15. $y(x+n)+a_{n-1}y(x+n-1)+\ldots+a_1y(x+1)+a_0y(x)=0.$

***n*th-order constant-coefficient linear homogeneous difference equation.**

Let us write out the characteristic equation:

$$\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_1\lambda+a_0=0. \qquad (1)$$

Consider the following cases.

1°. Suppose the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of equation (1) are all real and distinct. Then the general solution of the original finite-difference equation has the form

$$y(x)=\Theta_1(x)\lambda_1^x+\Theta_2(x)\lambda_2^x+\ldots+\Theta_n(x)\lambda_n^x, \qquad (1)$$

where $\Theta_1(x), \Theta_2(x), \ldots, \Theta_n(x)$ are arbitrary periodic functions with unit period, $\Theta_k(x)=\Theta_k(x+1)$, $k=1, 2, \ldots, n$.

For $\Theta_k(x)\equiv C_k$, it follows from (2) that there is a particular solution

$$y(x)=C_1\lambda_1^x+C_2\lambda_2^x+\ldots+C_n\lambda_n^x,$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

2°. Suppose there are m equal real roots, $\lambda_1=\lambda_2=\cdots=\lambda_m$ $(m\le n)$, the other roots being all real and distinct. Then the solution of the functional equation is given by

$$y=\bigl[\Theta_1(x)+x\Theta_2(x)+\cdots+x^{m-1}\Theta_m(x)\bigr]\lambda_1^x+ \\ +\Theta_{m+1}(x)\lambda_{m+1}^x+\Theta_{m+2}(x)\lambda_{m+2}^x+\cdots+\Theta_n(x)\lambda_n^x.$$

3°. Suppose there are m equal complex conjugate roots, $\lambda=\rho(\cos\beta\pm i\sin\beta)$ $(2m\le n)$, the other roots being all real and distinct. Then, if $\Theta_n(x)\equiv \text{const}_k$, the solution of the functional equation is expressed as

$$\begin{aligned} y&=\rho^x\cos(\beta x)(A_1+A_2x+\cdots+A_mx^{m-1})+\\ &+\rho^x\sin(\beta x)(B_1+B_2x+\cdots+B_mx^{m-1})+\\ &+C_{m+1}\lambda_{m+1}^x+C_{m+2}\lambda_{m+2}^x+\cdots+C_n\lambda_n^x,\end{aligned}$$

where $A_1, \ldots, A_m, B_1, \ldots, B_m, C_{2m+1}, \ldots, C_n$ are arbitrary constants.

References

Kuczma, M., *Functional Equations in a Single Variable*, Polish Scientific Publishers, 1968.

Mathematical Encyclopedia, Vol. 2 [in Russian], Sovetskaya Entsiklopediya, Moscow, 1979 (page 1030).

Mirolyubov, A. A., and Soldatov, M. A., *Linear Homogeneous Difference Equations* [in Russian], Nauka, Moscow, 1981.

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.

