3. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = 0.$

A model equation of gas dynamics. This equation is also encountered in hydrodynamics, multiphase flows, wave theory, acoustics, chemical engineering, and other applications.

1°. General solution:

$$y = xf(w) + \Phi(w),$$

where $\Phi(w)$ is an arbitrary function.

2°. The solution of the Cauchy problem with the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0$$

can be represented in the parametric form

$$y = \xi + \mathcal{F}(\xi)x, \qquad w = \varphi(\xi),$$

where $\mathcal{F}(\xi) = f\bigl(\varphi(\xi)\bigr)$.

3°. Consider the Cauchy problem with the discontinuous initial condition

$$w(0, y) = \begin{cases} w_1 & \text{for } y < 0, \\ w_2 & \text{for } y > 0. \end{cases}$$

It is assumed that $x \ge 0$, $f > 0$ and $f' > 0$ for $w > 0$, $w_1 > 0$, and $w_2 > 0$.

Generalized solution for $w_1 < w_2$:

$$w(x, y) = \begin{cases} w_1 & \text{for } y/x < V_1, \\ f^{-1}(y/x) & \text{for } V_1 \le y/x \le V_2, \\ w_2 & \text{for } y/x > V_2, \end{cases} \qquad \text{where} \quad V_1 = f(w_1), \ \ V_2 = f(w_2).$$

Here f^{-1} is the inverse of the function f, i.e., $f^{-1}\bigl(f(w)\bigr) \equiv w$. This solution is continuous in the half-plane $x > 0$ and describes a "rarefaction wave."

Generalized solution for $w_1 > w_2$:

$$w(x, y) = \begin{cases} w_1 & \text{for } y/x < V, \\ w_2 & \text{for } y/x > V, \end{cases} \qquad \text{where} \quad V = \frac{1}{w_2 - w_1}\int_{w_1}^{w_2} f(w)\,dw.$$

This solution undergoes a discontinuity along the line $y = Vx$ and describes a "shock wave."

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