12. $f_1(x)y(x)+f_2(x)y\left(\dfrac{ax-\beta}{x+b}\right)+f_3(x)y\left(\dfrac{bx+\beta}{a-x}\right)=g(x), \quad \beta=a^2+ab+b^2.$

In the equation, let us substitute first $\dfrac{ax-\beta}{x+b}$ for x and then $\dfrac{bx+\beta}{a-x}$ for x to obtain (the original equation comes first)

$$\begin{aligned} f_1(x)y(x)+f_2(x)y(u)+f_3(x)y(w)&=g(x),\\ f_1(u)y(u)+f_2(u)y(w)+f_3(u)y(x)&=g(u),\\ f_1(w)y(w)+f_2(w)y(x)+f_3(w)y(u)&=g(w), \end{aligned} \tag{1}$$

where

$$u=\frac{ax-\beta}{x+b}, \qquad w=\frac{bx+\beta}{a-x}.$$

Eliminating $y(u)$ and $y(w)$ from the system of linear algebraic equations (1), we arrive at a solution, $y=y(x)$, of the original functional equation.

Reference

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.

