5. $y(x+2) + a(x+1)y(x+1) + bx(x+1)y(x) = 0.$

This functional equation has particular solutions of the form

$$y(x;\lambda) = \int_0^\infty t^{x-1} e^{-t/\lambda}\,dt, \qquad (1)$$

where λ is a root of the quadratic equation

$$\lambda^2 + a\lambda + b = 0. \qquad (2)$$

For the integral on the right-hand side of (1) to converge, the roots of (2) that satisfy the condition $\operatorname{Re}\lambda > 0$ should be selected. If both roots, λ_1 and λ_2, satisfy this condition, the general solution of the original functional equation is expressed as

$$y(x) = \Theta_1(x)y(x,\lambda_1) + \Theta_2(x)y(x,\lambda_2),$$

where $\Theta_1(x)$ and $\Theta_2(x)$ are arbitrary periodic functions with unit period.

Reference

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.

