15. \( y(x + n) + a_{n-1} y(x + n - 1) + \ldots + a_1 y(x + 1) + a_0 y(x) = 0. \)

**nth-order constant-coefficient linear homogeneous difference equation.**

Let us write out the characteristic equation:

\[
\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0. \quad (1)
\]

Consider the following cases.

1°. Suppose the roots \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of equation (1) are all real and distinct. Then the general solution of the original finite-difference equation has the form

\[
y(x) = \Theta_1(x)\lambda_1^n + \Theta_2(x)\lambda_2^n + \ldots + \Theta_n(x)\lambda_n^n, \quad (1)
\]

where \( \Theta_1(x), \Theta_2(x), \ldots, \Theta_n(x) \) are arbitrary periodic functions with unit period, \( \Theta_k(x) = \Theta_k(x + 1) \), \( k = 1, 2, \ldots, n \).

For \( \Theta_k(x) = C_k \), it follows from (2) that there is a particular solution

\[
y(x) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_n\lambda_n^n,
\]

where \( C_1, C_2, \ldots, C_n \) are arbitrary constants.

2°. Suppose there are \( m \) equal real roots, \( \lambda_1 = \lambda_2 = \ldots = \lambda_m \) (\( m \leq n \)), the other roots being all real and distinct. Then the solution of the functional equation is given by

\[
y = [\Theta_1(x) + x\Theta_2(x) + \ldots + x^{m-1}\Theta_m(x)]\lambda_1^n + \Theta_{m+1}(x)\lambda_{m+1}^n + \Theta_{m+2}(x)\lambda_{m+2}^n + \ldots + \Theta_n(x)\lambda_n^n.
\]

3°. Suppose there are \( m \) equal complex conjugate roots, \( \lambda = \rho(\cos \beta \pm i \sin \beta) \) (\( 2m \leq n \)), the other roots being all real and distinct. Then, if \( \Theta_n(x) \equiv \text{const}_k \), the solution of the functional equation is expressed as

\[
y = \rho^x \cos(\beta x)(A_1 + A_2x + \ldots + A_m x^{m-1}) + \\
+ \rho^x \sin(\beta x)(B_1 + B_2x + \ldots + B_m x^{m-1}) + \\
+ C_{m+1}\lambda_{m+1}^n + C_{m+2}\lambda_{m+2}^n + \ldots + C_n\lambda_n^n,
\]

where \( A_1, \ldots, A_m, B_1, \ldots, B_m, C_{m+1}, \ldots, C_n \) are arbitrary constants.

**References**


