15. $f(t) + g(x) + h(x)Q(z) + R(z) = 0$, **where** $z = \varphi(x) + \psi(t)$.

Equations of this type often arise in functional separation of variables in nonlinear PDEs.

1°. Solution:

$$\begin{aligned} f &= -\tfrac{1}{2}A_1A_4\psi^2 + (A_1B_1 + A_2 + A_4B_3)\psi - B_2 - B_1B_3 - B_4,\\ g &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2,\\ h &= A_4\varphi + B_1,\\ Q &= -A_1z + B_3,\\ R &= \tfrac{1}{2}A_1A_4z^2 - (A_2 + A_4B_3)z + B_4, \end{aligned}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

2°. Solution:

$$\begin{aligned} f &= -B_1B_3e^{-A_3\psi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\psi - B_2 - B_4 - \frac{A_1A_4}{A_3^2},\\ g &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2,\\ h &= B_1e^{A_3\varphi} - \frac{A_4}{A_3},\\ Q &= B_3e^{-A_3z} - \frac{A_1}{A_3},\\ R &= \frac{A_4B_3}{A_3}e^{-A_3z} + \left(\frac{A_1A_4}{A_3} - A_2\right)z + B_4, \end{aligned}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

3°. In addition, the functional equation has the two degenerate solutions:

$$f = A_1\psi + B_1, \quad g = A_1\varphi + B_2, \quad h = A_2, \quad R = -A_1z - A_2Q - B_1 - B_2,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants, and

$$f = A_1\psi + B_1, \quad g = A_1\varphi + A_2h + B_2, \quad Q = -A_2, \quad R = -A_1z - B_1 - B_2,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $h = h(x)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants.

Reference

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations (Supplement S.5.5)*, Chapman & Hall/CRC Press, Boca Raton, 2004.

