (n-1) \lambda_1^{n-2} + \sum_{k=2}^{n} f_{n-k} (k-1) \lambda_1^{k-2}.$$

2°. In boundary value problems, a finite set of points, $x = 0, 1, \dots, N$, is often considered; the initial and terminal values of the unknown function, y_0 and y_N, are also prescribed. It is required to find $y_n \equiv y(x)|_{x=n}$ for $1 \le n \le N-1$.

If $\lambda_1 \neq \lambda_2$, the solution has the form

$$y_n = y_0 \frac{\lambda_1^N \lambda_2^n - \lambda_1^n \lambda_2^N}{\lambda_1^N - \lambda_2^N} + y_N \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^N - \lambda_2^N} + \sum_{k=2}^{n} f_{n-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2} - \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^N - \lambda_2^N} \sum_{k=2}^{N} f_{N-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}.$$

For $n = 1$, the first sum is zero.

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