1. $F\big(x,\ y(x),\ y(x+a)\big) = 0.$

We assume that $a > 0$. Let us rewrite the equation to solve for $y(x + a)$:

$$y(x+a) = f\big(x,\ y(x)\big). \qquad (1)$$

1°. First, we consider the equation on a discrete set of points $x = x_0 + ak$, where k is an integer. Given the initial data $y(x_0)$, we can use (1) to find successively $y(x_0 + a)$, $y(x_0 + 2a)$, etc.

On solving the original equation for $y(x)$, we obtain

$$y(x) = g\big(x,\ y(x+a)\big). \qquad (2)$$

Assuming here $x = x_0 - a$, we can find $y(x_0 - a)$ and then determine $y(x_0 - 2a)$ etc. likewise.

Thus, given initial data, the equation allows finding $y(x)$ at all points $x_0 + ak$, where $k = 0, \pm1, \pm2, \ldots$

2°. Consider the case where x in the equation varies continuously. We assume that $y(x)$ is a continuous function defined arbitrarily on the half-open interval $[0,\ a)$. On setting $x = 0$ in (1), we obtain $y(a)$.

Given $y(x)$ on $[0,\ a]$, one can use (1) to obtain $y(x)$ for $x \in [a,\ 2a]$, then for $x \in [2a,\ 3a]$, etc.

Remark. The case $a < 0$ is reduced with the substitution $z = x + a$ to an equation of the form $F\big(z + b,\ y(z + b),\ y(z)\big) = 0$, where $b = -a > 0$, which was considered above.

Reference

Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations: Exact Solutions (Supplement. Some Functional Equations)* [in Russian], Faktorial, Moscow, 1998.

