



## 2. $y_{n+2} + ay_{n+1} + by_n = f_n$ .

This is a **second-order linear nonhomogeneous difference equation** defined on a discrete set of points  $x = 0, 1, 2, \dots$ . The notation  $y_n = y(n)$ ,  $f_n = f(n)$  is adopted here.

Suppose  $\lambda_1$  and  $\lambda_2$  are roots of the characteristic equations

$$\lambda^2 + a\lambda + b = 0. \tag{1}$$

1°. If  $\lambda_1 \neq \lambda_2$ , the general solution of the difference equation is expressed as

$$y_n = y_1 \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} - y_0 b \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} + \sum_{k=2}^n f_{n-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}, \tag{2}$$

where  $y_1$  and  $y_0$  are arbitrary constants (the values of the unknown function at the first two points).

In the case of complex conjugate roots in solution (2), one should separate the real and imaginary parts.

If  $\lambda_1 = \lambda_2$ , the general solution of the difference equation is given by

$$y_n = y_1 n \lambda_1^{n-1} - y_0 b (n-1) \lambda_1^{n-2} + \sum_{k=2}^n f_{n-k} (k-1) \lambda_1^{k-2}.$$

2°. In boundary value problems, a finite set of points,  $x = 0, 1, \dots, N$ , is often considered; the initial and terminal values of the unknown function,  $y_0$  and  $y_N$ , are also prescribed. It is required to find  $y_n \equiv y(x)|_{x=n}$  for  $1 \leq n \leq N - 1$ .

If  $\lambda_1 \neq \lambda_2$ , the solution has the form

$$y_n = y_0 \frac{\lambda_1^N \lambda_2^n - \lambda_1^n \lambda_2^N}{\lambda_1^N - \lambda_2^N} + y_N \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^N - \lambda_2^N} + \sum_{k=2}^n f_{n-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2} - \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^N - \lambda_2^N} \sum_{k=2}^N f_{N-k} \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}.$$

For  $n = 1$ , the first sum is zero.

### References

**Doetsch, G.**, *Guide to the Applications of the Laplace and Z-Transforms* [in Russian], Nauka, Moscow, 1971 (pages 215, 218); English edition: Van Nostrand Reinhold Co., London, 1971.  
**Polyanin, A. D. and Manzhirov, A. V.**, *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.