3. $\boldsymbol{y(x+2)+ay(x+1)+by(x)=0.}$

Second-order constant-coefficient linear homogeneous difference equation.

The characteristic equation is written out as

$$\lambda^2+a\lambda+b=0. \tag{1}$$

Consider the following cases.

1°. The roots λ_1 and λ_2 of the quadratic equation (1) are real and distinct. Then the general solution of the original finite-difference equation is expressed as

$$y(x)=\Theta_1(x)\lambda_1^x+\Theta_2(x)\lambda_2^x, \tag{2}$$

where $\Theta_1(x)$ and $\Theta_2(x)$ are arbitrary periodic functions with unit period, $\Theta_k(x)=\Theta_k(x+1)$, $k=1,\ 2.$

If $\Theta_k\equiv\text{const}$, it follows from (2) that there are particular solutions

$$y(x)=C_1\lambda_1^x+C_2\lambda_2^x,$$

where C_1 and C_2 are arbitrary constants.

2°. The quadratic equation (1) has equal roots, $\lambda=\lambda_1=\lambda_2$. In this case, the general solution of the functional equation is given by

$$y=\bigl[\Theta_1(x)+x\Theta_2(x)\bigr]\lambda^x.$$

3°. In the case of complex conjugate roots, $\lambda=\rho(\cos\beta\pm i\sin\beta)$, the general solution of the functional equation is expressed as

$$y=\Theta_1(x)\rho^x\cos(\beta x)+\Theta_2(x)\rho^x\sin(\beta x),$$

where $\Theta_1(x)$ and $\Theta_2(x)$ are arbitrary periodic functions with unit period.

References

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