16. $f(t) + g(x)Q(z) + h(x)R(z) = 0$, **where** $z = \varphi(x) + \psi(t)$.

Equations of this type often arise in functional separation of variables in nonlinear PDEs.

1°. Solution:

$$\begin{aligned} g(x) &= A_2B_1e^{k_1\varphi} + A_2B_2e^{k_2\varphi}, \\ h(x) &= (k_1 - A_1)B_1e^{k_1\varphi} + (k_2 - A_1)B_2e^{k_2\varphi}, \\ Q(z) &= A_3B_3e^{-k_1z} + A_3B_4e^{-k_2z}, \\ R(z) &= (k_1 - A_1)B_3e^{-k_1z} + (k_2 - A_1)B_4e^{-k_2z}, \end{aligned} \tag{1}$$

where $B_1, \ldots, B_4$ are arbitrary constants and k_1 and k_2 are roots of the quadratic equation

$$(k - A_1)(k - A_4) - A_2A_3 = 0.$$

In the degenerate case $k_1 = k_2$, the terms $e^{k_2\varphi}$ and e^{-k_2z} in (1) must be replaced by $\varphi e^{k_1\varphi}$ and ze^{-k_1z}, respectively. In the case of purely imaginary or complex roots, one should extract the real (or imaginary) part of the roots in solution (1).

The function $f(t)$ is determined by the formulas

$$\begin{aligned} B_2 = B_4 = 0 \quad &\Longrightarrow \quad f(t) = [A_2A_3 + (k_1 - A_1)^2]B_1B_3e^{-k_1\psi}, \\ B_1 = B_3 = 0 \quad &\Longrightarrow \quad f(t) = [A_2A_3 + (k_2 - A_1)^2]B_2B_4e^{-k_2\psi}, \\ A_1 = 0 \quad &\Longrightarrow \quad f(t) = (A_2A_3 + k_1^2)B_1B_3e^{-k_1\psi} + (A_2A_3 + k_2^2)B_2B_4e^{-k_2\psi}. \end{aligned} \tag{2}$$

Solution (1), (2) involves arbitrary functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$.

2°. In addition, the functional equation has two degenerate solutions,

$$f = B_1B_2e^{A_1\psi}, \quad g = A_2B_1e^{-A_1\varphi}, \quad h = B_1e^{-A_1\varphi}, \quad R = -B_2e^{A_1z} - A_2Q,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = B_1B_2e^{A_1\psi}, \quad h = -B_1e^{-A_1\varphi} - A_2g, \quad Q = A_2B_2e^{A_1z}, \quad R = B_2e^{A_1z},$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $g = g(x)$ are arbitrary functions, and A_1, A_2, B_1, and B_2 are arbitrary constants.

Reference

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations (Supplement S.5.5)*, Chapman & Hall/CRC Press, Boca Raton, 2004.

